



Parametric study of the deformation of transversely isotropic discs under diametral compression

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ABSTRACT. The displacement field in a circular disc made of a transversely isotropic material is explored in a parametric manner. The disc is assumed to be loaded by a parabolic distribution of compressive radial stresses along two finite arcs of its periphery in the absence of any tangential (frictional) stresses. Advantage is here taken of a recently introduced closed-form analytic solution for the displacement field developed in an orthotropic disc under diametral compression which was achieved adopting the complex potentials technique for rectilinear anisotropic materials as it was formulated in the pioneering work of S.G. Lekhnitskii. The analytic nature of this solution permits thorough, in-depth exploration of the influence of some crucial parameters on the qualitative and quantitative characteristics of the deformation of transversely isotropic circular discs compressed between the jaws of the device suggested by the International Society for Rock Mechanics for the standardized implementation of the Brazilian-disc test. The parameters considered include the anisotropy ratio (i.e., the ratio of the two elastic moduli characterizing the disc material), the angle between the loading axis and the planes of transverse isotropy and the length of the loaded arcs. Strongly non-linear relationships between these parameters and the components of the displacement field are revealed.

KEYWORDS. Rectilinearly anisotropic materials; Transverse isotropy; Anisotropy ratio; Brazilian-disc test; Displacement field.



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INTRODUCTION

The configuration of a circular disc squeezed between two either plane or curved jaws is quite familiar to engineers, since it is widely used in practical applications (in the standardized form of the Brazilian-disc test [1, 2]) for the estimation of the tensile strength of brittle materials. Although the stress and displacement fields developed in a circular disc under the as above loading scheme are quite complicated and by no means uniaxial, it is widely accepted that, under specific restrictions and limitations [3-7], the fracture load recorded by such an experiment can be used to determine the material tensile strength (according to the simplified formulae proposed by Carneiro [8], Akazawa [9] and Hondros [10]), assuming that fracture starts from the disc center.



At this point, it is emphatically underlined that the analytic solutions, which have been used for the derivation of the as above classical formulae, are valid exclusively for linearly-elastic isotropic materials. In spite of this strict restriction, the as above formulae are quite often used, also, for the determination of the tensile strength of transversely isotropic materials, although it is well known that for non-isotropic materials the tensile strength is not a unique constant but rather it is a function of the orientation of the loading axis with respect to the anisotropy axes [11]. The main reason for this “compromise” is the fact that generally accepted analytic solutions for the stress and displacement fields developed in a finite circular disc made of a material characterized by even the simplest kind of anisotropy, i.e., transverse isotropy, are not as yet available. Indeed, the respective mathematical problem, i.e., that of a finite disc made of transversely isotropic material that is subjected to diametral pressure is extremely complicated and the analytic solutions available for the stress and displacement field are based on quite a few simplifying assumptions which unavoidably restrict seriously their applicability [12, 13]. Among these assumptions, the one perhaps most distanced from experimentally reality, is the simulation of the loading scheme by either a pair of diametral point forces or by a distribution of uniform radial pressure acting along two very “small” arcs of the disc periphery, symmetric with respect to the disc center, of arbitrarily predefined length.

In the direction of relieving some of these restrictions, an analytic, closed-form solution was recently introduced [14] considering an orthotropic disc loaded by a parabolic distribution of radial stresses acting along two finite arcs of the disc periphery. The specific loading scheme is very close to the actual distribution of radial stresses [15, 16] developed along the disc-jaw contact arcs, when an intact isotropic disc is compressed between the curved jaws of the device suggested by the International Society for Rock Mechanics (ISRM) [1], in which case the length of these arcs is not constant but rather it is a function of the load level imposed and the relative stiffness of the disc and jaws materials, as it is expressed by the ratio of the respective elastic moduli [17]. The problem of the orthotropic disc [14] was solved with the aid of Lekhnitskii [18] complex potentials technique whereas regarding a first approximation of the loading scheme and contact length previously mentioned formulae [16], based on Muskhelishvili formalism for isotropic bodies [19], were properly modified.

In the work described here, advantage is taken of the above mentioned solution [14], in an effort to enlighten and quantify the influence of some crucial material and geometric parameters on the displacement field developed in a disc made of a transversely isotropic material. Among the parameters considered here is the degree of anisotropy, or in other words the ratio δ of the two elastic moduli characterizing a transversely isotropic material (recall that $\delta=1$ corresponds to an isotropic material while increasing δ -values indicate stronger anisotropy). The role of the specific parameter is studied in-depth in order to quantify the limit of δ for which the use of analytic solutions developed for isotropic materials could be, perhaps, considered satisfactory approximations, also for transversely isotropic materials. The second parameter, the role of which is explored, is the inclination of the loading axis (in fact the axis of symmetry of the parabolic distribution) with respect to the axis of anisotropy. The role of the specific parameter is decisive since it governs the magnitude of shear deformation developed at the disc center in case the material layers are neither normal nor parallel to the loading direction. In other words, it dictates the shear stresses developed at the center of the disc, which in turn modify the fracture mechanism (with respect to that of the isotropic disc) rendering the validity of the Brazilian-disc test questionable in case of anisotropy. The last parameter studied here is the length of the loaded arcs. Although it is generally accepted that the specific quantity plays a rather minor role concerning the stress and strain fields at the center of the disc, it is known that it strongly influences the disc deformation in the immediate vicinity of the disc-jaw contact area [15, 16]. From this point of view it is definitely concluded that ignoring the length of the contact arc could yield erroneous results, since it is quite possible that fracture starts from the disc-jaw interface rather than from the center of the disc.

The results of the study indicate that the dependence of the strain and displacement fields on the as above parameters (and especially on the degree of anisotropy) is strongly non-linear and the room for maneuver (i.e., using solutions for isotropic discs to approximately describe the tensile strength of transversely isotropic materials) is rather limited. It is therefore concluded that there is a demanding need for novel standards for the determination of the tensile strength of the specific type of materials which should lie on analytical solutions capable to properly describe the stress and displacement fields in a transversely isotropic finite circular disc under parabolic diametral compression.

THE DISPLACEMENT FIELD IN A TRANSVERSELY ISOTROPIC DISC UNDER DIAMETRAL COMPRESSION

A circular disc of radius R and thickness d , made of a transversely isotropic material (or for brevity “transtropic” material, a term quoted in [18]) is in equilibrium, in the absence of any kind of friction, between two curved jaws, compressed against each other by a force P_{frame} (Fig.1). The radius of curvature of the jaws ranges from $1.5R$ (case corresponding to the ISRM standardization for implementing the Brazilian-disc test [1]) to infinity (case for the relevant ASTM standardized experimental procedure [2], i.e., plane loading platens). P_{frame} acts in the disc cross-section, which,

normally intersected by the planes of isotropy, is a plane of elastic symmetry. For the specific purely plane strain configuration, the present study is concerned with a thorough parametric investigation of the displacement and strain fields developed all over the disc cross-section.

