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Modelling Elastic–Plastic Crack Growth with the Dugdale Model

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ABSTRACT The Rice *et al.* analysis and the Wnuk–Dugdale model of a crack growing under elastic–plastic conditions are reviewed. It is shown that both analyses give essentially the same results for small-scale yielding, and that the latter provides a general criterion for growth under arbitrary yielding conditions. A more general expression for the Wnuk–Dugdale model is given in terms of a modified CTOD and material properties. The similarities with the modified J, J_M, approach are emphasised. These methods are applied to the Dugdale model for the CT specimen developed by Mall and Newman using experimental data from different materials. It is demonstrated that for some materials the crack growth step, r_C, cannot be considered constant, as in the past, but instead must be considered a function of the increasing applied deformation J. The methods developed provide insight into the way the different controlling variables affect the crack growth process.

Introduction

In recent years, the use of J-integral **(1)**–**(3)** related parameters to characterise elastic–plastic crack growth has become increasingly common, specially after the work of Hutchinson and Paris **(4)**. In their analysis they show that there is a J-controlled crack growth regime provided $\omega = (\mathrm{d}J/\mathrm{d}a)\,(b/J) \gg 1$, where b is the remaining ligament. Other investigators have also established limits on the amount of deformation that could be applied to a specimen and still have a one parameter description of the stress–strain fields **(5)**. As a result, ASTM established a standard for J–R curve determination in which a validity zone is defined: J cannot exceed the smaller of $b_0\sigma_0/20$ and $B\sigma_0/20$ where σ_0 is the yield strength and B is the specimen thickness, and Δa is limited to $0.1b_0$.

The problem is that the typical surveillance sub-size specimens used in testing reactor pressure vessel (RPV) material, welds, etc., consist of cracked bodies with an initial remaining ligament of about 0.5 in. (1 in. = 25.4 mm). For these dimensions the R curve is limited to crack extension values of about 0.05 in. or J values of about 2000 lb/in. (1 lb/in. = 1.75×10^{-4} MPa/m). The magnitude of these values is many times insufficient to predict maximum load and stability for large structural pieces, for which crack extensions up to 1 in. and J values ranging to 5000 lb/in. are common.

As a result, it is extremely important to understand the process of crack extension, in order to model its characteristics: describe the fracture phenomena, weigh the influence of the various parameters involved, so their number

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can be minimised, and obtain useful data for situations for which testing is limited or not feasible. As a special case, the possibility of extending *R*-curves beyond the available experimental data is of the biggest importance.

In this paper, different proposed models to characterise crack growth are critically examined and their limitations emphasised. The original models are then modified in different ways to allow a more general use of the equations. The different proposed routes are applied to experimental data.

Rice analysis

Rice *et al.* **(6)** analysed the problem of a crack growing under small-scale yielding, plain strain conditions in an elastic–perfectly-plastic material using finite elements and obtained an expression for the rate of displacement opening, $\dot{\delta}$, a distance r from the tip

$$\dot{\delta} = \alpha \dot{J}/\sigma_0 + \beta(\sigma_0/E)\dot{a} \ln (R/r) \tag{1}$$

where α and β are constants and R is a length parameter associated with the plastic zone size given by

$$R = 0.2JE/\sigma_0^2 \tag{2}$$

Using a constant angle growth criterion, equation (1) was integrated to give

$$\frac{\mathrm{d}J}{\mathrm{d}a} = \frac{\delta_c \sigma_0}{\alpha r_m} - \frac{\beta \sigma_0^2}{\alpha E} \ln \left(\frac{eR}{r_m}\right) \tag{3}$$

where δ_c and r_m are length parameters which describe the critical crack tip opening angle. Using the point of crack initiation under small-scale yielding conditions as the reference one, the above equation becomes

$$\frac{\mathrm{d}J}{\mathrm{d}a} = \frac{\mathrm{d}J}{\mathrm{d}a_0} - \frac{\sigma_0^2 \beta}{E\alpha} \ln \left(\frac{R}{\lambda E J_{\mathrm{Ic}}/\sigma_0^2}\right) \tag{4}$$

which eliminates the microscopic parameters. They also argue that a solution of a similar form would apply to large yielding conditions with a possible variation of the constants involved. That is, the above equation would be valid in general provided J is replaced by J_x, a J-like parameter that has been identified with J_M, **(7)(8)**. In particular, they proposed that in a bend specimen with a small ligament b, R should saturate to a fraction of the ligament, γ

$$R = \gamma b \quad \text{with } \gamma = 0.25 \tag{5}$$

In reference **(9)** a general formula was given that allows for a continuous description of the evolution of R as a function of J and b for any level of yielding.

$$R = (\gamma b)(1 - \exp(-0.2 J_M E/\gamma b \sigma_0^2)) \tag{6}$$

From their analysis it can be seen that $\mathrm{d}J/\mathrm{d}a$ is not a unique function of J but also of the specimen size, i.e., the amount of yielding for a given J, unless the logarithmic term is negligible with respect to the first one. That is linearity, i.e., constant slope, and uniqueness are mutually implied.

Wnuk–Dugdale model

Wnuk **(10)** proposed a crack growth criterion based on the increase in displacement realised by a material point a distance ahead of a crack during an incremental crack extension. When a load is applied to a cracked body, the material in the plastic zone, while not necessarily separating, is displaced or stretched. This displacement will increase with load until the crack extends. Wnuk focused on a material point a distance r_c ahead of the crack tip, i.e., at a position $x = a + r_c$ with respect to a fixed frame of reference, Fig. 1. He proposed that the crack will extend an amount r_c when that extension produces an increment δ_c in the displacement at the point of interest. This is known as Wnuk's final stretch criterion, and all that is required for its application is an expression for the displacement of points on the crack plane, i.e., crack surfaces and plastic zone, as a function of crack length and applied stress.

