A GENERAL MICROMECHANICAL APPROACH TO THE STUDY OF THE NEAR– SURFACE BUCKLING IN FIBROUS COMPOSITES

Yu. N. Lapusta*

This paper describes a general unified approach to the study of buckling- like phenomena and includes a statement of the problems, development of a method for their solution, and a discussion of numerical results. The study is carried out within the framework of the model of a piecewise-uniform medium with the involvement of the three-dimensional linearized theory of stability.

INTRODUCTION

Buckling phenomena as well as the interaction of reinforcing elements and free boundaries are very characteristic of unidirectional composites or fibre-reinforced outer layers of composites or products under compression. The most accurate results on these phenomena in composites can be obtained through employment of rigorous micromechanical approaches when three-dimensional linearized equations [1] are applied to each constituent of the internal structure. Here such problems are considered on the micro-level as applied to a system of fibres near the matrix boundary.

MAIN ASSUMPTIONS

We introduce local Lagrangian coordinates (r_q, θ_q, z_q) and (x_q, y_q, x_{3q}) coinciding in the initial state with a circular cylindrical coordinate systems and a rectangular coordinate systems, respectively.

* S.P. Timoshenko Institute of Mechanics, Nesterov Str.3, Kiev, 252057, Ukraine

The z_q axes are directed along the fibre axes while the x_{3q} axes lie at the free boundary of the matrix. The quantities corresponding to the fibres are denoted by the superscript "a" and the quantities relating to the matrix - by the superscript "m". "q" denotes the number of a certain fibre. In a general case, the fibres and matrix are assumed to consist of transversally isotropic compressible elastic materials. A compressive loading is applied in such a way that the deformations of the fibres and matrix along the axes z_q are equal to each other: $\varepsilon_{zz}^{0aq} = \varepsilon_{zz}^{0m}$. We assume that there exists a complete or a sliding contact at the interfaces between the fibres and the matrix. For the complete contact we should satisfy the continuity conditions for the forces and displacements at the cylindrical surface $r_q = R$ in the form

$$P_r^{aq} = P_r^m, \qquad P_\theta^{aq} = P_\theta^m, \qquad P_z^{aq} = P_z^m,$$

$$u_r^{aq} = u_r^m, \qquad u_\theta^{aq} = u_\theta^m, \qquad u_z^{aq} = u_z^m, \qquad (r_q = R) \tag{1}$$

For the sliding contact we have

$$P_r^{aq} = P_r^m, \qquad u_r^{aq} = u_r^m, \qquad P_\theta^{aq} = 0,$$

$$P_\theta^m = 0, \qquad P_z^{aq} = 0, \qquad P_z^m, = 0, \qquad (r_q = R) \tag{2}$$

The two cases ((1) and (2)) are treated in a similar manner. Therefore, we perform further consideration on the example of the case (1). We demand satisfaction of the following conditions on the free surface

$$P_{yy}^m = 0, \qquad P_{yx}^m = 0, \qquad P_{yz}^m = 0, \qquad (y_1 = 0). \tag{3}$$

We further assume that the precritical states of the fibres and matrix are homogeneous:

$$\sigma_{zz}^{0aq} \neq \sigma_{zz}^{0m}, \qquad \sigma_{rr}^{0aq} = 0, \qquad \sigma_{\theta\theta}^{0aq} = 0, \qquad \sigma_{rr}^{0m} = 0, \qquad \sigma_{\theta\theta}^{0m} = 0. \tag{4}$$

The assumption of the homogeneity of the precritical states of the fibres and matrix is based on the results presented in [2]. Formulae (4) are exact when the transverse expansion coefficients of the fibres and the matrix coincide.