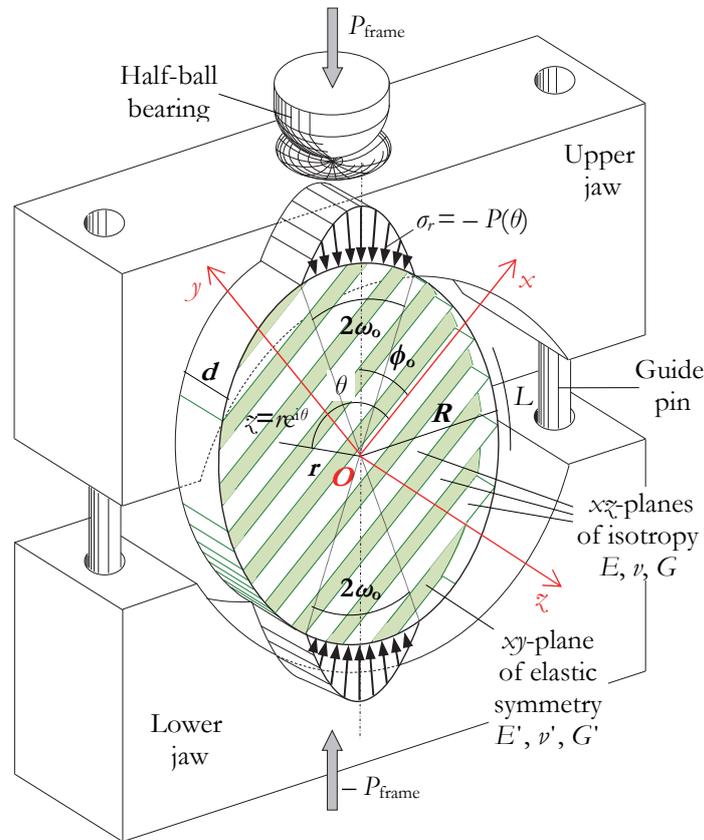


Figure 1: A disc made of a transversely isotropic material compressed between metallic jaws of radius ranging from $1.5R$ to infinity: Configuration of the analytic problem and definition of symbols.

In this context, a Cartesian co-ordinate system $\{O; x; y; z\}$ is considered at the disc center so that the xz -plane to be parallel to the planes of isotropy, the so-called strong direction, and the xy -plane to be parallel to the disc cross-section, known as the weak direction. The strong direction is characterized by a Young modulus E , a Poisson ratio ν and a shear modulus $G=E/(2(1+\nu))$; for the weak direction the respective engineering constants are E' , ν' and G' (actually, ν' characterizes dilatation/contraction in xz -planes of isotropy for compression/tension along y -direction). The force P_{frame} exerted by the loading frame to the disc through the metallic jaws forms an arbitrary angle ϕ_0 with respect to x -axis, i.e., the strong direction (Fig.1). In addition, and taking advantage of the isotropic disc-jaw contact problem, it is here assumed, in a first approximation, that P_{frame} is parabolically distributed along two finite arcs of the disc periphery L , according to the law [16]:

$$P(\theta) = P_c \left[1 - \sin^2(\phi_0 - \theta) / \sin^2 \omega_0 \right] \quad (1)$$

In Eq.(1) it is assumed that $P(\theta) > 0$ so that the pressure on the loaded rim will be $\sigma_r = -P(\theta)$. Moreover, it holds that $P_c = \max\{P(\theta)\}$ and angle ω_0 corresponds to the half loaded arc. These two quantities may equally well be defined in two different ways at one convenience. Namely, assuming without loss of generality that the jaw radius is equal to $1.5R$ and resorting to the isotropic disc-jaw contact problem, one may postulate [14]:

$$\omega_0(\phi_0) = \text{Arc sin} \sqrt{\frac{6K(\phi_0)P_{frame}}{\pi R d}}, \quad P_c(\phi_0) = \sqrt{\frac{3\pi P_{frame}}{32K(\phi_0)R d}}, \quad K(\phi_0) = \frac{\kappa(\phi_0) + 1}{4G'} + \frac{\kappa_J + 1}{4G_J} \quad (2)$$

where, assuming plane strain conditions, it holds that [14]:



$$\kappa(\phi_o) = 3 - 4\nu(\phi_o) = 3 - 4\nu' \sqrt{\frac{E^2}{E'^2} \cos^2 \phi_o + \sin^2 \phi_o} \tag{3}$$

In the above formulae, ν is Muskhelishvili constant [19] while index J indicates the jaw. Alternatively, ignoring the contact problem, one may fix arbitrarily the length of the loaded arc, i.e., the value of ω_o , in which case (keeping P_{frame} constant) [16] it would be written:

$$P_c = \frac{2P_{frame} \sin^2 \omega_o}{Rd(\sin 2\omega_o - 2\omega_o \cos 2\omega_o)} \quad \text{or} \quad P_c = \frac{3P_{frame}}{4Rd \sin \omega_o} \tag{4}$$

(the two expressions of Eq.(4) are equivalent, especially for small to moderate ω_o -values, with the latter, however, being the most adequate one in the complex form representation of the resultant force in [19]).

Under the above considerations and assuming, also, that both the displacement and rigid body rotation of the disc center are equal to zero, the displacement field on any point of the “transtropic” disc cross-section was recently obtained [14] by employing Lekhnitskiĭ formalism [18, 20]. According to that solution [14], the Cartesian components of the displacement field were found to be described by the following equations (where \Re indicates the real part):

$$u(x, y) = \left[\beta_{11} \sigma_x^{(0)} + \beta_{12} \sigma_y^{(0)} \right] x + \frac{\beta_{66}}{2} \tau_{xy}^{(0)} y + 2\Re \left\{ p_1 \left[A_3 P_{13}(\tilde{z}_1) + \sum_{m=5,7,9,\dots} A_m P_{1m}(\tilde{z}_1) \right] + p_2 \left[B_3 P_{23}(\tilde{z}_2) + \sum_{m=5,7,9,\dots} B_m P_{2m}(\tilde{z}_2) \right] \right\} \tag{5}$$

$$v(x, y) = \left[\beta_{12} \sigma_x^{(0)} + \beta_{22} \sigma_y^{(0)} \right] y + \frac{\beta_{66}}{2} \tau_{xy}^{(0)} x + 2\Re \left\{ q_1 \left[A_3 P_{13}(\tilde{z}_1) + \sum_{m=5,7,9,\dots} A_m P_{1m}(\tilde{z}_1) \right] + q_2 \left[B_3 P_{23}(\tilde{z}_2) + \sum_{m=5,7,9,\dots} B_m P_{2m}(\tilde{z}_2) \right] \right\} \tag{6}$$

In the above formulae, $\tilde{z}_j, j=1,2$, are the so-called complicated complex variables (in contrast with the ordinary complex variable $z=x+iy=re^{i\theta}$, Fig.1), defined as:

$$\tilde{z}_1 = x + i\beta_1 y, \quad \tilde{z}_2 = x + i\beta_2 y \quad \text{or} \quad \tilde{z}_1 = r \cos \theta + i\beta_1 r \sin \theta, \quad \tilde{z}_2 = r \cos \theta + i\beta_2 r \sin \theta, \quad 0 \leq r \leq R \tag{7}$$

where:

$$\left. \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \right\} = \sqrt{\frac{-(2\beta_{12} + \beta_{66}) \mp \sqrt{(2\beta_{12} + \beta_{66})^2 - 4\beta_{11}\beta_{22}}}{2\beta_{11}}} \in \mathbb{R}, \quad (\beta_1 > \beta_2) \tag{8}$$

and β_{ij} are the so-called reduced elastic constants defined as:

$$\beta_{11} = \frac{1-\nu^2}{E}, \quad \beta_{12} = -\nu' \frac{1+\nu}{E'}, \quad \beta_{22} = \frac{1}{E'} \left(1 - \frac{E}{E'} \nu'^2 \right), \quad \beta_{66} = \frac{1}{G'} \tag{9}$$

