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J Integral for Thin Shells

REFERENCE Sedmake, A., Berković, M., and Jarić, J., ***J* integral for thin shells**, *Defect Assessment in Components – Fundamentals and Applications*, ESIS/EGF9 (Edited by J. G. Blauel and K.-H. Schwalbe) 1991, Mechanical Engineering Publications, London, pp. 45–53.

ABSTRACT A thin shell *J* integral is introduced applying Gurtin's approach to the problem of quasi-static crack growth. It is shown that the thin shell *J* integral is path dependent, unless the middle surface is cylindrical and the crack is positioned along a generatrix. The finite element method has been employed for the evaluation of thin shell *J* integral. A cylindrical shell with an axial crack has been analysed, using the *J* integral as a measure for the crack driving force. The results are compared with the results from literature, indicating good agreement.

Introduction

The structural analysis of thin shells is a very important part of continuum mechanics, having in mind structures like pressure vessels, aircrafts and other thin walled parts, especially if containing cracks. There are many formulae for stress intensity factors for thin shells which are valid in the scope of linear elasticity but only few solutions for nonlinear fracture mechanics parameters, such as COD and *J* integral, **(1)–(4)**, exist. Nevertheless, these solutions are either restricted to special shell shapes (spherical or cylindrical), or not given in a suitable form for practical use.

Therefore, the aim of this paper is to present the *J* integral for thin shells, defined recently **(5)** as a general fracture mechanics parameter, valid for any shape of shell middle surface. Gurtin's approach **(6)** has been employed in order to define the energy release rate due to a unit crack growth and the appropriate integral expression, analogous to Rice's *J* integral, which is called thin shells *J* integral. Due to its physical meaning and path independence, the *J* integral for thin shells can be considered as a general fracture mechanics parameter, strictly valid in the scope of nonlinear elasticity. Therefore, we are actually dealing with a generalization of Rice's *J* integral to the problem of curved two-dimensional space, described here by Cosserat theory. We shall use an originally developed theory of thin shells here, despite of many already existing, e.g. **(7)**, because it is essential for further formulations. Therefore, we briefly present this theory, leaving out all details described elsewhere **(5)(8)**.

Thin shell theory

The notation follows Nagdi's classical work on the subject **(7)**: curvilinear coordinates in the middle surface are denoted by θ^α, $\alpha = 1, 2$, the third coordinate along the director by $\theta^3 = \xi$, Cartesian coordinates of a point in the middle surface by X^i, and any point of the shell by Y^i, $i = 1, 2, 3$. It is assumed

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that $\partial Y^i/\partial\theta^\alpha \neq 0$, i.e. transformations $Y^i = Y^i(\theta^\alpha)$ and $\theta^\alpha = \theta^\alpha(Y^i)$ are possible. The upper-case letters are used for the undeformed configuration and the lower-case letters for the deformed one. There are several essential assumptions in this theory, having in mind certain features of thin shells:

- the thickness is small compared with the minimal curvature radius ($H/R \ll 1$) and the characteristic length ($H/L \ll 1$), so that $O(H/R)$ and $O(H/L)$ are small quantities of the first order. Based on this assumption all thin shell quantities in this theory are given in the form of the first order approximations.
- the stress and strain tensor components (membrane, bending, and shear) are presented as Legendre polynomials in order to describe suitably boundary conditions on the shell faces.
- deformation–displacement, constitutive and equilibrium equations are derived independently (three-field theory) on the basis of three-dimensional elasticity theory.

We now quote all necessary equations of thin shell theory without any details of their derivation, which is given elsewhere, **(5)(8)**. The position vectors of any shell point in the undeformed and deformed configuration is given by

$$\boldsymbol{Y} = \boldsymbol{X} + \xi\,\frac{\partial \boldsymbol{Y}}{\partial \xi} = \boldsymbol{X} + \tfrac{1}{2}\zeta\boldsymbol{H} \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{x} + \xi\,\frac{\partial \boldsymbol{y}}{\partial \xi} = \boldsymbol{x} + \tfrac{1}{2}\zeta\boldsymbol{h} \tag{2}$$

where $\boldsymbol{H}$* denotes the director and $\xi = H\zeta/2$. The displacement vector is defined by

$$\boldsymbol{w} = \boldsymbol{y} - \boldsymbol{Y} = \boldsymbol{x} - \boldsymbol{X} + \tfrac{1}{2}\zeta(\boldsymbol{h} - \boldsymbol{H}) = \boldsymbol{u} + \tfrac{1}{2}\zeta\boldsymbol{k} \tag{3}$$

and its partial derivatives by

$$\boldsymbol{w}_\alpha = \frac{\partial \boldsymbol{w}}{\partial \theta^\alpha} = \boldsymbol{u}_\alpha + \tfrac{1}{2}\zeta\boldsymbol{k}_\alpha; \qquad \boldsymbol{w}_3 = \frac{\partial \boldsymbol{w}}{\partial \zeta} = \tfrac{1}{2}\boldsymbol{k} \tag{4}$$