FORMULATION OF THE PROBLEMS AND THEIR SOLUTION

Equilibrium equations for the fibres and for the matrix in terms of displacement disturbances have the form

$$[\omega_{it\alpha\beta}^{aq} u_{\alpha,\beta}^{aq}]_{,i} = 0, \qquad [\omega_{it\alpha\beta}^{m} u_{\alpha,\beta}^{m}]_{,i} = 0,$$

where coefficients $\omega_{it\alpha\beta}^{aq}$ and $\omega_{it\alpha\beta}^{m}$ depend on the precritical states and properties of the materials of the fibres and matrix, respectively. Expressions for these coefficients for various material models and different variants of the linearized three-dimensional theory of stability can be obtained from [1]. Employing the general solutions of the three-dimensional linearized equilibrium equations [1] for homogeneous precritical states to each component of the composite, we reduce the problem to the following formulation in terms of the potentials ψ^m, χ^m, ψ^{aq} and χ^{aq}

It is necessary to find the minimal nontrivial solutions of the equations

$$(\Delta_1 + \zeta_1^{m2}\frac{\partial^2}{\partial z_1^2})\psi^m = 0, \qquad (\Delta_1 + \zeta_2^{m2}\frac{\partial^2}{\partial z_1^2})(\Delta_1 + \zeta_3^{m2}\frac{\partial^2}{\partial z_1^2})\chi^m = 0, \tag{5}$$

written down for the matrix and the following equations for the fibres

$$(\Delta_1 + \zeta_1^{a2}\frac{\partial^2}{\partial z_1^2})\psi^{aq} = 0, \qquad (\Delta_1 + \zeta_2^{a2}\frac{\partial^2}{\partial z_1^2})(\Delta_1 + \zeta_3^{a2}\frac{\partial^2}{\partial z_1^2})\chi^{aq} = 0, \tag{6}$$

under conditions (1), (3) and the conditions of decay of the matrix displacements at infinity in the form

$$u_r^m \to 0, \qquad u_\theta^m \to 0, \qquad u_z^m \to 0 \qquad (y_q \to \infty) \tag{7}$$

Expressions for the coefficients ζ_j^{m2} and ζ_j^{a2}, depending on the $\omega_{it\alpha\beta}^m$ and $\omega_{it\alpha\beta}^{aq}$, respectively, have the form

$$\zeta_1^2 = \omega_{3113}\omega_{1221}^{-1}; \qquad \zeta_{2,3}^2 = c \pm (c^2 - \omega_{3333}\omega_{1111}^{-1}\omega_{3113}\omega_{1331}^{-1})^{0.5};$$

$$2c\omega_{1111}\omega_{1331} = \omega_{1111}\omega_{3333} + \omega_{1331}\omega_{3113} - \omega_{3131}\omega_{1133}(\omega_{1313} + \omega_{3311}) \tag{8}$$

It is necessary to add in (8) the appropriate superscripts "m" or "a", denoting the correspondence of a value to the matrix or the fibres. Usually, for elastic materials, $\zeta_2^{a2} \neq \zeta_3^{a2}$, $\zeta_2^{m2} \neq \zeta_3^{m2}$ and they are positive real numbers.

We search solutions in the most general form including all possible stability loss with allowance for the interaction between the fibres and the free surface. For the matrix, we have:

$$\psi^m = \gamma \sin\gamma z \sum_{q\in M}\{\sum_{n=1}^{\infty} A_{1n,1}^{mq} K_n(\zeta_1^m \gamma r_q)\sin n\theta_q + \sum_{n=0}^{\infty} A_{2n,1}^{mq} K_n(\zeta_1^m \gamma r_q)\cos n\theta_q +$$

$$+\sum_{n=1}^{\infty} A_{1n,1}^{mq}\int_{-\infty}^{+\infty} V_{11 1n}^q(t)\exp(-\zeta_1^m \cosh t\gamma y_q)\sin[\zeta_1^m \sinh t\gamma x_q]dt +$$

$$+\sum_{k=2}^{3}\sum_{n=0}^{\infty} A_{1n,k}^{mq}\int_{-\infty}^{+\infty} V_{11kn}^q(t)\exp(-\sqrt{\zeta_1^{m2} + \zeta_k^{m2}\sinh^2 t}\gamma y_q)\sin[\zeta_k^m \sinh t\gamma x_q]dt +$$

$$\sum_{n=0}^{\infty} A_{2n,1}^{mq}\int_{-\infty}^{+\infty} V_{211n}^q(t)\exp(-\zeta_1^m \cosh t\gamma y_q)\cos[\zeta_1^m \sinh t\gamma x_q]dt + \tag{9}$$

$$+\sum_{k=2}^{3}\sum_{n=0}^{\infty} A_{2n,k}^{mq}\int_{-\infty}^{+\infty} V_{21kn}^q(t)\exp(-\sqrt{\zeta_1^{m2} + \zeta_k^{m2}\sinh^2 t}\gamma y_q)\cos[\zeta_k^m \sinh t\gamma x_q]dt\}$$