Moreover,

$$p_1 = \beta_{11} \mu_1^2 + \beta_{12}, \quad p_2 = \beta_{11} \mu_2^2 + \beta_{12}, \quad q_1 = \beta_{12} \mu_1 + \beta_{22} / \mu_1, \quad q_2 = \beta_{12} \mu_2 + \beta_{22} / \mu_2 \tag{10}$$

with $\mu_j, j=1,2$ being the so-called complex parameters reading as:

$$\mu_1 = i\beta_1, \quad \mu_2 = i\beta_2 \tag{11}$$



In addition, with $m=3,5,7,9,\dots$,

$$A_m = \frac{-\beta_2(1-t_2^m)\Re a_m + (1+t_2^m)\Im b_m}{\beta_2(1+t_1^m)(1-t_2^m) - \beta_1(1-t_1^m)(1+t_2^m)} - i \frac{\beta_2(1+t_2^m)\Im a_m + (1-t_2^m)\Re b_m}{\beta_2(1-t_1^m)(1+t_2^m) - \beta_1(1+t_1^m)(1-t_2^m)} \quad (12)$$

$$B_m = \frac{\beta_1(1-t_1^m)\Re a_m - (1+t_1^m)\Im b_m}{\beta_2(1+t_1^m)(1-t_2^m) - \beta_1(1-t_1^m)(1+t_2^m)} + i \frac{\beta_1(1+t_1^m)\Im a_m + (1-t_1^m)\Re b_m}{\beta_2(1-t_1^m)(1+t_2^m) - \beta_1(1+t_1^m)(1-t_2^m)}$$

where

$$t_{1,2} = (1 + i\mu_{1,2}) / (1 - i\mu_{1,2}) \quad (13)$$

and $\Re a_m, \Re b_m, \Im a_m, \Im b_m$ are the real and imaginary parts of the following constants (as coefficients of the Fourier series representation of the boundary conditions on L):

$$\begin{aligned}
 \left. \begin{matrix} a_3 \\ b_3 \end{matrix} \right\} &= \left\{ \begin{matrix} -1 \\ +i \end{matrix} \right\} \frac{P_c R}{6\pi} \left[\sin 2\omega_0 - \frac{1}{2\sin^2 \omega_0} \left(\sin 2\omega_0 - \omega_0 - \frac{\sin 2\omega_0 \cos 2\omega_0}{2} \right) \mp \frac{\sin 4\omega_0}{4} e^{-i2\phi_0} \right. \\
 &\quad \left. \pm \frac{e^{-i2\phi_0}}{2\sin^2 \omega_0} \left(\frac{\sin 4\omega_0}{2} - \frac{2\cos 2\omega_0 \sin 4\omega_0 - \sin 2\omega_0 \cos 4\omega_0}{3} \right) \right] e^{-i2\phi_0} \\
 \left. \begin{matrix} a_m \\ b_m \end{matrix} \right\} &= \left\{ \begin{matrix} -1 \\ +i \end{matrix} \right\} \frac{P_c R}{2\pi m} \left[\frac{\sin(m-1)\omega_0}{m-1} - \frac{1}{2\sin^2 \omega_0} \left[\frac{\sin(m-1)\omega_0}{m-1} \right. \right. \\
 &\quad \left. \left. + \frac{(m-1)\cos 2\omega_0 \sin(m-1)\omega_0 - 2\sin 2\omega_0 \cos(m-1)\omega_0}{4-(m-1)^2} \right] \right\} (1 - \cos m\pi) e^{-i(m-1)\phi_0} \\
 &\quad + \left\{ \begin{matrix} 1 \\ +i \end{matrix} \right\} \frac{P_c R}{2\pi m} \left[\frac{\sin(m+1)\omega_0}{m+1} - \frac{1}{2\sin^2 \omega_0} \left[\frac{\sin(m+1)\omega_0}{m+1} \right. \right. \\
 &\quad \left. \left. + \frac{(m+1)\cos 2\omega_0 \sin(m+1)\omega_0 - 2\sin 2\omega_0 \cos(m+1)\omega_0}{4-(m+1)^2} \right] \right\} (1 - \cos m\pi) e^{-i(m+1)\phi_0} \quad (m=5,7,\dots)
 \end{aligned} \quad (14)$$

What is more, and again for $m=3,5,7,9,\dots$,

$$P_{1m,2m}(\tilde{x}_{1,2}) = \frac{-1}{R^m (1 - i\mu_{1,2})^m} \left\{ \left[\tilde{x}_{1,2} + \sqrt{\tilde{x}_{1,2}^2 - R^2(1 + \mu_{1,2}^2)} \right]^m + \left[\tilde{x}_{1,2} - \sqrt{\tilde{x}_{1,2}^2 - R^2(1 + \mu_{1,2}^2)} \right]^m \right\} \quad (15)$$

Finally,

$$\sigma_x^{(0)} = (\bar{b}_1 - b_1) / (Ri), \quad \sigma_y^{(0)} = (a_1 + \bar{a}_1) / R, \quad \tau_{xy}^{(0)} = (a_1 - \bar{a}_1) / (Ri) = -(b_1 + \bar{b}_1) / R \quad (16)$$

indicate constant stresses throughout the disc cross-section [20], where the respective coefficients of Fourier series representation of the boundary conditions on L read as [14]:

$$\left. \begin{matrix} a_1 \\ b_1 \end{matrix} \right\} = \left\{ \begin{matrix} -1 \\ +i \end{matrix} \right\} \frac{P_c R}{2\pi} \left[2\omega_0 - \frac{2\omega_0 - \sin 2\omega_0}{2\sin^2 \omega_0} \mp \sin 2\omega_0 e^{-i2\phi_0} \pm \frac{e^{-i2\phi_0}}{2\sin^2 \omega_0} \left(\sin 2\omega_0 - \omega_0 - \frac{\sin 2\omega_0 \cos 2\omega_0}{2} \right) \right] \quad (17)$$



THE ROLE OF CRITICAL PARAMETERS ON THE DEFORMATION OF “TRANSTROPIC” DISCS

Taking now advantage of the closed-form expressions for the displacements presented in previous section it is possible to explore crucial features of the deformation induced on a “transtropic” disc under parabolic pressure. As it was mentioned the parameters that will be considered include the anisotropy ratio δ , the inclination of the loading with respect to the planes of transverse isotropy and the length of the two loaded arcs. Before analytic data are presented attention is drawn to the following issues:

- The anisotropy ratio δ is defined as $\delta=E/E'$ and it is assumed that $\delta>1$ since E corresponds to the strong anisotropy direction.
- The term “material” represents, in fact, a specific combination of the values assigned to the material properties characterizing a transversely isotropic material, namely E, ν, E', ν' and G' . Various combinations of these values (mainly of E and E') are considered, in order for the analysis to cover the broadest possible range of rocks and rock-like materials. These materials, when do not correspond to a specific natural material, are denoted as “fictitious” materials. It is underlined that when considering various combinations of the values assigned to E, ν, E' and ν' , attention must be paid to fulfill the necessary condition:

$$\nu_{xy} = \frac{E}{E'}\nu' \tag{18}$$

characterizing the magnitude of dilatation in y -direction due to compression in x -direction, dictated by the symmetry of the strain tensor

- Under the term “actual material” a serpentinous schist with $E=58$ GPa, $\nu=0.34, E'=27$ GPa, $\nu'=0.12$ [11] is implied.
- In all applications use is made of an approximating formula between the material properties which was proposed by Lekhnitskii [18] and is valid for a relatively wide class of rocks:

$$G' \simeq \frac{EE'}{E(1+2\nu')+E'} \tag{19}$$

- Although Eqs.(5, 6) refer to the points z_1 and z_2 , all results obtained on the disc cross-section and on L , refer to the point (x, y) or (r, θ) , through Eqs.(7).

The role of the anisotropy ratio, δ and the load inclination angle ϕ_o

In order for a global overview of the deformation imposed on the “transtropic” disc under parabolic radial pressure to be obtained, it was decided as a first step to pay attention at the immediate vicinity of the disc center, given that this is the point where fracture is expected (or where fracture “must” start), for the results of the Brazilian-disc test to be reliable. In this direction, the components of the strain tensor are plotted in Figs.2 and 3 versus the anisotropy ratio δ , for a series of “fictitious” “transtropic” materials. The values assigned to δ vary from $\delta=1$ (which corresponds to a disc made of an isotropic material) to $\delta=5$, covering well the range of anisotropy ratios that can be met in real rocks and rock-like materials. The mechanical properties of the materials considered are recapitulated in Tab. 1. The radius of the disc was set equal to 50 mm and its thickness (length) equal to 10 mm. A constant overall load equal to $P_{frame}=20$ kN was exerted on the discs. Angle ϕ_o , i.e., the inclination of the axis of symmetry of the parabolic distribution of radial stresses with respect to x -axis, was set equal to $\phi_o=0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$ and 90° .