Introducing the base vectors in an usual manner one can define metric tensor components and determinant in the undeformed configuration as follows

$$G_{\alpha\beta} = \boldsymbol{X}_\alpha \cdot \boldsymbol{X}_\beta + \tfrac{1}{2}\zeta(\boldsymbol{X}_\alpha \cdot \boldsymbol{H}_\beta + \boldsymbol{X}_\beta \cdot \boldsymbol{H}_\alpha) = A_{\alpha\beta} - \zeta H B_{\alpha\beta};$$

$$G_{\alpha 3} = 0; \qquad G_{33} = H^2/4 \tag{5}$$

$$G = H^2(A - \zeta H B)/4 \tag{6}$$

* Bold letters denote vectors.

where A and B denote the first and second fundamental form. Using Galerkin's procedure it is now possible to define strain–displacement equations as an approximation of the three-dimensional theory:

$$\gamma^0_{\alpha\beta} = 2\bar{\boldsymbol{x}}_\alpha \cdot \boldsymbol{u}_\beta; \qquad \gamma^1_{\alpha\beta} = \tfrac{1}{3}(\bar{\boldsymbol{h}}_\alpha \cdot \boldsymbol{u}_\beta + \bar{\boldsymbol{x}}_\alpha \cdot \boldsymbol{k}_\beta) \tag{7}$$

$$\gamma^0_{\alpha 3} = \bar{\boldsymbol{x}}_\alpha \cdot \boldsymbol{k} + \bar{\boldsymbol{h}} \cdot \boldsymbol{u}_\alpha; \qquad \gamma^1_{\alpha 3} = \tfrac{1}{6}(\bar{\boldsymbol{h}}_\alpha \cdot \boldsymbol{k} + \bar{\boldsymbol{h}} \cdot \boldsymbol{k}_\alpha) \tag{8}$$

where $\gamma^0_{\alpha\beta}$, $\gamma^1_{\alpha\beta}$, $\gamma^0_{\alpha 3}$ and $\gamma^1_{\alpha 3}$ denote the coefficients of Legendre polynomials, and the bar denotes so-called middle configuration, defined by $2\bar{\boldsymbol{x}} = \boldsymbol{X} + \boldsymbol{x}$.

Since the strain energy density is a function of $\boldsymbol{x}_\alpha$, $\boldsymbol{h}$, and $\boldsymbol{h}_\alpha$, $W = W(\boldsymbol{x}_\alpha, \boldsymbol{h}, \boldsymbol{h}_\alpha)$, the equilibrium equations can be written as

$$\left.\frac{\partial W}{\partial \boldsymbol{x}_\alpha}\right|_\alpha = 0 \tag{9}$$

$$\left.\frac{\partial W}{\partial \boldsymbol{h}_\alpha}\right|_\alpha = \frac{\partial W}{\partial \boldsymbol{h}} \tag{10}$$

where | denotes the covariant derivative. Equations (9) and (10) are in fact the membrane and bending equilibrium equations in the absence of inertial and volume forces. Now, one can define nonlinear constitutive equations

$$\boldsymbol{N}^\alpha = \frac{\partial W}{\partial \boldsymbol{x}_\alpha}; \qquad \boldsymbol{M}^\alpha = \frac{\partial W}{\partial \boldsymbol{h}_\alpha}; \qquad \boldsymbol{s} = \frac{\partial W}{\partial \boldsymbol{h}} \tag{11}$$

where $\boldsymbol{N}^\alpha$, $\boldsymbol{M}^\alpha$ and $\boldsymbol{s}$ denote membrane, bending and shear stress vectors. If the explicit (linear elastic) constitutive equations are required, Galerkin's procedure can be applied resulting in an approximation of the three-dimensional constitutive equations **(8)**.