The purpose of the Dugdale model was to develop a relationship between plastic zone size, applied load, and crack length. It was originally developed for the case of a crack in an infinite plate subjected to remote tension, under plane stress conditions in an elastic–perfectly-plastic material. It also provides the complete displacement profile of the crack line to the end of the plastic

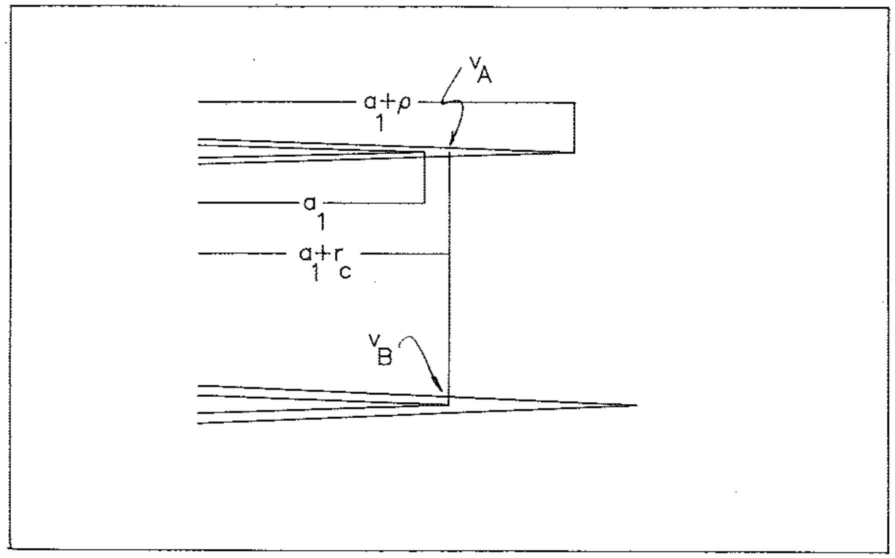

Fig 1 Wnuk's growth criterion

zone, and is therefore a good candidate to be used in the Wnuk growth model. As compared to finite element analysis (FEA) the Dugdale model requires very little numerical calculation and, within its limitations, it can provide insight into the fracture mechanisms of real specimens.

For the case of the centre crack panel mentioned the crack line displacement field is given in **(11)** by

$$v(x) = \frac{(1-\nu^2)\sigma_0}{\pi E'}((x-a)\Gamma(x, a) - (x+a)\Gamma(x, -a)) \tag{7}$$

where

$$\Gamma(x, a) = \ln \frac{l^2 - xa - \sqrt{\{(l^2-x^2)(l^2-a^2)\}}}{l^2 - xa + \sqrt{\{(l^2-x^2)(l^2-a^2)\}}}$$

x is the position coordinate measured from the middle point of the crack, a is half the crack length, l is the length of the crack plus plastic zone size ρ, and E' is the generalised Young's modulus.

Smith **(12)** examined the implications of the Wnuk criterion by expressing it in terms of the displacement profile given by the Dugdale model. Crack extension will either be stable or unstable depending on whether r_c is smaller or larger than the plastic zone ρ. If r_c is larger than the plastic zone there is no initial displacement at a distance r_c from the crack tip and then the critical displacement (previous displacement + incremental displacement δ_c) will be δ_c for every increment of growth. This implies that crack growth will occur at a constant J or CTOD value, i.e., unstable.

For r_c within the plastic zone, the condition for crack extension from the displacement equation can be expressed as **(12)**

$$\frac{dJ}{da} = \frac{\sigma_0 \delta_c}{r_c} - \frac{l}{S} \ln \left\{\frac{2ea}{r_c}(1 - \exp(-SJ/a))\right\} \tag{8}$$

where $S = \pi E'/4\sigma_0^2$. This equation can also be expressed in terms of ρ and a

$$\frac{dJ}{da} = \frac{\sigma_0 \delta_c}{r_c} - \frac{1}{S} \ln \left\{\frac{2ea}{r_c}\left(1 - \frac{a^2}{(a+\rho)^2}\right)\right\} \tag{9}$$

These are general equations, valid for any extent of yielding. Note that the second term is a function not only of J but also of a, so that a unique relation between dJ/da and J (independent of the amount of yielding, or crack length a in this case) is guaranteed only if the second term is negligible compared to the first. This is emphasised if the two extreme cases of very large- and small-scale yielding are considered: for very large-scale yielding, i.e., $\rho = 1 - a \gg a$

$$\frac{dJ}{da} = \frac{\sigma_0 \delta_c}{r_c} - \frac{4\sigma_0^2}{\pi E'} \ln \left(\frac{2ea}{r_c}\right) \tag{10}$$

and for small-scale yielding ($\rho = 1 - a \ll a$)

$$\frac{dJ}{da} = \frac{\sigma_0 \delta_c}{r_c} - \frac{4\sigma_0^2}{\pi E'} \ln \left(\frac{\pi e E' J}{2\sigma_0^2 r_c}\right) \tag{11}$$

It is clear then that dJ/da will be a unique function of J, or independent of specimen size only if the second term is negligible compared to the first one. Then dJ/da will be primarily a function of the two Wnuk parameters (r_c and δ_c) and the yield strength. The J–R curve will also be linear, and then as for the Rice analysis, linearity and uniqueness are mutually implied. Since the magnitude of the logarithmic term is of order unity, the condition for linearity can be expressed in terms of the nominal strain $\varepsilon_0 = \sigma_0/E$ as

$$\varepsilon_0 \ll \frac{\delta_c}{r_c} \tag{12}$$

In terms of material properties, the expression above indicates that low strength and high toughness will favour unique and linear R curves. Smith proposed that the second term is negligible if ε_0 is about 2 percent of δ_c/r_c. So for low strength/high toughness materials, J exhibits a unique (independent of extent of yielding) and linear dependence on the Wnuk parameters, namely

$$\frac{dJ}{da} \cong \frac{\sigma_0 \delta_c}{r_c} \tag{13}$$

It is clear then that except for the value of some constants the result from the Wnuk–Dugdale model for the small-scale yielding case is the same as that obtained from the Rice analysis. Moreover, it seems possible to use the former to extend the analysis of crack growth beyond the results of the latter.

