ANALOG OF δ_c-MODEL FOR ELASTO-PLASTIC PIECEWISE-HOMOGENEOUS PLATES AND SHELLS WITH CRACKS

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A method is proposed for investigation of the stressed-strained state and limit equilibrium of piecewise-homogeneous thin-walled structural elements with cracks considering plastic strains in the vicinity of cracks. The problem is reduced to a system of singular equations with unknown limits of integration and discontinuous right-hand parts containing the unknown forces and moments in the plastic zone.

An algorithm for numerical solution of the systems obtained is developed simultaneously with conditions of plasticity and finiteness of stresses in the vicinity of a crack tip. The influence of geometric and mechanical parameters is investigated for the crack tip opening in the given piecewise-homogeneous element.

INTRODUCTION

For a number of constructive materials the plastic strains, developing in the vicinity of a crack, may occupy considerable zones commensurable, for example, with the crack sizes. In such cases to estimate the limit equilibrium of material using the Griffits-Irvin conception is not acceptable and it's necessary to solve the elasto-plastic problem.

The appearance of plastic strips on extension to the crack line in the elasto-plastic materials with particularly pronounced yield surface arises already at the first stages of strain. Such strips, by analogy with δ_c-model, can be considered as the surfaces of discontinuity of elastic displacements and rotation angles that are under the unknown forces and moments. That is, the elasto-plastic problem on limit equilibrium for a piecewise-homogeneous plate or shell with cracks of certain length reduces to the problem on elastic equilibrium of the same structure with a crack of unknown length, the faces of which are under certain loading (forces and moments) that satisfy the plasticity condition. According to the δ_c-model, the fracture of material is assumed to begin when the maximum opening of a real crack's tip reaches the critical value.

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FINITE CRACK PERPENDICULAR TO THE INTERFACE OF A PIECEWISE-HOMOGENEOUS PLATE

Consider two elasto-plastic half-planes (μ_1, k_1) and (μ_2, k_2) joint along the line $\theta = \pm\pi/2$. One of its parts contains a finite crack $a < r < b$ with a length $2l_0$ along $\theta = \pi$, i.e. perpendicular to the interface. And let the distance be $a \neq 0$. The whole structure is under the expansive force p. According to the δ_c-model conception, the zones of plasticity appear in the vicinity of a crack, that can be considered as its continuation $a^p < r < a;\ b < r < b^p;\ a^p \neq 0$. In addition, taking into account the piecewise-homogeneous structure of a plate, the plastic zones length is unknown and unsymmetric about the local centre of a crack. So, we have a crack of unknown length $2l\ (2l = b^p - a^p)$ the faces of which (within the limits of plasticity zones) are under the constant stress σ_0, for which the following dependences are to be fulfilled: $\sigma_0 = \sigma_T$ for elasto-plastic materials; $\sigma_0 = (\sigma_T + \sigma_B)/2$ for materials with strengthening.

Such a problem has to be solved under the following boundary conditions:

$$
\begin{aligned}
&\tau_{2r\theta} = 0, \qquad u_{2\theta} = 0, \qquad \theta = 0, \qquad 0 < r < \infty, \\
&\left.\begin{aligned} \tau_{1\theta\theta} = \tau_{2\theta\theta}, \quad \tau_{1r\theta} = \tau_{2r\theta} \\ u_{1r} = u_{2r}, \quad u_{1\theta} = u_{2\theta} \end{aligned}\right\} \quad \theta = \frac{\pi}{2}, \qquad 0 < r < \infty, \\
&\tau_{1r\theta} = 0, \qquad \tau_{1\theta\theta}(r, \pi - 0) = \begin{cases} -p, & a < r < b \\ \sigma_0 - p, & b < r < b^p \cup a^p < r < a \end{cases}
\end{aligned} \tag{1}
$$

Let the unknown function of displacement jump $f(r)$ be defined as:

$$
f(r) = -2\frac{\partial}{\partial r}\left[u_{1\theta}(r, \pi - 0)\right] \tag{2}
$$

under the conditions $f(r) = 0, \quad 0 \le r < a^p \cup b^p < r < \infty \ ; \quad \int\limits_{a^p}^{b^p} f(r)dr = 0$.