Thin shell J integral

Only a brief description of the thin shell J integral deduction will be given here, since all necessary details are given elsewhere **(5)**. The energy release rate due to the unit quasi-static crack growth has been defined using Gurtin's approach **(6)**, giving a sound physical meaning of the final integral expression. With regard to this, we first introduce thin shell with an edge crack, represented in Fig. 1, with the following notation:

$S(l) = \{z(s):0 \leqslant s \leqslant l\}$ – set of points comprising the crack of length l
$z_l = z(l)$ – crack tip position vector
$\boldsymbol{r}$ – position vector of any material point relative to the crack tip
$\boldsymbol{e} = \partial z_l/\partial l$ – unit crack growth vector
$\boldsymbol{v}$ – unit outward normal vector to the boundary
$\boldsymbol{m}$ – unit outward normal vector to the crack faces.

Crack length l is used as a time scale and fracture fields of C^n order are introduced under the same assumptions as in Gurtin's paper **(6)**. Therefore, in

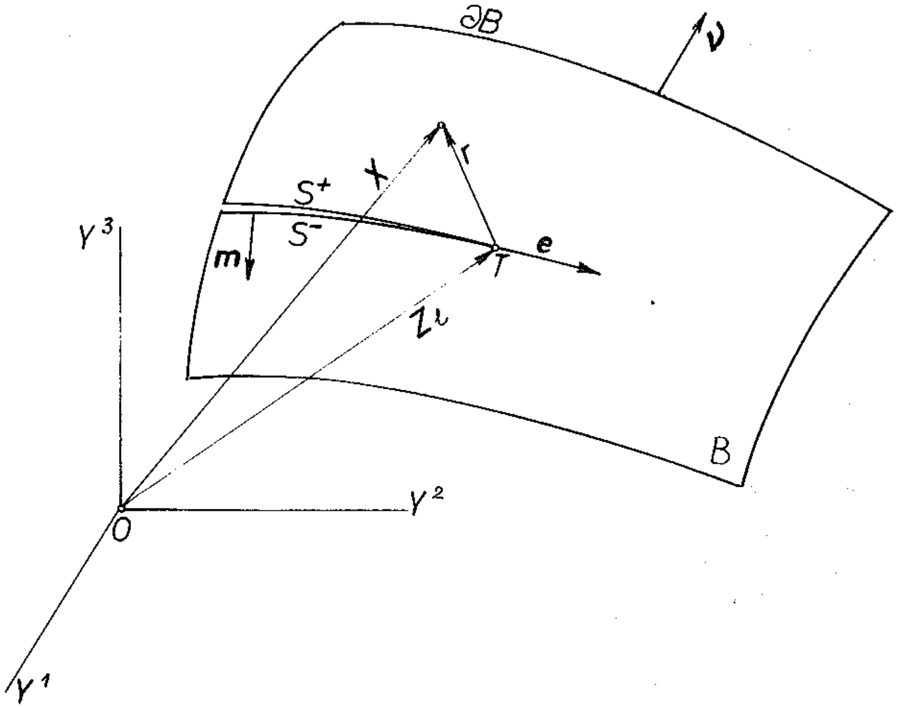

Fig 1 Thin shell with an edge crack

order to define any singular field at the crack tip, it is necessary to introduce an isolated region around crack tip. Here we have used a sphere of the radius δ, centred at the crack tip whose intersection with the surface B defines an isolated region D_δ (Fig. 2). Now, one can apply the divergence theorem over the regular domain B_δ (Fig. 3)

$$\int_{B_\delta} \dot{W}\,\mathrm{d}S = \int_{\partial B} (\boldsymbol{N}^\alpha \cdot \dot{\boldsymbol{x}} + \boldsymbol{M}^\alpha \cdot \dot{\boldsymbol{h}})\nu_\alpha\,\mathrm{d}L - \int_{\partial D_\delta} (\boldsymbol{N}^\alpha \cdot \dot{\boldsymbol{x}} + \boldsymbol{M}^\alpha \cdot \dot{\boldsymbol{h}})n_\alpha\,\mathrm{d}L \tag{12}$$

where

$$\dot{W} = (\boldsymbol{N}^\alpha \cdot \dot{\boldsymbol{x}} + \boldsymbol{M}^\alpha \cdot \dot{\boldsymbol{h}})|_\alpha \tag{13}$$

