CRACKS AND STRAIN ENERGY

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During an increment of crack extension, changes occur in the strain energy content of both near-field and far-field regions. Although an elastic system containing a crack must obey a global energy equation, some of the energy quantities within the system may have very little effect on the environment of the crack tip, which is controlled by near-field stresses. The present paper sets out to confirm this argument by showing calculations of the energy available for crack extension in two-dimensional situations.

BASIS OF METHOD

For the purpose of examining energy quantities associated with a crack, an infinitely large elastic plate is considered, containing a slit or internal crack of length $2a$. Within a distance somewhat less than $0.5a$ from the crack tip, stresses vary inversely as the square root of distance, values at the crack tip being infinitely large. Inside a distance approximately $3a$ from the centre of the crack, stresses vary in a complicated way, but can be exactly specified. Beyond this distance lies an annulus in which a fairly simple stress distribution prevails. On a contour drawn within this annulus, simple expressions for stress and displacement can be used for calculating work transmitted through the contour. It is not convenient to place the contour at the outer boundary of the plate, as here, stresses and displacements cannot be simultaneously specified to the required degree of accuracy. The outer boundary is therefore assumed to be well outside the chosen contour.

Three exercises may be carried out on the plate. When load is applied to a plate that does not contain any crack, parameters are given subscript zero. Load is then applied to a plate containing a crack while the crack can open freely. Work

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W_1 is supplied at the contour and strain energy U_1 is generated. These quantities ought to be equal, but this can be verified by calculating each separately. Finally, loading is applied and maintained while the crack is prevented from opening. The crack is then allowed to open gradually. Some kind of mechanism is imagined to be attached to the double-ended crack to absorb work $2R$ during the process of opening. While loading at the remote boundary is maintained at its full value, additional displacements occur at the contour, so that additional work W_2 is supplied. During this time, strain energy of material within the contour increases by an amount U_2. However, the final state of the plate is the same whether the crack opens during loading or after loading, so the final strain energy is $U_1 = U_0 + U_2$. During the process of gradual crack opening, energy quantities follow the physical principle that no energy is lost. Hence the incremental strain energy of the defined portion of the plate must be equal to the positive work supplied both at its inner boundary at the crack surface and on the contour, giving $U_2 = W_2 - 2R$ where $2R$ is the amount of energy delivered to the site of the crack as the crack is allowed to open.

Boundary conditions of stress can be satisfied by using elliptical co-ordinates (α, β) defined by $x = a \, Cosh \, \alpha \, Cos \, \beta$, $y = a \, Sinh \, \alpha \, Sin \, \beta$. Lines of constant α are ellipses encircling the crack. For a plate subjected to an all-round tensile stress S at an infinitely remote outer boundary, the stress components may be found by differentiating an Airy function F, as given by Inglis (1):

$$F = \frac{1}{4} S a^2 (Sinh \, 2\alpha - 2\alpha) \tag{1}$$

For a state of plane stress, the strain energy density D is given by:

$$D = \frac{1-\nu}{E}\left(\frac{\sigma_\alpha + \sigma_\beta}{2}\right)^2 + \frac{1+\nu}{E}\left[\left(\frac{\sigma_\beta - \sigma_\alpha}{2}\right)^2 + \tau_{\alpha\beta}^2\right] \tag{2}$$

In this expression, stress components in any orthogonal co-ordinates may be used. Integration of strain energy density can now be carried out over the surface within a selected elliptical contour to obtain an expression for strain energy U_1. Details of these calculations are not given here, as a simpler approach will be explored.

INTERMEDIATE REGION

For values of α larger than about 2, ellipses of constant α tend towards circles, and this allows the stress function, Eq. (1), to be expanded in terms of polar co-ordinates (r, θ), shown in Figure 1:

$$F = \frac{1}{4} S\left[2r^2 - a^2(2 \, \ell n \, r + Cos \, 2\theta)\right] \tag{3}$$