A more general expression for the Wnuk model

The Wnuk's criterion can be expressed in a more general way, i.e., without making specific use of the Dugdale model. Looking at the displacement at the same material point ($x = a_1 + r_c$), but at two different crack lengths a_1 and $a_1 + r_c$, v_A and v_B respectively, one can write

$$v_A(x = a_1 + r_c, a = a_1, \sigma = \sigma_1) + \tfrac{1}{2}\delta_c = v_B(x = a_1 + r_c, a = a_1 + r_c, \sigma = \sigma_2)$$

or

$$v_A(x = a_1 + r_c, a = a_1, \sigma = \sigma) + \tfrac{1}{2}\delta_c = \tfrac{1}{2}\mathrm{CTOD}_B \tag{14}$$

v_A can be expressed as

$$v_A = \tfrac{1}{2}\mathrm{CTOD}_A + \left.\frac{\delta v}{\delta x}\right|_{\sigma=\sigma_1,\, a=a_1} r_c$$

so that

$$\tfrac{1}{2}\mathrm{CTOD_B} - \tfrac{1}{2}\mathrm{CTOD_A} - \left.\frac{\delta v}{\delta x}\right|_{\sigma,\,a} r_c = \tfrac{1}{2}\delta_c \tag{15}$$

In differential form

$$\frac{\mathrm{d(CTOD)}}{\mathrm{d}a} = \frac{\delta_c}{r_c} + 2\left.\frac{\delta v}{\delta x}\right|_{\sigma,\,a} \tag{16}$$

$$\frac{\mathrm{d\,(CTOD)^*}}{\mathrm{d}a} = \frac{\mathrm{d\,(CTOD)}}{\mathrm{d}a} - 2\left.\frac{\delta v}{\delta x}\right|_{\sigma,\,a} = \frac{\delta_c}{r_c} \tag{17}$$

This general description of the Wnuk criterion highlights its dependence not only on r_c and δ_c, but also on the displacement profile near the crack tip, i.e., the rate of change of displacement with respect to position at constant σ and a. Moreover, the criterion can be regarded as an equality linking a modified version of the CTOD, CTOD*, and a function of the microstructure. In fact the CTOD* can be seen as a way of accounting for the intermediate step in crack growth between two 'equilibrium points', that one occurring at constant displacement profile without any additional expense of energy. The resemblance of equation (17) to the modified J, J_M, is obvious.

$$\mathrm{d}J_M/\mathrm{d}a = \mathrm{d}J/\mathrm{d}a - (\partial J_{pl}/\partial a)_{v_{pl}} \tag{18}$$

where again the original J was modified to account for that intermediate step in the crack growth between two points: that one taken place at constant applied plastic displacement, resulting in no change in J_{pl}, **(7)(8)**. The second term of equation (18) offsets the deformation theory results to guarantee exactly that. Although a mathematical proof was not intended here, it is important to realise the similarity in the concept behind the second terms of equations (17) and (18).

On the other hand, as will be seen later, the right hand side of equation (17) does not need to be necessarily a constant but, more generally, it can be a function of the process zone size, or CTOD, triaxiality, etc.

The Dugdale model for the CT specimen

In the above section the similarity of form between the rate of change of J_{pl} with respect to crack length at constant v_{pl} and the rate of change of the Dugdale displacement with position in the same profile has been established. It is also important to see how the irreversibility of plasticity can be incorporated in deformation theory type of results.

Mall and Newman **(13)** developed plastic zone size and CTOD equations of the Dugdale model for the CT specimen. Their analysis involves superposition of stresses and displacements due to loading at the pin holes and the Dugdale internal 'compensating' stress, Fig. 2. The displacement relations were developed following a method by Tada *et al.* **(14)** which uses virtual forces applied

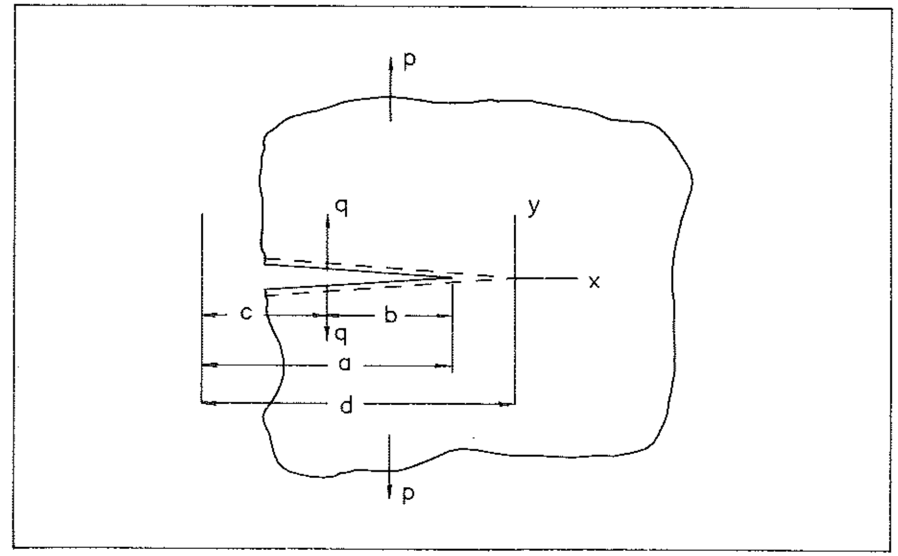

Fig 2 Crack in arbitrarily shaped plate

at the point of interest. For the loaded cracked body, the method gives the displacement at $x = c - d$ as

$$v(x = c - d) = \frac{1-\eta^2}{E'}\int_c^d K_p \frac{\partial K_q}{\partial q}\,\mathrm{d}a \tag{19}$$

where $\eta = 0$ for plane stress and $\eta = v$ for plane strain, P is an applied force and q is a virtual force, with K_p and K_q being the stress intensity factors due to p and q. The resulting equations for displacement due to pin-loading and uniform pressure on the crack surface are

displacement due to pin-loading

$$v_p(c) = \frac{2P}{E'}\int_{c/W}^{\alpha} \frac{F(\alpha)G(\alpha, \Delta)}{\sqrt{\{2\pi(\alpha - c/W)\}}}\,\mathrm{d}\alpha \tag{20}$$

displacement due to uniform pressure on crack surface

$$v_\sigma(x) = \frac{\sigma W}{2\pi E'}\int_{c/W}^{\alpha} \frac{H(\alpha, B_1, B_2)G(\alpha, \Delta)}{\sqrt{(\alpha - c/W)}}\,\mathrm{d}\alpha \tag{21}$$

where $\alpha = a/W$, $\Delta = b/(W - a + b)$, c is as shown in Fig. 2, and B_1 and B_2 refer to the range over which the uniform pressure acts. Equation (20) was also fitted to a polynomial expression:

polynomial expression for displacement due to pin-loading

$$v_p = 4\sqrt{\left(\frac{\xi}{2\pi W}\right)}\left[1 + C_1\left(\frac{\xi}{a}\right) + C_2\left(\frac{\xi}{a}\right)^2\right]F \tag{22}$$

where $\xi = a - x$. These displacement equations can be used to find the crack profile once the plastic zone size is known. Mall and Newman's final results were equations for the plastic zone size and the CTOD

plastic zone size

$$\rho = \frac{\pi W}{8}\left(\frac{PF(\alpha)}{W\sigma_0}\right)^2 F_0 \tag{23}$$

crack tip opening displacement

$$\text{CTOD} = \sqrt{\left(\frac{\beta}{2\pi}\right)} F_1 F_2 - \frac{W\sigma_0}{\pi P}\sqrt{(\beta H_1 H_2)} \tag{24}$$

where F_0 is a function of P, W and σ_0. β, F_1, F_2, H_1, and H_2 are functions of the plastic zone size.