The integral representation of solution in terms of the unknown displacement jumps of crack faces, constructed using the fundamental solution of a resolving system of equations and the Mellin transform, is obtained:

$$
\frac{1}{\pi}\int\limits_{a^p}^{b^p}\frac{f(s)}{s - r}ds + \frac{1}{\pi}\int\limits_{a^p}^{b^p} H(r, s) f(s) ds = \frac{1 + k_1}{2\mu_1}\begin{cases} -p, & a < r < b \\ \sigma_0 - p, & b < r < b^p \cup a^p < r < a. \end{cases} \tag{3}
$$

The equation obtained for elasto-plastic problem differs from a corresponding equation for elastic structure (Cook and Erdogan (1)) in this way that the limits of integration are unknown and the right-hand parts containing the unknown loading in the plastic zone are discontinuous. The solution of equation (3) is sought in the form of sum:

$$
f(r) = h(r) + \Psi(r) \tag{4}
$$

Here $h(r)$ is the solution of canonical singular integral equations with the discontinuous right-hand parts

$$\frac{1}{\pi}\int_{a^p}^{b^p}\frac{h(s)}{s-r}ds=\frac{1+k_1}{2\mu_1}\begin{cases}-p, & a<r<b\\ \sigma_0-p, & b<r<b^p \cup a^p<r<a,\end{cases} \tag{5}$$

that satisfies the condition $\int_{a^p}^{b^p} h(s)ds=0$.

Then the function $\Psi(r)$ is represented as the solution of the following equation

$$\frac{1}{\pi}\int_{a^p}^{b^p}\frac{\Psi(s)}{s-r}ds+\frac{1}{\pi}\int_{a^p}^{b^p}H(r,s)\,\Psi(s)\,ds=\frac{1}{\pi}\int_{a^p}^{b^p}H(r,s)\,h(s)\,ds, \tag{6}$$

that satisfies the condition $\int_{a^p}^{b^p}\Psi(s)ds=0$.

For the described crack we'll take the local orthogonal coordinates (x,y). The origin is placed in the middle of a crack. Then $a^p \le r \le b^p \mapsto -1\le x\le 1$; $a\mapsto \tau_1$; $b\mapsto \tau_2$; $s\mapsto t$. The solution of this equation (5) is found using the inversion formula for the Cauchy-type integrals. We obtain:

$$h(t)=\frac{1+k_1}{2\mu_1}\frac{1}{\pi^2\sqrt{1-t^2}}\left\{\pi D^0\cdot t-\sum_{i=1}^{2}D\cdot z_i(t)\right\}, \tag{7}$$

$$D^0=-p,\qquad D=\sigma_0-p,\qquad q_1=\sqrt{1-\tau_1^2},\qquad q_2=\sqrt{1-\tau_2^2},$$

$$z_i(t)=\arccos\tau_i+(-1)^i q_i-\frac{1}{2}\sqrt{1-t^2}\,L^{(i)}(t),$$

$$L^{(i)}(t)=\ln\left|\left(1-(-1)^i\left(\tau_i t-q_i\sqrt{1-t^2}\right)\right)\Big/\left(1+(-1)^i\left(\tau_i t+q_i\sqrt{1-t^2}\right)\right)\right|.$$

The function $\Psi(t)$ will be represented as:

$$\Psi(t)=\frac{\varphi(t)+c}{\sqrt{1-t^2}}. \tag{8}$$

Here $\varphi(t)$ is found using the mechanical quadrature method by expansion in all Chebyshev's polinomials of the first kind $T_\nu(t)$ of degree from 0 to (n_c-1), similarly as it is in Osadchuk (4).

$$\varphi^{(i)}(t)=\sum_{\nu=0}^{n_c-1}A_\nu^{(i)}T_\nu(t),\; i=1,2;\; \varphi(t)=\varphi^{(1)}(t)+\varphi^{(2)}(t).$$

For the crack faces opening δ we have

$$\delta^{(i)}=2u_{1\theta}(\tau_i,0),\; \text{i=1,2}. \tag{9}$$

Having taken the critical value of the crack faces opening, we come to the criterial equation that determines the relation between the applied load, the length of a crack $2l$, the physico-mechanical and geometric parameters of plate under the conditions of limit equilibrium.