In obtaining this expression, terms in a^2 are retained but terms of higher powers are discarded, and the stress function then has sufficient accuracy at any radius larger than about $3a$. It may be differentiated in the usual way to obtain the stress components. These, with the displacements are:

$$\sigma_r = S\left[1-\frac{1}{2}\left(a^2/r^2\right)(1-2Cos\ 2\theta)\right], \qquad \sigma_\theta = S\left[1+\frac{1}{2}\left(a^2/r^2\right)\right] \tag{4}$$

$$\tau_{r\theta} = \frac{1}{2}S\left(a^2/r^2\right)Sin\ 2\theta$$

$$Eu_r = Sr\left[1-\nu+\frac{1}{2}\left(a^2/r^2\right)(1+\nu-2Cos\ 2\theta)\right], \qquad Eu_\theta = \frac{1}{2}S\left(a^2/r\right)(1-\nu)\ Sin\ 2\theta$$

As an alternative, these equations may be obtained from stresses in (x, y) co-ordinates derived from the stress function of Westergaard (2).

The work W_1 transmitted at the contour while loading is being applied is now found by integrating products of stress and displacement around a contour of radius r, to obtain the expression $W_1 = \pi\left(S^2/E\right)\left[(1-\nu)r^2 + \nu a^2\right]$. This work is converted to strain energy. The second term represents the increase due to the presence of the crack and confirms the value found by using elliptical co-ordinates. The above stresses can be further used for finding the work done while the crack is permitted to open after loading has been applied. An additional displacement then occurs at the contour, equal to the displacement of the cracked plate minus that of the uncracked plate. When this increment of displacement is multiplied by the radial stress acting on the contour, the additional work is found to be $W_2 = \pi a^2(1+\nu)S^2/E$. Part of this work goes to increasing strain energy. The rest is delivered to the site of the crack, this part being $2R = W_2 - U_2 = \pi a^2 S^2/E$.

ALL-ROUND AND UNIAXIAL LOADING

The situation considered here is where stresses $\sigma_r = S$ and $\sigma_y = T$ act together in some combination at the remote boundary. In the intermediate region at some considerable radius r, the stress and the displacement can be represented in polar co-ordinates in the manner already described. Results are given in summary here. Incremental work transmitted through the contour as the crack opens amounts to:

$$W_2 = \pi\left(a^2/4E\right)\left[4(1+\nu)S^2 + (9+5\nu)ST + (5+\nu)T^2\right] \tag{5}$$

Part of this supplies a gain in strain energy, of an amount

$$U_2 = \pi\left(a^2/4E\right)\ \left[4\nu S^2 + (1+5\nu)ST + (1+\nu)T^2\right] \tag{6}$$

It is seen that this gain is always positive. The rest of the incremental work is delivered to the site of the crack, $2R = \pi(a^2/E)(S+T)^2$. From this expression it can be deduced that the energy delivered to the crack depends only on the component of the applied boundary stress acting in the *y*- direction. Strain energy density *D* for a uniaxial boundary stress is shown in Figure 2, expressed as a multiple of its value at infinity. Features include a region of high energy near the tip of the crack and a depleted region along the vertical axis of symmetry. A value of unity is approached asymptotically as shown in Figure 3.

J-INTEGRALS AND NEAR-FIELD STRESSES

The J-integral directly gives the rate at which energy is supplied to a single-ended crack as the crack extends, that is, $J = \partial R / \partial a$, and is given by:

$$J = \int_{\Gamma} D\, dy - \sigma_{ij} \frac{\partial u_i}{\partial x} n_j\, ds \qquad (7)$$

This line-integral is taken around a contour Γ surrounding the crack tip, as shown in Figure 4. Stresses and displacements in (*x, y*) co-ordinates are used and n_j is the direction-cosine of the outward-facing normal to segment *ds* of the contour. This integral can be envisaged as the result of a movement of stress and displacement fields as the crack gains a small increment of length, this movement being relative to a contour fixed to the plate. When the first term is positive, the lateral sides of the contour produce a positive contribution of strain energy to the energy available for crack growth, though this does not necessarily mean that the whole of the region enclosed has lost strain energy. The total value of the J-integral is independent of the contour chosen, as pointed out by Rice (3). However, the proportions of the total contributed by each of the two terms may vary depending on the contour selected, so these proportions can be of no significance. For determining the J-integral for near-field stresses around a Mode I crack, a circular contour centred on the crack tip can be used, the radius *r* being not more than about $0.25a$. Displacement gradients may be expressed in terms of the stress intensity factor K_I:

$$E\, \partial u_x/\partial x = \frac{1}{2} K_I (2\pi r)^{-\frac{1}{2}} Cos\frac{1}{2}\theta \left[1 - 3v - (1+v)(1 - 2Cos\theta)Cos\theta\right]$$
$$E\, \partial u_y/\partial x = -\frac{1}{2} K_I (2\pi r)^{-\frac{1}{2}} Sin\frac{1}{2}\theta \left[(5+v) - (1+v)(1 + 2Cos\theta)Cos\theta\right] \qquad (8)$$