The dependence of the normal radial strain component ϵ_{rr} (i.e., that along the loading axis) and the normal transverse strain component $\epsilon_{\theta\theta}$ (i.e., that normal to the loading axis) on the anisotropy ratio δ is shown in Figs.2(a, b) for all ϕ_o -values. It is observed from Fig.2a that the normal radial strain ϵ_{rr} is negative (contractive) for all δ -values, as it is obviously expected. For relatively low δ -values ($\delta<1.3$) ϵ_{rr} decreases rapidly, reaching a minimum value for $\delta\approx 1.3$ for all ϕ_o -values without any exception. For higher δ -values ϵ_{rr} increases smoothly. The dependence of ϵ_{rr} on the inclination ϕ_o of the anisotropy planes with respect to the loading line is smooth and monotonous.

The respective behaviour of the transverse strain component $\epsilon_{\theta\theta}$ is somehow “symmetric” to that of ϵ_{rr} (Fig.2b): It is positive (dilatative) for all δ - and ϕ_o -values, it exhibits a clear maximum (again for $\delta\approx 1.3$) and its dependence on angle ϕ_o is also smooth and monotonous.

Of higher importance is the variation of the shear strain $\gamma_{r\theta}$ against the anisotropy ratio δ which is shown in Fig.3. Contrary to what happens at the center of an isotropic ($\delta=1$) disc (where $\gamma_{r\theta}$ is constantly zero for obvious symmetry reasons),



Materials	E [GPa]	E' [GPa]	ν' ($=\nu_{yx}$)	ν_{xy} (<0.5)	δ
“Fictitious” isotropic	75	75	0.30	0.30	1
1	20	15	0.30	0.12	1.33
2	25			0.10	1.67
3	30			0.10	2.00
4	35			0.10	2.33
“Fictitious” “transtropic”	5	40		0.10	2.67
6	45			0.10	3.00
7	50			0.10	3.33
8	55			0.10	3.67
9	60			0.10	4.00
10	65			0.10	4.33
11	70			0.10	4.67
12	75			0.10	5.00

Table 1: The mechanical properties of the materials considered.

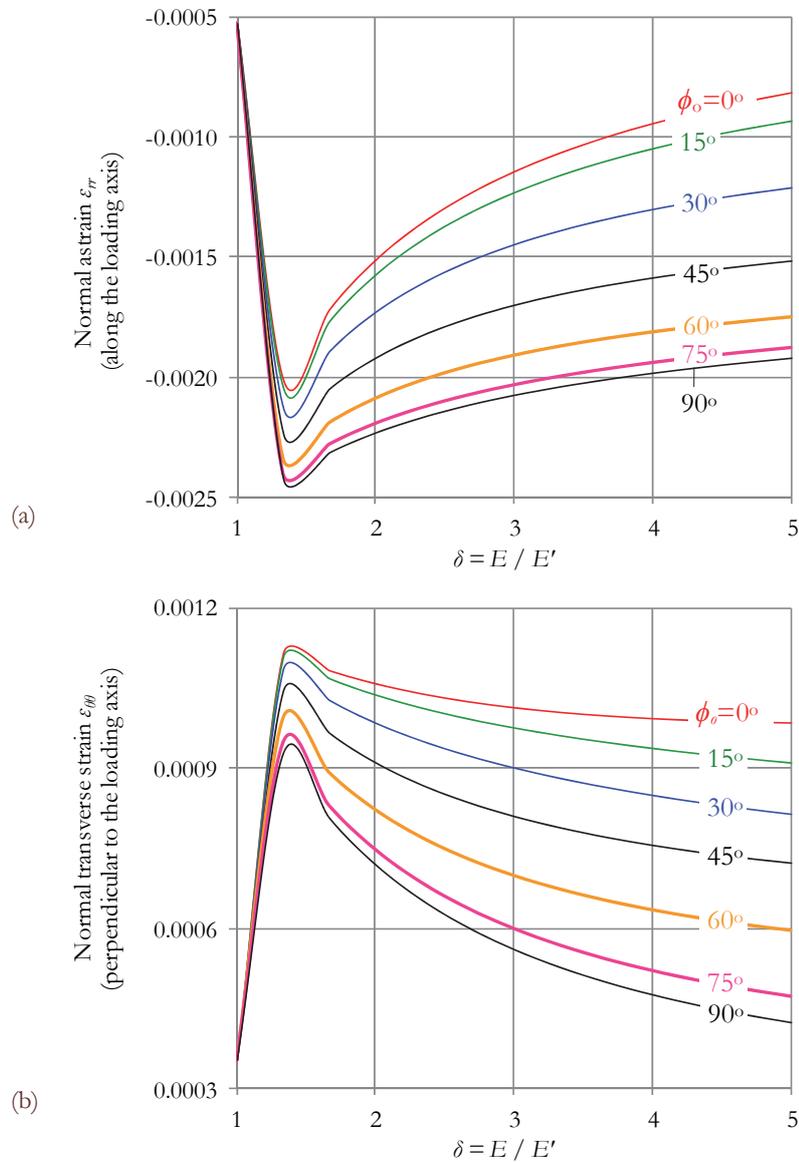


Figure 2: The dependence of the normal radial ε_{rr} (a) and transverse $\varepsilon_{\theta\theta}$ (b) strain on the anisotropy ratio δ for a series of ϕ_0 -values.

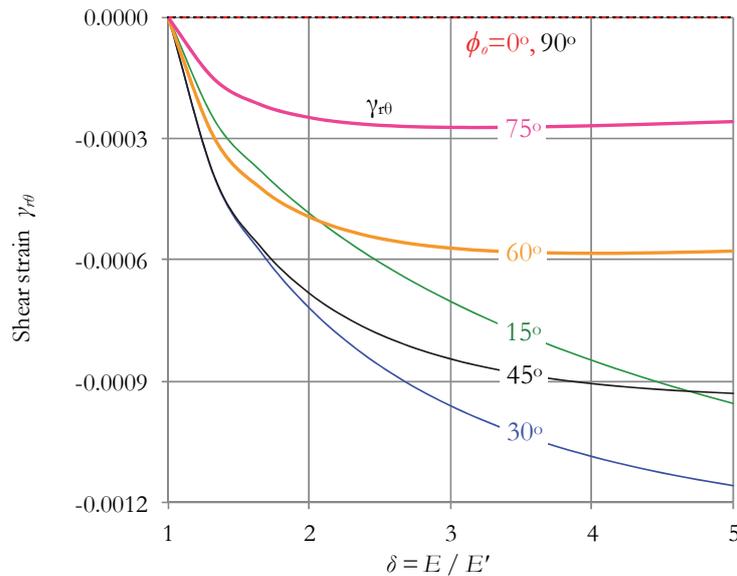


Figure 3: The dependence of the shear strain $\gamma_{r\theta}$ on the anisotropy ratio δ for a series of ϕ_0 -values.

in the case of “transtropic” discs $\gamma_{r\theta}$ is non-zero (excluding the limiting values $\phi_0=0^\circ$, $\phi_0=90^\circ$, for which symmetry imposes zeroing of $\gamma_{r\theta}$) and the deformation in the immediate vicinity of the disc center becomes both distortive and dilatational rather than purely dilatational. The magnitude of $\gamma_{r\theta}$ increases constantly with increasing δ for ϕ_0 -values in the $0^\circ < \phi_0 < 45^\circ$ range. On the contrary, for ϕ_0 -values in the $45^\circ < \phi_0 < 90^\circ$ range $\gamma_{r\theta}$ increases for low δ -values and then it reaches a plateau. The dependence of the magnitude of $\gamma_{r\theta}$ on the angle ϕ_0 is non-monotonous exhibiting its maximum value around $\phi_0 \approx 30^\circ$. For the differences of the deformation between an isotropic and a “transtropic” disc to become more evident, and in order to highlight the loss of some symmetries which are “familiar” in case of isotropic materials, the deformed shape of an element $(\Delta r, \Delta \theta) = (15\pi R/180, 15\pi R/180)$ of the disc cross-section, oriented along the loading line, is drawn in Figs.4(a, b). This element is located in the immediate vicinity of the disc-jaw contact arc, i.e., in the area where the deformation field is extremely amplified due to the “direct” action of the external loading scheme (see the sketches embedded in Figs.4). Both the undeformed element (black dashed line) and the respective deformed ones are drawn: The deformed element for the isotropic configuration is drawn with green line while that for the “transtropic” one with red line. Two geometries were considered, one for $\phi_0=20^\circ$ (Fig.4a) and a second one for $\phi_0=60^\circ$ (Fig.4b).