Restrictions on the CT model

As pointed out by Mall and Newman, the model developed has some limitations. On one hand the equations are not valid for small a/W ratios and, on the other hand, the applied deformation should be limited to moderate values. If it is too large, the extent of the plastic zone will exceed a critical fraction of the ligament resulting in yielding of the backface. Obviously the equations do not account for this behaviour.

This model, like the original Dugdale model, does not allow for hardening effects or for a non-uniform Dugdale stress distribution. Increasing the value of yield strength results in a smaller plastic zone size for a given load or a given displacement. From another perspective, increasing the yield allows greater displacements to be reached because the elastic contribution is greater and the limitations imposed by plasticity are not realised as early.

Validation of Mall and Newman's equations

The growing crack curve is completely defined by P, v, and a. Mall and Newman's equations provide a relationship between these three so that there are only two independent parameters. To verify the proper implementation of the equations, two parameters from actual test data **(15)** were input and the third parameter calculated and compared to the actual test value. The test data had been used by Newman *et al.* **(16)** in a subsequent study which made use of the equations. The data consists of three test specimens (one 1T, one 2T, and one 4T) each for two materials, 2024-T351(LT) and 7075-T651(TL). Reference Table 1 for material properties.

Use of P and a as independent parameters resulted in values of displacement which were 25 percent–75 percent less than the actual values, Fig. 3. This trend is most evident in the 4T specimens of the 2024 material. The J

Table 1 Material properties for validation data

Material	*Yield (ksi)*	*Young's modulus (ksi)*	J_{Ic} *(lb/in)*
2024	45.65	10 400	—
7075	76.85	10 400	55

values were lower than those calculated from the original P, v, and a data as shown in Fig. 4. It should be noted that the use of P and a did not restrict the extent of growth to which the model could be used.

Use of v and a as independent parameters resulted in values of P close to, but higher than the actual values, Fig. 3. The J values from this method were higher than the actual values, Fig. 4. This method was limited in that final displacement values for some specimens could not be reached. One method of attaining a particular displacement was to increase the yield. This has the effect of decreasing the plastic zone size and allowing solutions for displacements in regions where the equations are valid.

Since the combination of P and a and the combination of v and a bracketed the actual data, it seems reasonable that using a and the area under the P–v diagram for a non-growing crack would give better results. The P–v curve from this method compares well with the actual P–v record for both materials, Fig. 5. Of course, J_D values are identical to the actual J_D, however, J_M is lower than the actual J_M. This implies that Mall and Newman's equations, while good for total displacement, do not properly proportion elastic and plastic contributions.

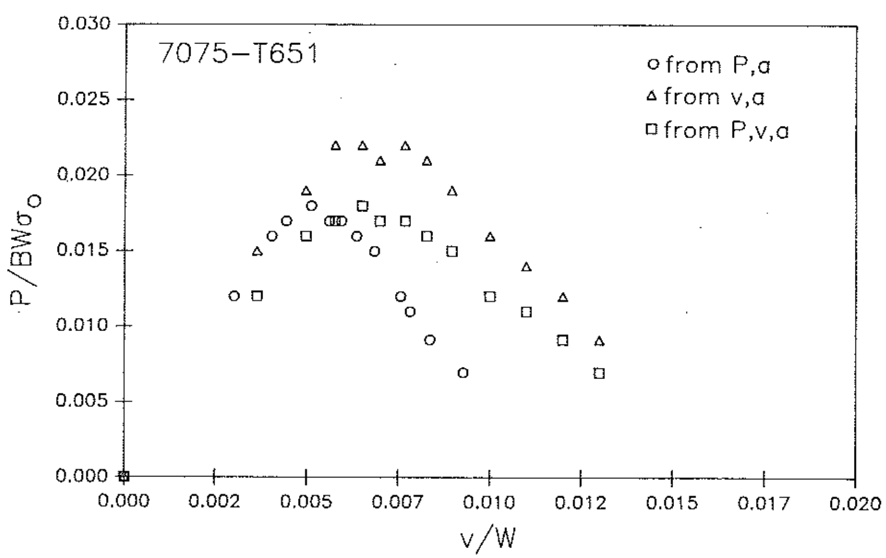

Fig 3 Actual P–v record vs calculated values

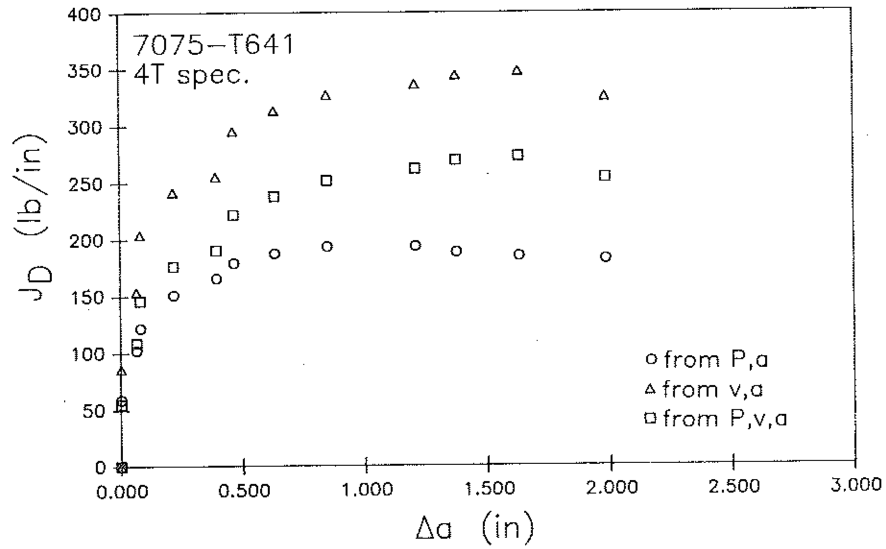

Fig 4 **Actual J_D–R curve vs calculated values (1 in. = 25.4 mm; and 1 lb/in. = 1.75 10^{-4} MPa/m)**