When these values are used in Eq. (7), one obtains:

$$J = \left(K_I^2/4E\right)\left[(1-v) + (3+v)\right] = K_I^2/E \qquad (9)$$

The proportions of the two terms depends on the choice of a contour, but their sum correctly gives the energy release rate.

DISCUSSION AND CONCLUSIONS

It has often been remarked that under fixed-grip testing conditions, no further energy can be supplied at the outer boundary of the plate after the crack begins to extend. So energy for crack extension must come from the strain energy stored in the plate. However, it has been shown here that the crack is embedded in a core in which strain energy increases as the crack extends. The explanation is that a small and approximately uniform reduction in strain energy density occurs in the outer part of the plate. Detailed calculations suggest that this region extends from a radius of about $2a$, outwards to the fixed grips. The effect of this small reduction is magnified by the large area over which it acts. The crack continues to be surrounded by an annulus in which the average strain energy increases.

It is true that the system as a whole must satisfy an energy balance. There is no doubt that the growing crack leaves behind it a localized area of depleted strain energy, as can be appreciated from Figures 2 and 3. However, the general idea that energy for crack extension comes from depletion of strain energy seems to be of limited usefulness. Transfer of strain energy from one place to another in regions remote from the crack can have no effect on conditions prevailing in the immediate vicinity of the crack tip. On the other hand, a J-integral taken around a contour encircling the crack tip provides a rational basis for discussing an energy criterion for crack extension. It disregards irrelevant happenings in places remote from the crack tip.

REFERENCES

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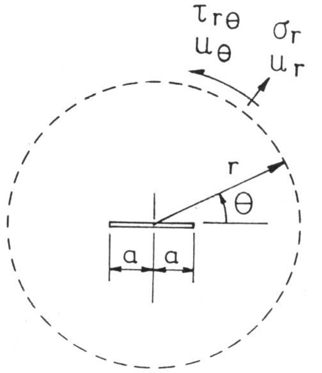

Figure 1. Circular contour in the intermediate region

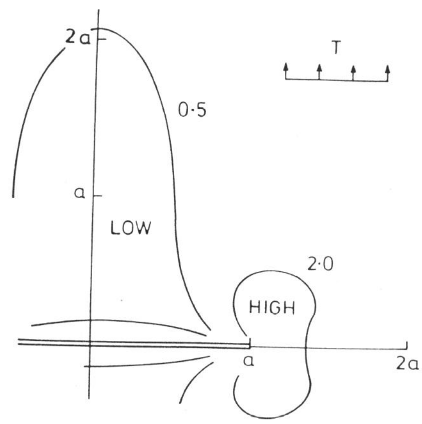

Figure 2 Strain energy density D

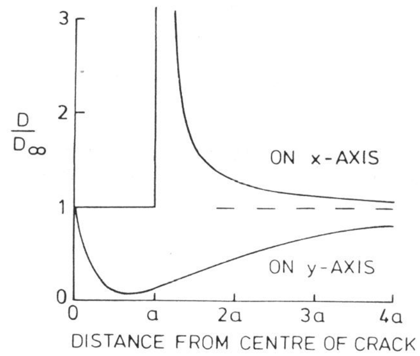

Figure 3 Variation of strain energy density

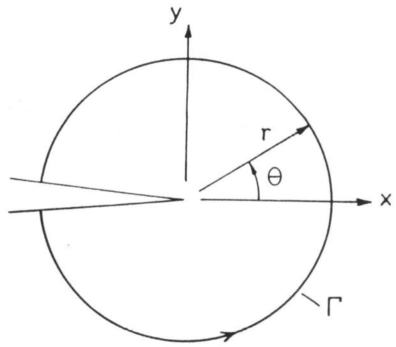

Figure 4 Contour around the tip of a crack