The materials considered correspond to rocks with relatively low stiffness in order for the deformation to be clearly visible. For the “fictitious” isotropic material it is assumed that $E=E'=4$ GPa, $\nu=\nu'=0.34$ while for the “fictitious” “transtropic” one it is assumed that $E=4$ GPa, $\nu=0.34$, $E'=2$ GPa, $\nu'=\nu_{yx}=0.2$ (and it follows that $\nu_{xy}=0.40 < 0.50$). Again for clarity reasons the value of the load imposed is rather “unrealistic” ($P_{\text{frame}}=100$ kN). Given that the analysis introduced is linearly elastic, it is obvious that the magnitude of the load imposed will only quantitative influence the results; from a qualitative point of view the conclusions drawn are not influenced by the specific parameter.

Finally, it is mentioned that the length of the loading arc ($2\omega_0$) is not constant but rather it is obtained with the aid of the solution of the respective isotropic disc-jaw contact problem. As a result it depends both on the inclination of the loading axis with respect to the anisotropy planes, ϕ_0 , and also on the properties of the disc materials. In this context, for the configuration with $\phi_0=20^\circ$ (Fig.4a) it is calculated that $\omega_0(\phi_0=20^\circ)=23.05^\circ$ for the “fictitious” “transtropic” disc while for the same configuration for the “fictitious” isotropic material it is calculated that $\omega_0(\phi_0=20^\circ)=24.52^\circ$. Similarly for the configuration with $\phi_0=60^\circ$ (Fig.4b) it is determined that $\omega_0(\phi_0=60^\circ)=22.29^\circ$ for the “fictitious” “transtropic” disc and $\omega_0(\phi_0=60^\circ)=24.52^\circ$ for the “fictitious” isotropic disc (obviously for the isotropic disc the length of the contact arc does not depend on the inclination angle ϕ_0 of the loading axis with respect to x -axis of the reference system (see Fig.1)).

The most important conclusion drawn from Figs.4(a, b) (besides the expected differences between the “transtropic” and the isotropic disc in the “rigid body” translation and rotation of the element in question, as intermediate steps from the unstrained to its final strained state), is the evident asymmetry detected in the variation of shear deformation in case of “transtropic” materials, at symmetric points, i.e., of the corner points of the element in question, even when located along the loaded radius. Indeed, while for the isotropic material $\gamma_{r\theta}(r, \theta) = \gamma_{r\theta}(r, \theta + \Delta\theta)$ all along the loaded radius, this is by no means

the case for the “transtropic” material for which clearly $\gamma_{r\theta}(r, \theta) \neq \gamma_{r\theta}(r, \theta + \Delta\theta)$. Moreover, it is observed that the differences in deformation between the particular “fictitious” isotropic and “transtropic” discs are more pronounced for $\phi_o=60^\circ$.

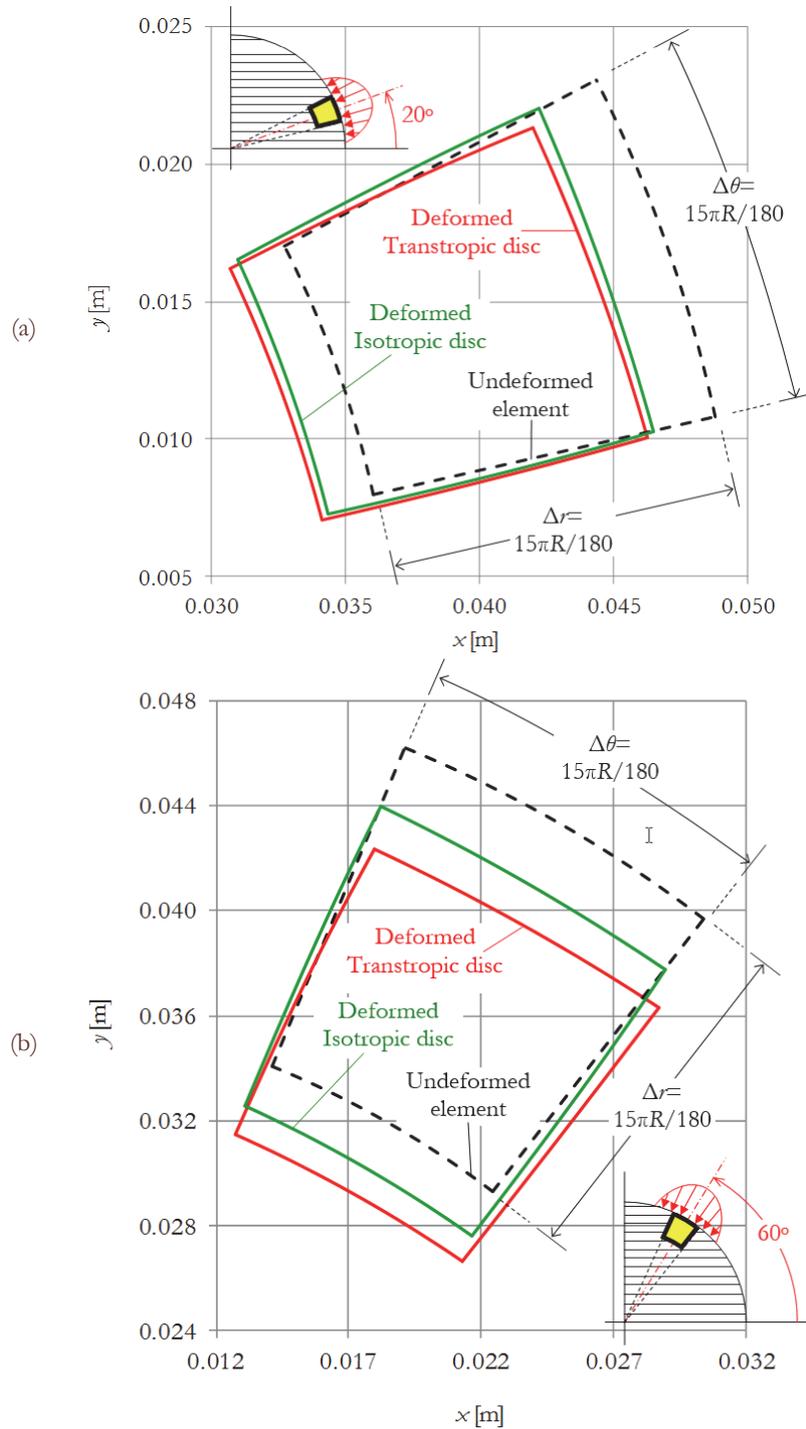


Figure 4: Deformed vs. undeformed (black dashed line) configuration of an element (Δr ; $\Delta \theta$) (oriented along the loading axis and normally to it), for an isotropic (green line) and a “transtropic” (red line) disc, when the loading direction is $\phi_o=20^\circ$ (a) and $\phi_o=60^\circ$ (b).