On the other hand, the transport theorem for thin shells can be written as

$$\frac{\mathrm{d}}{\mathrm{d}l}\int_{B_\delta} W\,\mathrm{d}S = \int_{B_\delta} \dot{W}\,\mathrm{d}S - \int_{\partial D_\delta} W\boldsymbol{\nu}\cdot\boldsymbol{n}\,\mathrm{d}L \tag{14}$$

Having in mind Fig. 2, $\boldsymbol{\nu}\cdot\boldsymbol{n}$ can be expressed as

$$\boldsymbol{\nu}\cdot\boldsymbol{n} = [\boldsymbol{e} - (\boldsymbol{e}\cdot\boldsymbol{N})\boldsymbol{N}]\cdot\boldsymbol{n} = \boldsymbol{e}\cdot\boldsymbol{n} \tag{15}$$

since $\boldsymbol{N}\cdot\boldsymbol{n} = 0$.

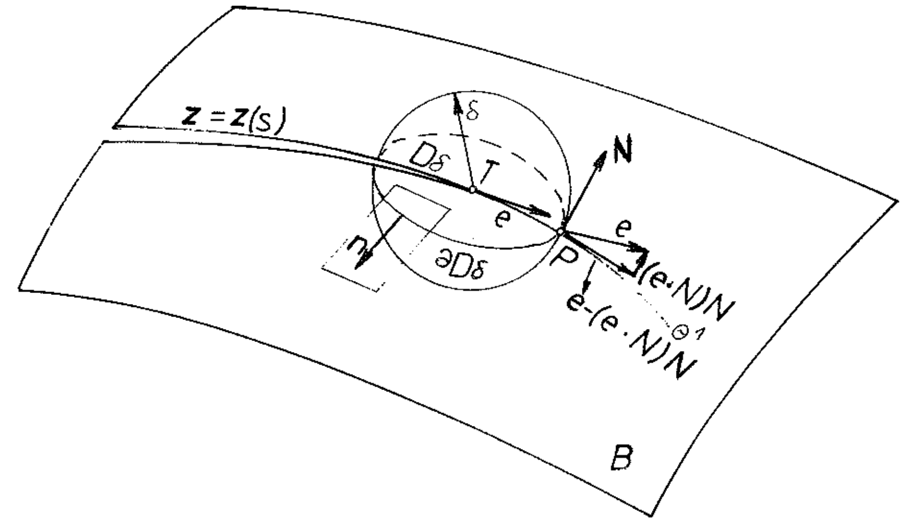

Fig 2 Definition of the regions and boundaries

In order to define the energy release rate, we write down the global energy balance law for the cracked thin shell

$$\frac{\mathrm{d}}{\mathrm{d}l}\int_{B} W\,\mathrm{d}S + \varepsilon(l) = \int_{\partial B} (\boldsymbol{N}^\alpha \cdot \dot{\boldsymbol{x}} + \boldsymbol{M}^\alpha \cdot \dot{\boldsymbol{h}})\nu_\alpha\,\mathrm{d}L \tag{16}$$

where $\varepsilon(l)$ denotes the energy release rate due to unit crack growth. After some simple algebra and several physically sound assumptions, one can conclude

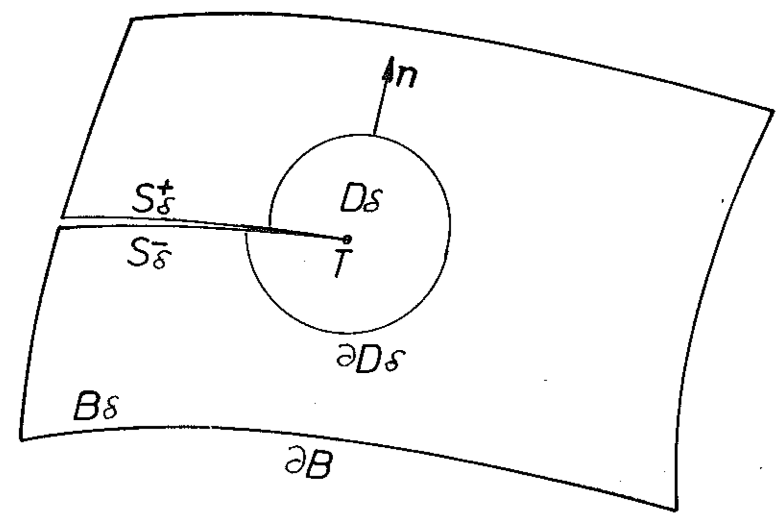

Fig 3 Sphere isolating the crack tip

that

$$\varepsilon(l) = \lim_{\delta \Rightarrow 0} \int_{\partial D_\delta} (W\boldsymbol{e} \cdot \boldsymbol{n} + \boldsymbol{N}^\alpha \cdot \dot{\boldsymbol{u}}\boldsymbol{n}_\alpha)\, \mathrm{d}L \tag{17}$$

It should be noted here that bending components are neglected since they are second order small quantities compared with the membrane components.