Correction of the CTOD

Mall and Newman's equations provide a sort of 'state variable', i.e., history independent, solution to the crack growth parameters and do not necessarily take into account the irreversibility of plastic deformation. For example, if a

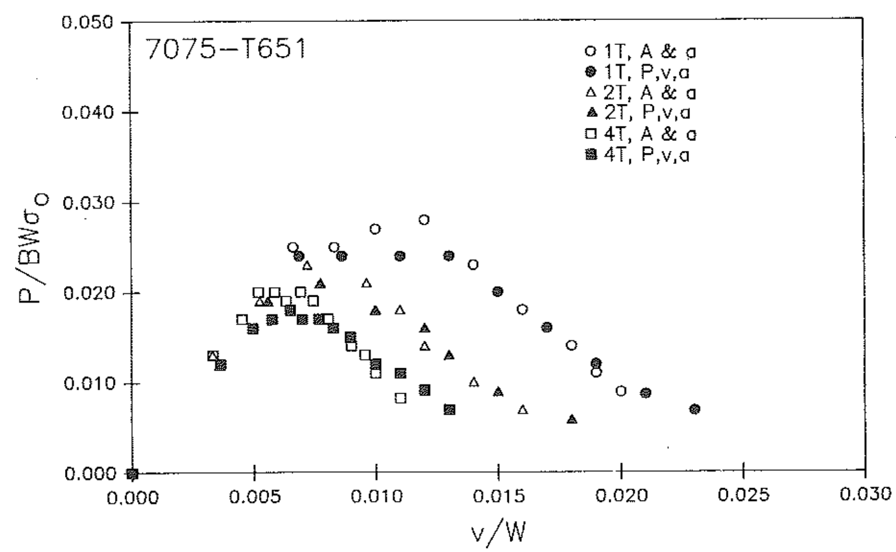

Fig 5 **Actual P–v record vs calculated values**

crack of length a extends incrementally to $a + \delta a$ and the plastic part of the COD, COD_{pl} (with COD being the displacement v taken at the load line) is held constant, the plastic part of the CTOD, $CTOD_{pl}$, is found to decrease, which is not realistic. To compensate for this, the state variable CTOD ($CTOD_{sv}$) can be adjusted to a real CTOD ($CTOD_r$)

$$\mathrm{CTOD_r} = \mathrm{CTOD_{sv}} - \int_{a_0}^{a} \frac{\partial \mathrm{CTOD_{plsv}}}{\partial a}\bigg|_{v_{pl}} \mathrm{d}a \tag{25}$$

or

$$\frac{\mathrm{dCTOD}}{\mathrm{d}a}\bigg|_{r} = \frac{\mathrm{dCTOD}}{\mathrm{d}a}\bigg|_{sv} - \frac{\partial \mathrm{CTOD_{plsv}}}{\partial a}\bigg|_{v_{pl}} \tag{26}$$

Note that the value of the integral is negative so that $CTOD_r > CTOD_{sv}$. This correction is related to the difference between J_D and J_M, emphasised in equations (17) and (18). This correction was applied to 1T, 2T, and 4T specimen test data of 7075 Al. The CTOD's were determined by using area and a as the two independent parameters. In fact, $J_D/CTOD_{sv}$ compares well with $J_M/CTOD_r$, as shown in Fig. 6.

A Dugdale model with variable strength distribution

One shortcoming of the Dugdale model is the assumption of a constant stress acting at the plastic zone. However, due to plastic constraint and strain hardening, the effective yield stress may have a peak of several times the nominal

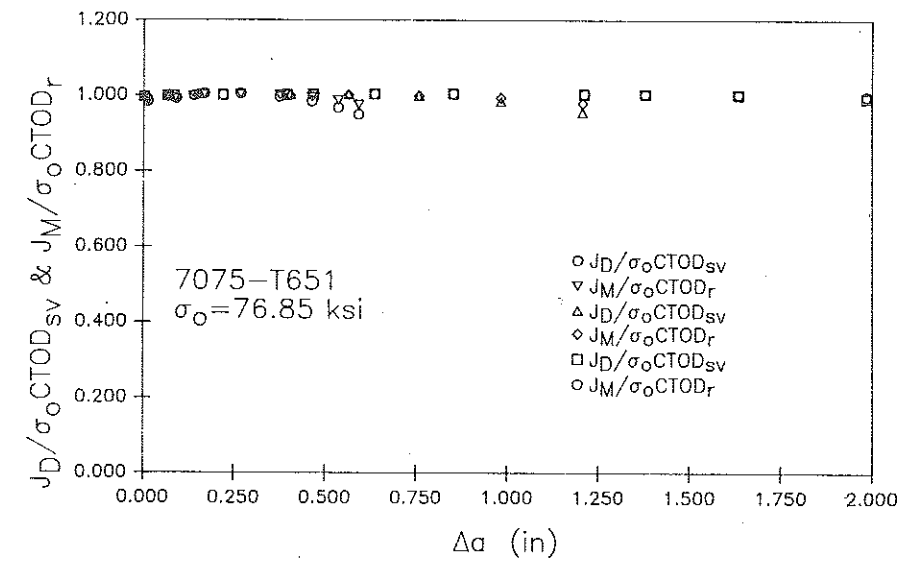

Fig 6 **Relationship between J_D, J_M, $CTOD_{sv}$, and $CTOD_r$ (1 in. = 25.4 mm; and 1 ksi = 6.895 MPa)**

value, thus a non-uniform stress distribution may be in order. The actual stress distribution normal to the crack plane is thought to be zero at the crack tip because of the free surface, and to rise sharply to a maximum at a small distance ahead of the tip. From the maximum it decreases approximately inversely proportional to the square root of the distance from the crack tip until it reaches the nominal value of yield at the end of the plastic zone.