The role of the length of the contact arc, $2\omega_o$

The specific parameter, i.e., the length of the loaded arc, is in general disregarded in practical applications of the Brazilian-disc test, since it is considered that the stress state at the disc center can be simulated as a biaxial stress field with a con-



stant biaxiality ratio k , equal to $k = \sigma_{\text{compressive}} / \sigma_{\text{tensile}} = 3$, independently of the actual boundary conditions which prevail along the loaded arcs of the disc. This was suggested, perhaps for the first time, by Peltier [21], who arrived at the conclusion that, as long as the loaded arc is not “too large” (indicatively, he mentioned as an upper limit for the length of the loaded arcs a value less than one fifth of the disc diameter), the tensile stresses remain uniform almost all along the length of the loaded diameter. Barenbaum and Brodie [22] as well as Rudnick et al. [24] drew similar conclusions. However, the fact that the stress field at the disc center is more or less insensitive to the actual conditions along the loaded arcs is by no means a necessary and sufficient condition for the validity of the results provided by the Brazilian-disc test: It has been indicated by many researchers that depending on the actual stress field in the immediate vicinity of the loaded arcs it is quite possible that fracture could start far from the disc center, unless certain precautions are taken, rendering the results of the test erroneous [3-7, 24-26]. In the direction of minimizing the as above risk, Mellor and Hawkes [5] suggested, for the first time, the use of curved jaws (instead of plane loading platens) of radius exceeding that of the specimens, suggestions which were the seed for the respective ISRM standard [1]. It is therefore evident that, in spite of the insensitivity of the stress field at the disc center to the actual boundary conditions, the reliability of the final outcomes of the Brazilian-disc test strongly depends on the actual conditions prevailing along the disc loaded arcs and on their length. As a result, one should consider, at least qualitatively, the role of the specific parameter (length of the contact arc), independently of whether the disc is made of isotropic or “transtropic” material. Given that the problem for isotropic materials was recently considered thoroughly [17], a first attempt is described here to explore the role of the loaded arcs’ length on the displacement field in case of “transtropic” discs. This attempt is possible because the displacement field in a “transtropic” disc under parabolic pressure is, for the first time, provided here in terms of analytic, closed-form expressions.

In this direction, a disc made of a serpentinous schist (with $E=58$ GPa, $\nu=0.34$, $E'=27$ GPa, $\nu'=0.12$) [11] was considered, compressed between the ISRM jaws, by an overall load equal to $P_{\text{frame}}=20$ kN, which is kept constant for all ω_o -values assigned to the contact semi-arc. Given now that the length of the two contact arcs ($2\omega_o$) is arbitrarily prescribed (rather than being determined from the respective contact problem) the amplitude of the parabolic distribution must be determined through Eqs.(4) (it is here recalled that the two formulae of Eq.(4) are almost equivalent, especially for relatively small ω_o -values). As a first step, it was decided to plot the components of the displacement field along the disc loaded diameter, which is, perhaps, the locus of utmost importance for practical applications.

A relatively wide interval of ω_o -values was studied, ranging from a very small one equal to $\omega_o=2^\circ$ (which in fact approaches the diametral compression by a pair of “point” forces, or equivalently to compression of very stiff discs which are not significantly deformed at the disc-jaw interface) to a rather very high one equal to $\omega_o=30^\circ$ (which corresponds to diametral compression of discs with extremely low stiffness).

Concerning angle ϕ_o , three characteristic values were studied, namely $\phi_o=0^\circ$ (loading parallel to the planes of isotropy, or equivalently x -axis parallel to the loading line), $\phi_o=90^\circ$ (loading normal to the planes of isotropy, or equivalently x -axis normal to the loading line) and $\phi_o=30^\circ$ (corresponding to the geometry for which the shear strain at the disc center is maximized independently of the anisotropy ratio, δ (Fig.3)).

The results of the analysis are shown in Figs.5(a, b, c), in which the polar components u_r and u_θ of the displacement field are plotted along the loaded diameter from $r=0$ (the center of the disc) to $r=R$ (or in other words to the point where the disc is in contact to the loading jaw). As it is expected, both for $\phi_o=0^\circ$ and $\phi_o=90^\circ$ the transverse (tangential) component of the displacement field is zero for obvious symmetry reasons. In addition, as it was also expected, the magnitude of the non-zero component of the displacement, i.e., u_r , increases with increasing ϕ_o (for the same point and the same ω_o -value) since for $\phi_o=0^\circ$ loading is exerted along the strong anisotropy axis (characterized by high value of the elastic modulus or equivalently by increased stiffness) while for $\phi_o=90^\circ$ loading is exerted along the weak anisotropy axis (characterized by low value of the elastic modulus or equivalently by decreased stiffness).

In accordance to similar conclusions for isotropic discs [15, 16], it is also here concluded that the influence of angle ω_o on the displacement (and therefore on the strain- and stress-fields) is definitely negligible at the disc center. The role of ω_o becomes significant only for $r \rightarrow R$. The lower limit of r for which ω_o plays some role on the displacement field appears very sensitive to the value of angle ϕ_o . From a quantitative point of view, it can be seen from Fig.5a that for $\phi_o=0^\circ$ and $r=R$ it holds that $u_r=6.5 \times 10^{-5}$ for $\omega_o=2^\circ$ while for $\omega_o=30^\circ$ the respective value is $u_r=3.5 \times 10^{-5}$. In other words, increasing ω_o from 2° to 30° results to a decrease of the radial displacement in the vicinity of the loaded arc by more than 45%. Similar conclusions are drawn also for $\phi_o=90^\circ$ and $\phi_o=30^\circ$. In general, increasing the length of the loading arc weakens the intensity of the displacement field.

Concerning the influence of angle ω_o on the transverse (tangential) component of the displacement field, it can be said that it is almost negligible all along the loaded radius. Indeed, as it can be seen from Fig.5b only for $r > 0.45R$ some differences can be detected, which are much lower compared to the respective ones of u_r .