Finally we introduce thin shell J integral so that

$$\varepsilon(l) = \lim_{\delta \Rightarrow 0} J(\partial D_\delta) \tag{18}$$

i.e. as an integral expression:

$$J(\Gamma) = \int_\Gamma (W\boldsymbol{e} \cdot \boldsymbol{n} + \boldsymbol{N}^\alpha \cdot \dot{\boldsymbol{u}}\boldsymbol{n}_\alpha)\, \mathrm{d}L = \int_\Gamma (W\delta^\alpha_\beta - \boldsymbol{N}^\alpha \cdot \boldsymbol{u}_\beta) e^\beta n_\alpha \, \mathrm{d}L \tag{19}$$

which is path dependent for general shape of middle surface. This can be concluded from the expression

$$J(\Gamma) = J(\partial D_\delta) + \int_D (W\delta^\alpha_\beta - \boldsymbol{N}^\alpha \cdot \boldsymbol{u}_\beta) b^\alpha_\beta \boldsymbol{e} \cdot \boldsymbol{N}\, \mathrm{d}S + \int_{S_\delta -} W\boldsymbol{e} \cdot \boldsymbol{m}\, \mathrm{d}L - \int_{S_\delta +} W\boldsymbol{e} \cdot \boldsymbol{m}\, \mathrm{d}L \tag{20}$$

derived using the divergence theorem over region D (Fig. 4). Symbols S_δ^+ and S_δ^- are defined in Fig. 4.

It is obvious that thin shell J integral would be path independent if $\boldsymbol{e} \cdot \boldsymbol{m} = 0$ and $\boldsymbol{e} \cdot \boldsymbol{N} = 0$. These conditions are fulfilled only for cylindrical shells with an axial crack, but this does not mean that the expression (20) is

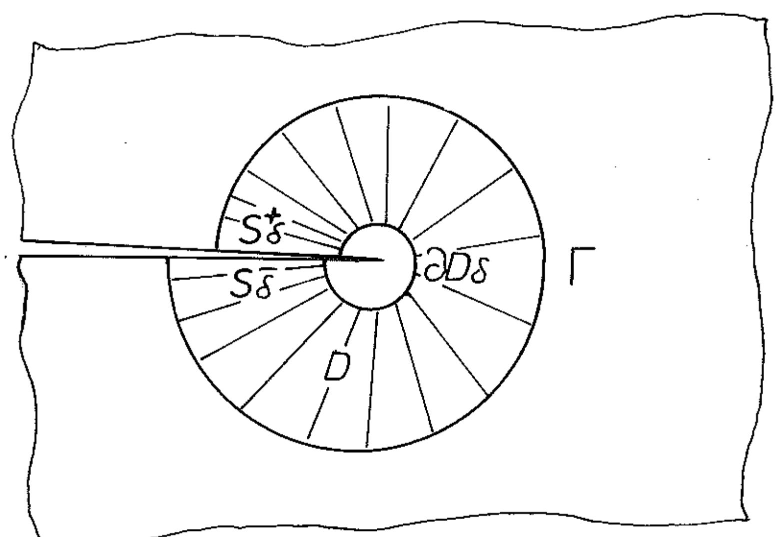

Fig 4 *J*-integration paths and area

useless otherwise. On the contrary, for the general shape of thin shell and arbitrary crack position there is still an integral expression which is path independent and has physical interpretation as an energy release rate, containing additional terms which recover path independence of the J integral, lost due to the shell curvature.

Finally, if the usual assumption is made that θ^1 coordinate is positioned along the crack ($e^1 = 1$, $e^2 = 0$), and taking (18) into account one can get the following expression

$$J(\Gamma) = \varepsilon(l) + \int_D (W\delta^\alpha_1 - \boldsymbol{N}^\alpha \cdot \boldsymbol{u}_1) b^\alpha_1 N_1 \, \mathrm{d}S + \int_{S_\delta -} W m_1 \, \mathrm{d}L - \int_{S_\delta +} W m_1 \, \mathrm{d}L \tag{21}$$

which has been used here in order to evaluate the energy release rate as the measure of crack driving force.