Functions χ^m are constructed in the form similar to (9). Here $\gamma = \pi l^{-1}$, l is length of a halfwave of a stability loss form. $A^{mq}_{1n,j}, A^{mq}_{2n,j}$ are unknown coefficients and $V^q_{1ijn}(t), V^q_{2ijn}(t)$ are unknown functions to be defined from the boundary conditions, M is a set of the fibre numbers.

For the fibres we have

$$\psi^{aq} = \gamma \sin \gamma z \{ \sum_{n=1}^{\infty} A^{aq}_{1n,1} I_n(\zeta^a_1 \gamma r_q) \sin n\theta_q + \sum_{n=0}^{\infty} A^{aq}_{2n,1} I_n(\zeta^a_1 \gamma r_q) \cos n\theta_q \}$$

$$\chi^{aq} = \cos \gamma z \{ \sum_{s=2}^{3} A^{aq}_{1n,s} \sum_{n=0}^{\infty} A^{aq}_{1n,s} I_n(\zeta^a_s \gamma r_q) \cos n\theta_q + \sum_{n=1}^{\infty} A^{aq}_{2n,s} I_n(\zeta^a_s \gamma r_q) \sin n\theta_q \} \quad (10)$$

Solutions thus constructed exactly satisfy equations (5), (6). We further distinguish between two main cases. If the system fibres- matrix has some planes of symmetry (as in problems for a single fibre, a symmetric pair of neighbouring fibres in a matrix or a periodic series of fibres) this can simplify further mathematics and implementation of the solution techniques. In this case, as a rule, we can consider individually possible modes of stability loss. In the second case (an arbitrary system of fibres in a matrix without any symmetry) we should use solutions in their most general form, as they appear in (9) and (10). For the latter case, we proceed as described below: 1) We represent solutions for the matrix in one of the rectangular coordinate systems, say (x_1, y_1, z_1) in the form of improper integrals and introduce them into (3). As a result, we obtain systems of equations for the determination of the unknown functions under the integral sign. 2) Then, substituting the solutions for the matrix represented in the coordinate system (r_q, θ_q, z_q) together with the solutions for the fibres into (1), after a change of variables , we obtain an infinite homogeneous system of equations in the form

$$B^{mq}_{\alpha k} X^{mq}_k + B^{aq}_{\alpha k} X^{aq}_k + \sum_{\eta \in M}^{2} (\sum_{n=0}^{\infty} Q^{\eta q 1}_{\alpha k n} X^{m\eta}_n + \sum_{n=0}^{\infty} Q^{\eta q 2}_{\alpha k n} Y^{m\eta}_n) = 0,$$

$$D^{mq}_{\alpha k} Y^{mq}_k + D^{aq}_{\alpha k} Y^{aq}_k + \sum_{\eta \in M}^{2} (\sum_{n=0}^{\infty} Q^{\eta q 3}_{\alpha k n} X^{m\eta}_n + \sum_{n=0}^{\infty} Q^{\eta q 4}_{\alpha k n} Y^{m\eta}_n) = 0, \quad (11)$$

$$(\alpha = 1, 2; \quad k = 0, 1, 2, 3, ...)$$

3) From the condition of existence of non–trivial solutions we derive a characteristic equation

$$\Delta(p, \kappa) = 0 \quad (12)$$

where $\Delta(p, \kappa)$ is the determinant of (11). On the stage of calculations we perform the following actions: 4) we fix the values of the stiffness and geometrical parameters of the problem; 5) solving equation (12), we obtain all possible relations (dependences) $p^j = p^j(\kappa), (j \in J)$, where p is a loading parameter and $\kappa = \gamma R$ is a wave formation parameter; 6) we determine the minimal values of parameter p for each function $p^j(\kappa)$: $p^j_m = \min_{\kappa \neq 0} p^j(\kappa)$; 7) we determine the critical value of this parameter: $p_{cr} = \min_{j \in J}(p^j_m)$. The value of κ_{cr}, corresponding to p_{cr}, determines the length of the halfwave of the realizable stability loss mode by the formula $l_{cr} = \pi R \kappa^{-1}_{cr}$.