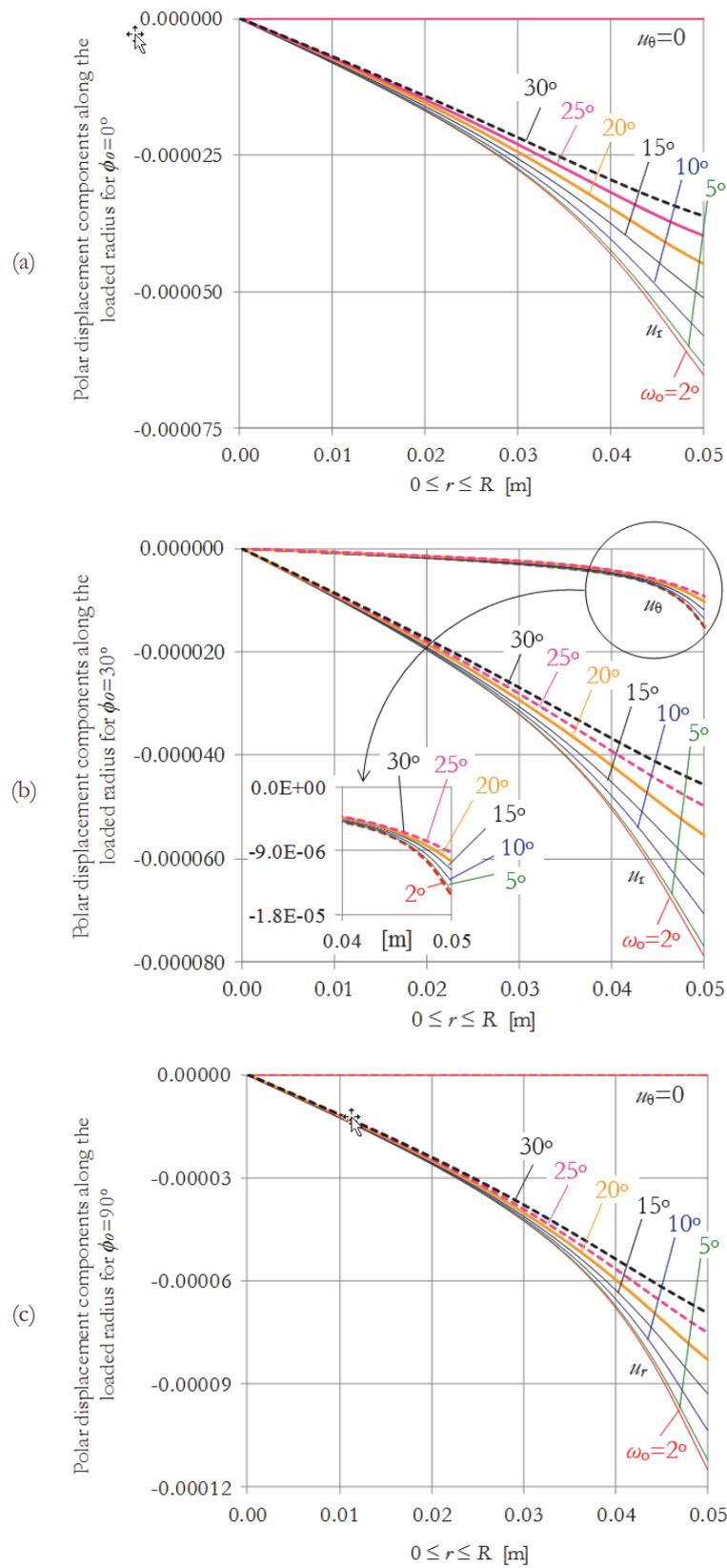


Figure 5: The distribution of the polar-displacement components, u_r , u_θ , along the loaded radius $0 \leq r \leq R$, for $\phi_0=0^\circ$ (a), $\phi_0=30^\circ$ (b) and $\phi_0=90^\circ$ (c) for various values of the contact semi-arc ω_0 equal to $\omega_0=2^\circ, 5^\circ, 10^\circ, 15^\circ, 20^\circ, 25^\circ$ and 30° .

DISCUSSION AND CONCLUDING REMARKS

The displacement field developed in a transversely isotropic disc under diametral compression was determined analytically and in closed form. The main advantage of the present solution is the nature of the analytic expressions, which are relatively easily programmable, providing an overview of the deformed shape either of an element of the disc (see Figs.4(a, b)) or for the disc as a unit. To highlight the specific feature of the solution the deformed shape of a “fictitious” “transtropic” disc is plotted in Figs.6(a, b, c), in juxtaposition to the respective isotropic one, for three characteristic configurations, i.e., $\phi_o=30^\circ$ (Fig.6a), $\phi_o=45^\circ$ (Fig.6b), and $\phi_o=60^\circ$ (Fig.6c). In all cases the transtropic disc is characterized by $E=40$ GPa, $\nu=0.30$, $E'=10$ GPa, $\nu'=v_{yx}=0.12$, $v_{xy}=0.48$ and the isotropic one by $E=E'=40$ GPa, $\nu=\nu'=0.30$. Again the length of the contact arcs is determined from the respective contact problem and the magnitude of the load imposed is “unrealistically” high for the deformed shape to be clearly distinguished from the undeformed one. It is again easily concluded that the deformation of the “transtropic” disc is asymmetric with respect to the loading line. This “degree of asymmetry” depends both on the anisotropy ratio, δ , and also on the value of angle ϕ_o , although this dependence is extremely complicated and by no means monotonous. Moreover, it is interesting to emphasize the fact that for the “transtropic” disc the diameters parallel and normally to the loading direction do not remain linear segments after deformation, but rather they become curved segments, contrary to what happens for the isotropic disc, for which the specific diameters (as expected) remain linear segments. The curvature of the deformed diameters depends (according to an extremely complicated manner) on quite a few parameters, the most important of which are: The orientation of the diameter with respect to the loading line, the distance from the disc center, and (obviously) on the orientation of the load with respect to the planes of isotropy (angle ϕ_o). Analogous conclusions are drawn for the diameters (symmetrically placed with respect to the loaded diameter) connecting the starting and ending points of the loaded arcs. Namely, while in the case of the isotropic disc these diameters exhibit completely symmetric deformation, for the “transtropic” disc the deformation is asymmetric following the general distortion of the deformed configuration as a whole. Moreover, worth mentioning is the diversification in the values of ω_o (corresponding to the half-contact length) between the isotropic and the “transtropic” disc, as well as, among the three deformed configurations of the “transtropic” disc itself; namely, while for the isotropic disc ω_o is constantly equal to 40.04° for all three values of angle ϕ_o (30° , 45° , 60°), this is not the case for the “transtropic” disc where the relevant values of ω_o read respectively as 58.46° , 63.89° and 72.22° (due to the successive decrease of the “transtropic” disc stiffness for increasing ϕ_o -values).

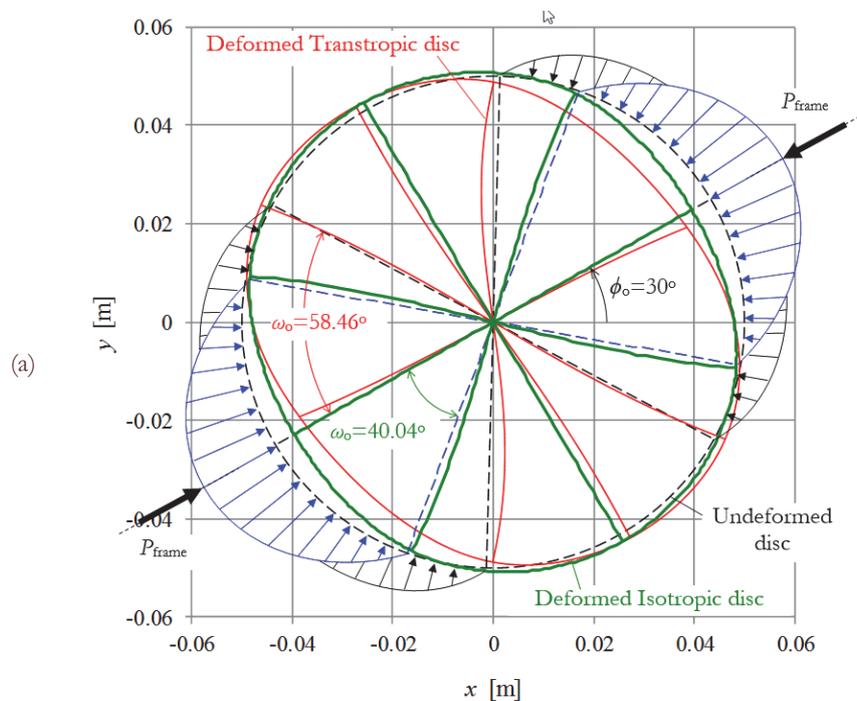


Figure 6: (a) Deformed versus undeformed (black dashed line) configuration of a circular disc under parabolic diametral compression, for an isotropic- (green line) and a “transtropic”- (red line) material, for $\phi_o=30^\circ$.