Harrop **(17)** extended the Dugdale model for a crack in a infinite plate under remote tension to the case of a closure stress distribution, σ_{cl} in the plastic zone of the form

$$\sigma_{cl} = nr^2 + mr + c \tag{27}$$

where r is the distance from the plastic zone tip. Using complex variable analysis, he solved for the crack opening displacement as a function of position, and for the remotely applied stress, σ, as a function of crack length and plastic zone size. The equation for the crack opening displacement at x is

$$\begin{aligned}2v(x) = \frac{8}{\pi E}\Bigg[& n\left\{\frac{a^3}{3}\coth^{-1}\sqrt{\left(\frac{l^2-x^2}{l^2-a^2}\right)} - \frac{x^3}{3}\coth^{-1}\frac{a}{x}\sqrt{\left(\frac{l^2-x^2}{l^2-a^2}\right)}\right. \\ & \left. + \frac{a\sqrt{\{(l^2-x^2)(l^2-a^2)\}}}{3} + \frac{(l^2-x^2)^{3/2}}{3}\cos^{-1}\frac{a}{l}\right\} \\ & + m\left\{\frac{\sqrt{\{(l^2-x^2)(l^2-a^2)\}}}{2} - \frac{x^2-a^2}{2}\coth^{-1}\sqrt{\left(\frac{l^2-x^2}{l^2-a^2}\right)}\right\} \\ & + c\left\{a\coth^{-1}\sqrt{\left(\frac{l^2-x^2}{l^2-a^2}\right)} - x\coth^{-1}\frac{a}{x}\sqrt{\left(\frac{l^2-x^2}{l^2-a^2}\right)}\right\}\Bigg]\end{aligned} \tag{28}$$

The equation for the remote stress is

$$\sigma = \frac{2}{\pi}\left[\frac{n}{a}\left\{l^2\cos^{-1}\frac{a}{l} + a\sqrt{(l^2-a^2)}\right\} + m\sqrt{(l^2-a^2)} + c\cos^{-1}\frac{a}{l}\right] \tag{29}$$

Unfortunately, any second order stress distribution with the conditions $\sigma_{cl} = 0$ at $r = 0$ and $\sigma_{cl} = \sigma_0$ at $r = \rho$ will result in either a maximum located at the middle of the plastic zone or negative stresses at the end of the tip of the plastic zone. To match the actual stress distribution an expression for σ_{cl} of at least third order is needed, resulting in a much more complicated analysis which is beyond the scope of this work. A closer approximation would be a linear stress distribution with $\sigma_{cl} = 3\sigma_0$ at $r = 0$ and $\sigma_{cl} = \sigma_0$ at $r = \rho$.

Determination of r_c

The Wnuk–Dugdale model was shown to give the same result as the FEA for small-scale yielding conditions. It also provides an equation for the length parameter R for different specimen geometries and level of yielding.

However, the preceding analyses fail to account for the behaviour of some materials which have unique but non-linear resistance curves. This may be due in part to the assumption that r_c and δ_c are constant material properties. Models of crack growth based on growth and coalescence of voids about microstructural features generally rely on r_c and δ_c being constant. However, there exists at the crack tip a zone of intense deformation, i.e., the process zone, whose size depends on the loading. It seems reasonable that the increment of crack extension be proportional to this process zone size. The triaxiality of the stress field varies with the distance from the tip and the level of applied deformation. It may induce 'preferred' locations for void growth within the process zone that depend on the applied load.

It has been shown **(18)** that this process zone is about 2–3 times the CTOD. It is proposed that r_c is some fraction of the process zone, i.e., r_c is proportional to J and the CTOD. Now if $\mathrm{d}J/\mathrm{d}a$ is approximated by $\sigma_0\delta_c/r_c$, equations (8) or (17), then the R curve will be non-linear, but maintain uniqueness. Of course, unique but non-linear R curves may be the product of materials that are not low strength/high toughness, i.e., when simplifications made possible by neglecting the second term of equation (8) may not be applicable.

Implementation of the Wnuk–Dugdale growth model

As a first step, a sensitivity analysis was conducted for the Wnuk–Dugdale model to determine the influences of a constant r_c, and r_c proportional to the CTOD and non-uniform stress distributions. The relationships between δ_c, r_c, and J were explored by comparing the growth model to real test records. These relationships were then used to model the crack growth of real specimens.

Sensitivity analysis

For the sensitivity analysis, this model was run with values of $\sigma_0 = 50$ ksi and $E = 30\,000$ ksi, which is typical of steel. For the first runs, r_c and δ_c were kept constant during growth. Since this model is of a semi-infinite plate, large and small-scale yielding were modelled by small and large initial crack lengths, respectively. Changing the initial crack length had little effect on the results, as shown in Fig. 7 which implies independence of extent of yielding.

Figure 8 shows the effect of varying the initial values of r_c and δ_c independently, but keeping them constant during growth. As r_c increases, less J is required for crack growth. This trend is reversed for δ_c increasing. Figure 8 also shows the dependence of $\mathrm{d}J/\mathrm{d}a$ on the δ_c/r_c ratio, as previously discussed. It seems that the ratio of δ_c and r_c is more important than their absolute values.

The effect of the non-uniform stress profile is much the same as that of increasing the r_c/δ_c ratio, as seen in Fig. 9. First order stress profiles were used

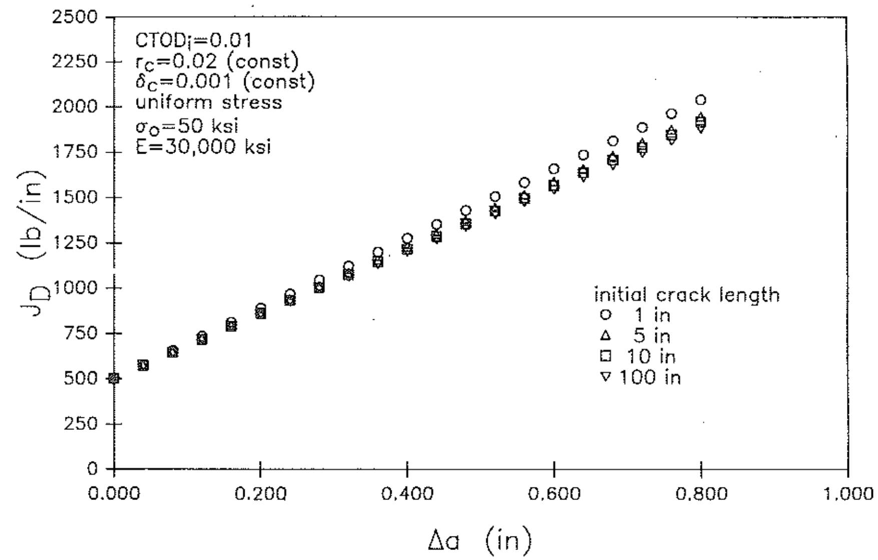

Fig 7 **Effect of initial crack length on J_D (1 in. = 25.4 mm; and 1 lb/in. = 1.75 10^{-4} MPa/m)**

since no suitable second order distribution could be found. The stress was specified as the yield at the end of the plastic zone ($x = 1$) and 2 and 3 times the yield at the tip ($x = a$). Because the same effect can be produced by changing the r_c/δ_c ratio and because these stress profiles tended to limit the extent of