RESULTS AND DISCUSSION

Numerical investigation is carried out with $\nu_a = \nu_m = 0.3$ and different values of the geometrical and stiffness parameters of the problem. Selection of the same value for ν_a and ν_m is justified by the fact that the value of p_{cr} in the problems of instability of fibrous composites under compression is not substantially influenced by the difference between ν_a and ν_m in a wide range from 0.1 to 0.4. We also note that for the values of ν_a and ν_m considered here, formulae (4) for the precritical state are satisfied exactly. Figure 1 demonstrates values of contraction p_m^1 (curve1) and p_m^2 (curve 2) for the case $E_a E_m^{-1} = 1000$ in relation to the parameter δR^{-1}. Calculations have been carried out for a series of fibres parallel to the matrix boundary with different thicknesses of the bridge between neighbouring fibres (parameter δ) and the thickness of the bridge between the series and the free surface (parameter β) equal to 0.5R. p_m^1 corresponds to the stability loss mode for which all fibres lose stability in the same phase and the fibre axes remain in the planes perpendicular to the matrix boundary. p_m^2 corresponds to the stability loss mode for which all fibres lose stability in the same phase but the fibre axes go out of these planes. Curve 3 corresponds to the limiting case $\beta R^{-1} \to \infty$. Figure 2 shows the dependence of p_{cr} on $E_a E_m^{-1}$ calculated with allowance for three periodical series of fibres near the matrix boundary (curve 1). The thickness of the bridge between neighbouring fibres and the thickness of the bridge between the fibres from the first series and the free surface have been taken equal to 0.5R. The curve No.2 corresponds to the limiting case when the thicknesses of these bridges is sufficiently large (tends to infinity).

It follows from the results obtained that the mode of stability loss of a system of fibres that is realized near the free boundary of a binder depends on the parameter β. This may be a mode which does not occur in the case of an infinite matrix for the same combination of the geometric and stiffness parameters. We consider this as an evidence of the need to rigorously account for the effect of the free surface of the matrix on the stability of a system of fibres. The effect of the free surface is always magnified by the mutual interaction of fibres during stability loss (the latter significantly depends on the parameter δ). Some quantitative conclusions can be derived directly from Figures 1 and 2.

REFERENCES

1. A.N.Guz', "Fundamentals of the three–dimensional theory of stability of deformable bodies", (in Russian), Vishcha Shkola, Kiev, 1986, 512 p.
2. Micromechanics of composite materials: Focus on Ukrainian research // Applied Mechanics Reviews. Special Issue. – 1992. – **45**, No. 2. – P. 13–101.

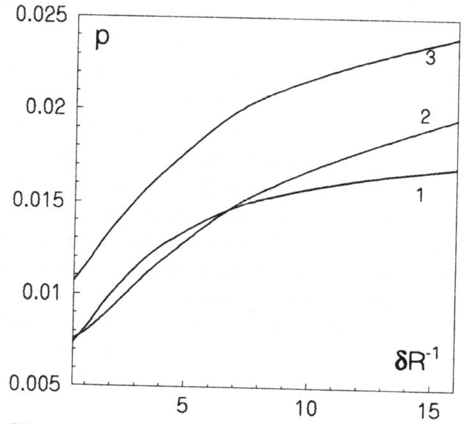

Figure 1. Loading parameter p versus δR^{-1}

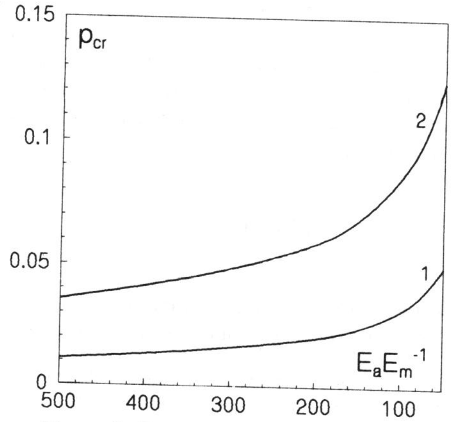

Figure 2. Parameter p_{cr} versus $E_a E_m^{-1}$