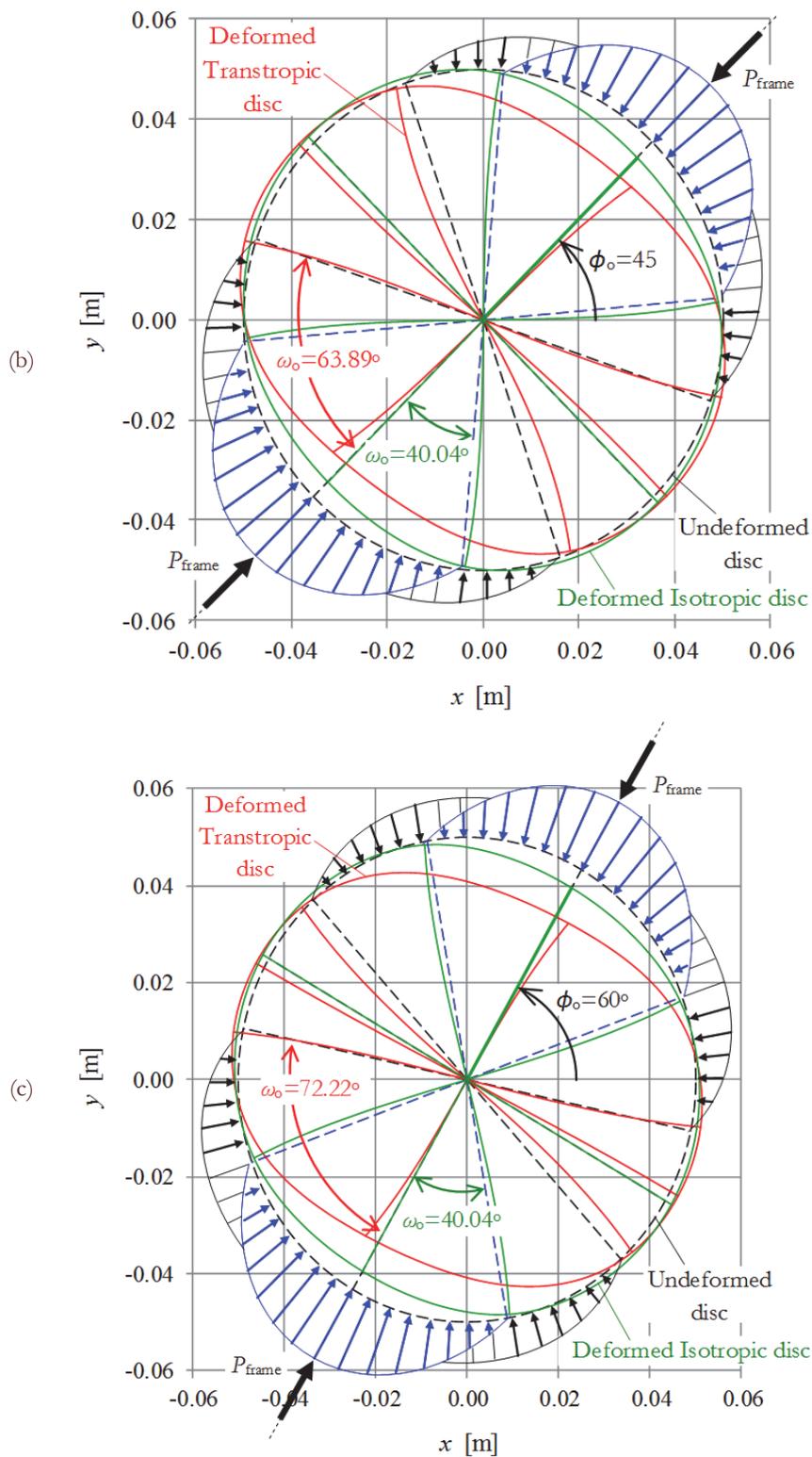


Figure 6 (continued): Deformed versus undeformed (black dashed line) configuration of a circular disc under parabolic diametral compression, for an isotropic- (green line) and a "transtropic"- (red line) material, for $\phi_o = 45^\circ$ (b) and $\phi_o = 60^\circ$ (c).

Besides the above mentioned characteristic, related to the "convenient" and "easy-to-use" (or "easy-to-program") nature of the expressions for the displacement components, the solution introduced cures some critical drawbacks of existing



solutions. The first one is related to the kind of the loading scheme considered (i.e., the parabolic distribution of radial stresses), which approaches closely the actual distribution developed along the disc-jaw interface when the Brazilian-disc test is implemented according to the ISRM standard [1]. In fact the specific distribution is of “cyclic” nature [15, 27], however, it is quite accurately simulated by a parabolic one, like the one adopted here. In any case the parabolic distribution is much closer to experimental reality compared to either a “uniform” one or to a loading scheme consisting of “point” diametral forces.

The second advantage of the present solution is the fact that the loaded arc is not arbitrarily prescribed but rather it is determined from the solution of the respective disc-jaw contact problem. As a result, the present solution takes into account the relative stiffness of the disc and jaw materials, providing, in a first approximation, realistic values for the length of the contact arc. To make the importance of the specific issue clear, it is mentioned characteristically that, depending on modulus of elasticity of the disc material, the length of the contact arc varies within extremely broad limits, ranging from 2° to 3° (for very stiff discs made, for example, of Dionysos marble) [16, 28] to 20° or even 30° (for discs of low stiffness made, for example, of shell-stones) [16, 29].

Another issue that should be thoroughly discussed (also in future studies) is the “suitability” of the anisotropy ratio, δ , to act as a proper parameter that could properly describe the dependence of the strain-field, developed in a “transtropic” disc, on the anisotropy degree of the disc material. To make this point clear, the variation of the shear strain $\gamma_{r\theta}$, developed at the disc center, $\theta = \phi_o = 30^\circ$, is plotted in Fig.7 against an “average stiffness” of the “transtropic” disc material. This “average stiffness” is here defined as the mean value $E_{av} = (E + E')/2$ of the two elastic moduli characterizing a “transtropic” material. To draw Fig.7 a series of fictitious “transtropic” materials were considered with constant anisotropy δ equal to $\delta = 2$. The modulus of elasticity along the “strong” anisotropy axis ranges from $E = 4$ GPa to $E = 80$ GPa, while that along the “weak” anisotropy axis ranges from $E' = 2$ GPa to $E' = 40$ GPa in such a way that δ is kept constant equal to $\delta = 2$. For the Poisson ratios it is considered (for all combination of E and E') that $\nu = 0.30$, and $\nu' = 0.15$. An average external load equal

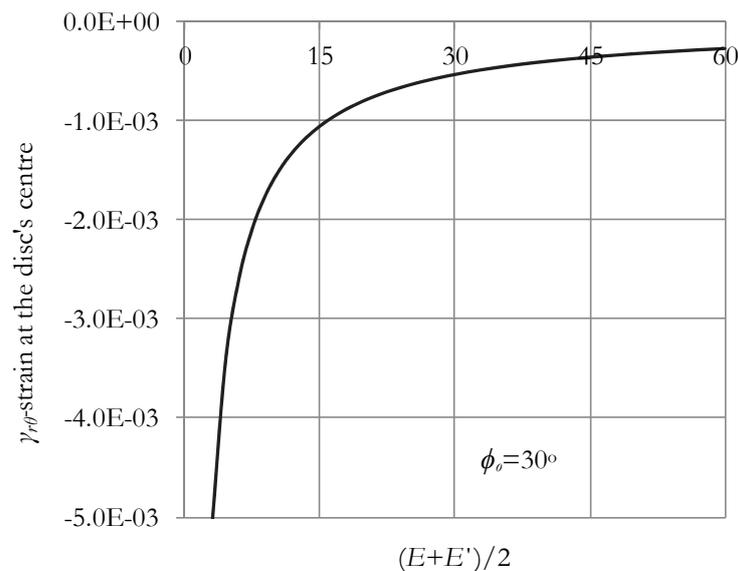


Figure 7: The dependence of the shear strain $\gamma_{r\theta}$ at the disc center on the “average stiffness” $(E+E')/2$, i.e., on the mean value of the two elastic moduli characterizing “transtropic” materials.

to $P_{frame} = 20$ kN is imposed on the discs for all materials’ combinations. The non-linear nature of the dependence shown in Fig.7 is quite striking, indicating that, even for materials with the same δ -value, the shear strain does not decrease linearly with linearly increasing “average stiffness” of the disc material (the term “average stiffness” is a “neologism” and is introduced in an attempt to somehow quantify the stiffness of materials which are characterized by more than one elastic moduli).

Before concluding some limitations, restricting the general applicability of the present solution, should be mentioned. The assumption of zero shear stresses along the loaded arc (in other words the assumption of perfectly smooth interface) is, perhaps, the most critical one. It is widely accepted that these stresses do not influence the stress field at the disc center, however, under specific conditions they could be responsible for premature fracture in the vicinity of the loaded arc, jeopardizing, thus, the validity of the results of the Brazilian-disc test [30-32]. Although it is an extremely complicated



problem, it is absolutely necessary for the role of these stresses to be thoroughly explored. Along the same lines, and despite a first attempt made here to approximately obtain the length of the contact arc of the “transotropic” disc compressed between the ISRM jaws [1] (by properly modifying the relevant formulae of the isotropic disc-jaw contact problem (Eqs.(2), (3))), use of the latter should be also made with consciousness, until further analytical and experimental results are available.

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