Figure 1 show the dependence of a relative crack's tip opening $\Delta=\delta^{(i)}/\delta$, ($\delta=8l_0\sigma_0\ln\sec(\pi p/2\sigma_0)/(\pi E)$ is the crack face opening in a homogeneous plate) on the geometric ($\eta = (b - l_0)/l_0$) and mechanical (μ_1, μ_2) parameters. The curves 1 and 2 correspond to the value $r = a$ and $r = b$, respectively.

CASE OF TWO BOUNDED CRACKS IN A PIECEWISE-HOMOGENEOUS PLATE

In this case each half plane of composite medium contains a finite crack ($a_1 < r < b_1$ and $a_2 < r < b_2$) along the lines $\theta = \pi$, $\theta = 0$, respectively, perpendicular to the interface $\theta = \pm\pi/2$. All of conditions for the problem with one crack are satisfied for the given case. The planes ($r > 0, \theta = \pi$) and ($r > 0, \theta = 0$) are free from shear and are acted upon by the normal stresses p_1 and p_2, respectively. Under deformation the zones of plastic strains with unknown length (a_1^p, a_1), (a_2^p, a_2), (b_1, b_1^p), (b_2, b_2^p) arise. Let the distances be $a_i^p \neq 0$, (i = 1,2). The surfaces are subjected to some stresses $\sigma_0^{(i)}$, $i = 1,2$. The values of $\sigma_0^{(i)}$ are dependent on the mechanical parameters of the half-plates. The boundary conditions are expressed in similar way. Such problem, when using the aforesaid procedure, can be reduced to the following system of two singular integral equations

$$\frac{1}{\pi}\int_{a_i^p}^{b_i^p}\frac{f_i(s)}{s-r}ds + \frac{1}{\pi}\sum_{j=1}^{2}\int_{a_i^p}^{b_i^p}H_{ij}(r,s)f_j(s)ds = \frac{1+k_i}{2\mu_i}\begin{cases}-p_i, & a_i < r < b_i\\ \sigma_0^{(i)} - p_i, & b_i < r < b_i^p \cup a_i^p < r < a_i\end{cases},$$

$$f_1(r) = -2\frac{\partial}{\partial r}\left[u_{1\theta}(r, \pi - 0)\right], \qquad f_2(r) = -2\frac{\partial}{\partial r}\left[u_{2\theta}(r, +0)\right], \quad i = 1,2, \tag{10}$$

$k_i = 3 - 4\nu_i$ - for the plane strain, $k_i = (3 - \nu_i)/(1 + \nu_i)$ - for the generalized plane stress.

It should be noted that the system obtained differs from the corresponding equations for elastic structure (Erdogan and Biricikoglu (2)) in that way that the limits of integration are unknown and the right-hand parts are discontinuous. The subsequent solution of the problem above is constructed quite analogously as in the previous case.

PIECEWISE-HOMOGENEOUS CYLINDRICAL SHELL WITH A CRACK

Consider the problem for a piecewise-homogeneous cylindrical shell, when one part contains a through crack with a length $2l_0$(Fig.2). The shell is referred to three orthogonal coordinates (α, β, γ) with origin in the middle of the crack at a distance l^* from the interface of materials $\alpha = \alpha^*$ ($\alpha^* = l^*/R$, R is the radius of the median surface). The crack $|\alpha| \leq \alpha_0$ is located in the plane $\beta = 0$. The shell is under certain external load. According to the δ_c-model, the zones of plastic strains are substituted by the surfaces of discontinuity of elastic displacements and angels of rotation; l_p is the length of plastic zone on the crack line extension located nearer to the interface, l^p is the corresponding length near the opposite crack tip. The reaction of plastic zones is modelled by the unknown normal forces $N^{(i)}$ and bending moments $M^{(i)}$, that must satisfy the yield condition for a thin-walled shells. For example, the Treska yield condition for a plastic layer or a plastic hinge

$$\frac{N^{(i)}}{2h\sigma_T} + \frac{3\left|M^{(i)}\right|}{2h^2\sigma_T} = 1 \quad \text{or} \quad \left[\frac{N^{(i)}}{2h\sigma_T}\right]^2 + \frac{\left|M^{(i)}\right|}{h^2\sigma_T} = 1 . \tag{11}$$