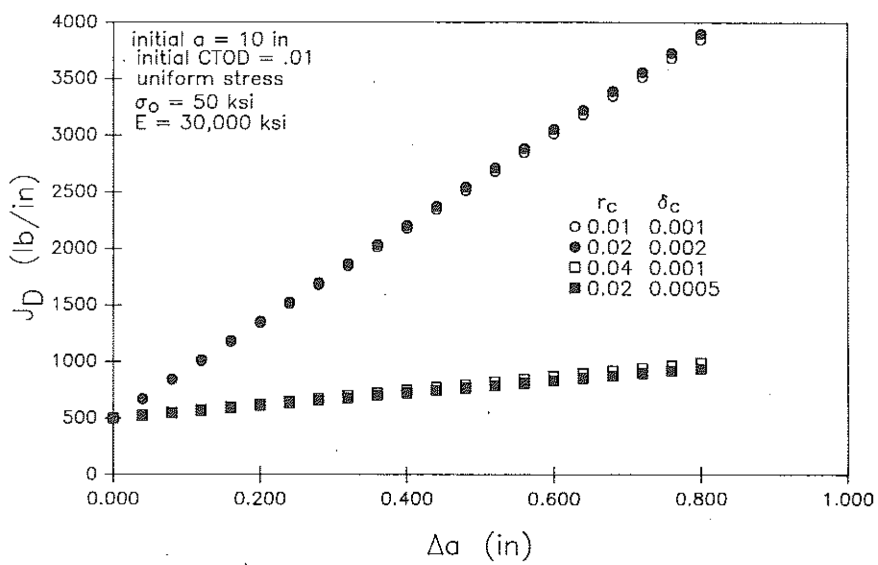

Fig 8 **Effect of r_c/δ_c on J_D (1 in. = 25.4 mm; and 1 lb/in. = 1.75 10^{-4} MPa/m)**

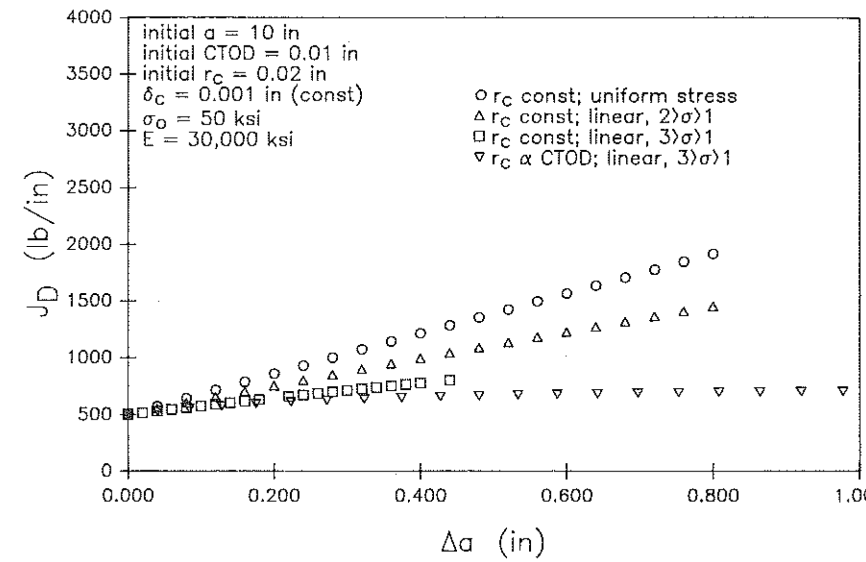

Fig 9 **Effect on non-uniform Dugdale stress (1 in. = 25.4 mm; and 1 lb/in. = 1.75 10^{-4} MPa/m)**

crack growth that could be modelled, non-uniform stress profiles were not studied further.

If the process zone is assumed to be proportional to the CTOD, or J, then as J increases dJ/da should decrease and non-linear curves can be generated

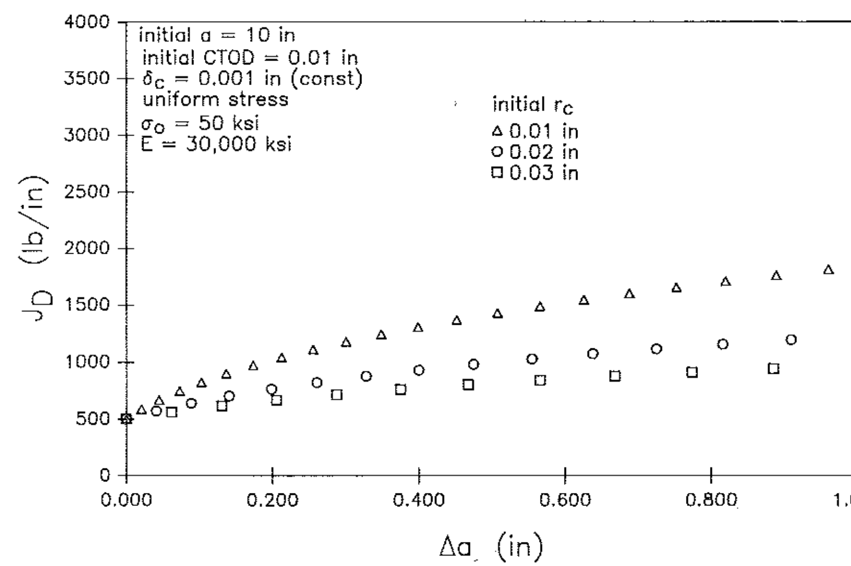

Fig 10 **Effect of r_c proportional to J_D on J_D (1 in. = 25.4 mm; and 1 lb/in. = 1.75 10^{-4} MPa/m)**

while maintaining uniqueness, as shown in Fig. 10. Initial crack length had little effect on this growing process zone model.