Thin shell *J* integral evaluation

To solve the problem of cracked thin shells, the finite element method has been employed, as described in **(5)(9)**. As it has been shown, the thin shell J integral loses path independence due to the shell curvature and/or the crack position, but there are additional integral expressions (surface term due to curvature and line terms due to the crack position) which recover path independence of the complete integral expression. The integral expression (20) is derived under the assumption of nonlinear elasticity, including both geometrical and material nonlinearity. We shall deal here only with material nonlinearity, i.e. with ductile material behaviour. Very simple procedure is used here, based on linear elastic material behaviour, described by constitutive relations

$$S_0^{\alpha\beta} = \frac{E}{1 - \nu^2} [\nu A^{\alpha\beta} A^{\theta\psi} + (1 - \nu) A^{\alpha\theta} A^{\beta\psi}] \gamma^0_{\theta\psi} \tag{22}$$

$$S_0^{\alpha 3} = \frac{E}{1 + \nu} \left[\frac{2}{H}\right]^2 A^{\alpha\beta} \gamma^0_{\beta 3} \tag{23}$$

where E and ν stand for Young's modulus and Poisson's ratio, respectively. For the plastically deformed material we simply use parameters E_s and ν_s instead of material constants E and ν, which are defined according to the experimental stress–strain curve and by the relation **(10)**

$$\nu_s = \frac{1}{2}\left[1 - \frac{E_s}{E}(1 - 2\nu)\right] \tag{24}$$

Such a procedure can be defined as a modified secant method and can be very useful for many real problems.

Results

The example chosen for testing of the procedure is a cylindrical shell (Radius R, thickness t), with an axial crack (length $2a$). The results are given in the

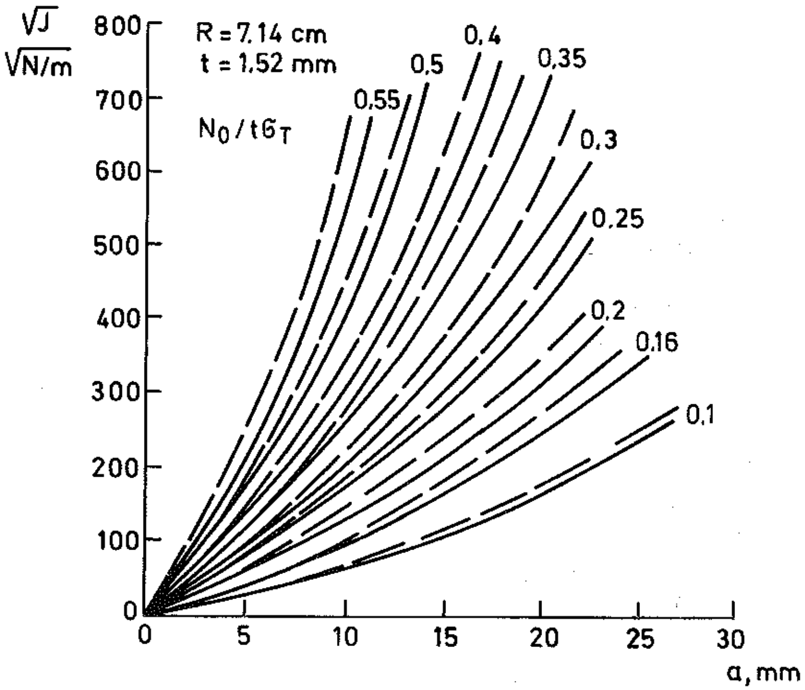

Fig 5 Crack driving forces for the cylindrical shell with an axial crack

form of crack driving force (CDF) curves (Fig. 5 – full lines), obtained using the described procedure for thin shell J integral evaluation. In the same figure, the results obtained by Ratwani *et al.* **(11)** are presented (dotted lines) in order to make a comparison. It should be noticed that the procedure applied in **(11)** is based on an analytical elastic solution (integral equations without transverse shear) extended to the plastic behaviour of material, using Dugdale's model of plastic zone. Having in mind that such a procedure is the conservative one and that the procedure used here underestimates the exact solution, one can say that the results presented in Fig. 5 are in good agreement. Of course, further analysis for the other important problems, such as cylindrical shell with a circumferential crack and a spherical shell with a meridian crack, as well as the problem of part-through cracks, should be performed.

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