Thus, the elasto-plastic problem for the shell with a crack of a length $2l_0$ is reduced to the problem on elastic equilibrium of the shell with a crack of unknown length $2l_\alpha$ $(2l_\alpha = 2l_0 + l_p + l^p)$, the faces of which satisfy the following conditions

$$N_2 = \begin{cases} N^{(2)} - N^{(0)}, & -(\alpha_0 + \alpha^p) \le \alpha \le -\alpha_0 \\ -N^{(0)}, & |\alpha| < \alpha_0 \\ N^{(1)} - N^{(0)}, & \alpha_0 \le \alpha \le \alpha_0 + \alpha_p \end{cases}, \tag{12}$$

$$M_2 = \begin{cases} M^{(2)} - M^{(0)}, & -(\alpha_0 + \alpha^p) \le \alpha \le -\alpha_0 \\ -M^{(0)}, & |\alpha| < \alpha_0 \\ M^{(1)} - M^{(0)}, & \alpha_0 \le \alpha \le \alpha_0 + \alpha_p \end{cases}.$$

$N^{(0)}, M^{(0)}$ are the forces and moments acting on the crack line in the shell without a crack,

$$\alpha^p = l^p / R, \quad \alpha_p = l_p / R, \quad \alpha_0 + \alpha_p < \alpha^* .$$

Using the resolving system of differential equations in displacements for the homogeneous shells with cracks and the method of generalized coupling problems (under the boundary conditions (12) and the conditions of ideal mechanical contact on the interface) we obtain a system of six singular integral equations in terms of the unknown displacement jump functions (Kushnir et al.(3)).

To ensure the system's completeness we add one of the conditions (11), and the conditions of zero stress intensity factors at both crack tips:

$$K_N^{(i)}(l_\alpha) = 0, \qquad K_M^{(i)}(l_\alpha) = 0 .$$

The qualitative change of crack face opening is quite analogous as in the case of a composite plate, but the value of it in a shell is always larger than in a plate under the same conditions. If the face opening of a through crack is less than a critical one, the fracture process stops, otherwise the crack propagates along the shell. In different materials of the shell the presence of similar cracks has different influence on its strength.

SYMBOLS USED

ν_i = the Poisson's ratio

μ_i = shear moduli (Pa)

σ_T, σ_B = yield and strength limits of the material (Pa).

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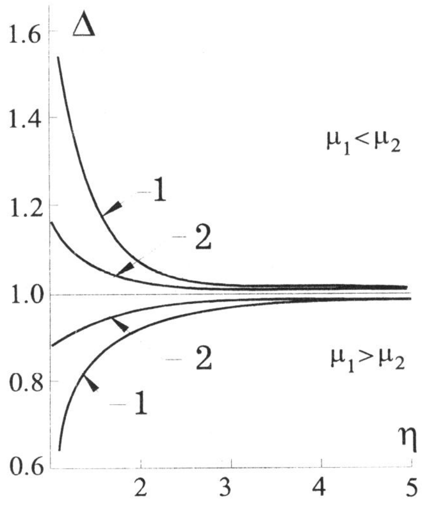

Figure 1 Influence of geometric and mechanical parameters on the crack face opening

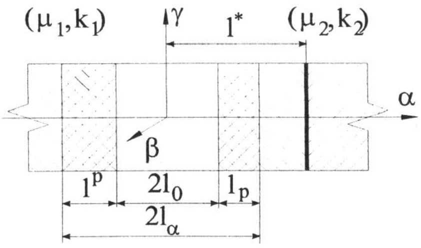

Figure 2 Illustration of a shell model