Determination of r_c–δ_c for real materials

This model was compared to actual experimental results from specimens of stainless steel (303SS) and aluminum (7075-T651) **(15)**. Reference Table 2 for material properties.

To study the r_c–δ_c–J relationship, the model was implemented with a constant r_c value and then forced to follow the same J_M–Δa path as the real specimen. The necessary values of δ_c for each increment were calculated. For this model (tension in a semi-infinite plate) J_D and J_M are approximately equal, but in the real data (bending with a finite ligament) J_M exhibits less size dependence and was used for comparisons **(7)(8)**, δ_c was used as the free variable for simplicity of computation. Because r_c/δ_c is a more significant parameter, it should be possible to vary either δ_c or r_c and obtain the same results so long as the proper ratio is maintained. A general expression can be written for the relationship between r_c/δ_c and J as

$$\frac{r_c}{\delta_c} = k\left(\frac{J}{J_{Ic}}\right)^n \tag{30}$$

Once the r_c/δ_c ratios were determined for the whole curve, the constants k and n were evaluated from log–log plots. Combining this equation with equation (13), the condition for crack growth becomes

$$\frac{dJ}{da} J^n = \frac{\sigma_0}{k} J_{Ic}^n = \text{constant} \tag{31}$$

This type of relationship between r_c/δ_c and J is analogous to that proposed by Paris, Saka *et al.*, Wilson, and others **(19)–(22)** where for different alloys they determined n to range between 1 and 3.

For the 303 SS, k ranged from 4.6 to 4.9 and n from 0.56 to 0.62 (see Table 3). Substitution of the r_c/δ_c equation and these constants into the growth model resulted in good agreement, as seen in Fig. 11. The ratio of ε_0 to the average initial value of δ_c/r_c was 0.0049, well below Smith's suggested value of 0.02.

Following the same procedures, the 7075 aluminum yielded values of k from 14 to 18 and n from 0.01 to -0.06. Substitution of these values into the model did not result in good agreement. Plots of $\ln(r_c/\delta_c)$ versus $\ln(J/J_{Ic})$ for the 7075 showed better straight line correlation than the 303 SS, so that one would expect results at least as good. However, the ratio of ε_0 to the average initial value of δ_c/r_c was 0.07, almost four times the suggested value indicating that the logarithmic term (reference equation (8)) is not negligible, which was the assumption for these computations.

Table 2 Material properties for modelling data

Material	*Yield (ksi)*	*Young's modulus (ksi)*	J_{Ic} *(lb/in)*
303 SS	35	30 000	125
7075 Al	76.85	10 400	55

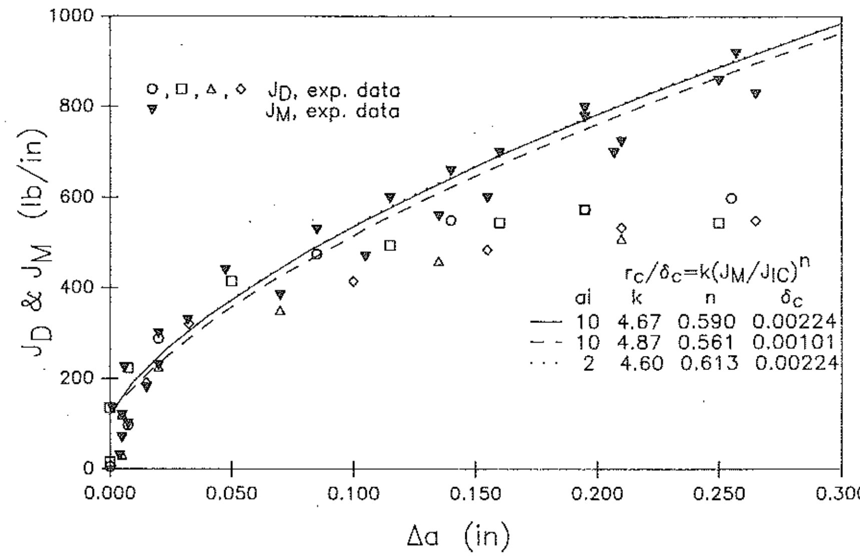

Fig 11 Modelling of 303 SS

Conclusions

The following conclusions can be drawn.

(1) The Wnuk–Dugdale model (WDM) was shown to give similar results to those obtained from FEA by Rice *et al.* for small-scale yielding conditions.
(2) The WDM also provides ways to determine the relationship between R load and crack length.
(3) The Wnuk model was expressed in a general way: in one side of the equation emphasising the need to alter the original definition of the CTOD, i.e.,

Table 3 Parameters for Wnuk/Dugdale growth model

Material	a_0 *(in)*	r_{c0} (10^{-3} *in*)	δ_{c0} (10^{-3} *in*)	*K*	*n*
303 SS	10	10.5	2.24	4.67	0.590
303 SS	10	3.50	1.01	4.87	0.561
303 SS	2	10.5	2.24	4.60	0.613
7075 Al	10	0.700	0.0839	14.7	0.0116
7075 Al	10	1.40	0.137	16.7	−0.0396
7075 Al	1	1.40	0.136	16.8	−0.0376

CTOD*, and identifying the other with material parameters that can also be a function of the applied deformation, through the process zone size, stress triaxiality, etc.

(4) The similarity of the CTOD* and J_M was highlighted.

(5) Experimental data were used to model crack growth in CT specimens. It was demonstrated that the adjustment of the $CTOD_{pl}$ to account for the irreversibility of the process yields a direct connection to J_M.

(6) Unique and yet non-linear resistance curves were discussed and explained in terms of an r_c that depends on the extent of the process zone.

(7) The coefficient and exponent of the relationship between r_c and J were obtained from different sets of experimental data. It was shown that this dependence is very strong in some cases, but for others the exponent is very close to zero indicating a virtual constant r_c.

(8) It is believed that these methods can provide insight in the interplay of the different variables controlling the crack process, such as distribution and mean distance of voids or particles, together with process zone size, degree of triaxiality, etc.

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