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J Integral Calculations in Mixed Mode Elastic–Plastic Crack Problems

REFERENCE Diekmann, P., Buchholz, F.-G., Grebner, H., and Richard, H.-A., ***J* integral calculations in mixed mode elastic–plastic crack problems**, *Defect Assessment in Components – Fundamentals and Applications*, ESIS/EGF9 (Edited by J. G. Blauel and K.-H. Schwalbe) 1991, Mechanical Engineering Publications, London, pp. 35–44.

ABSTRACT For the treatment of elastic–plastic problems the use of the *J* integral as fracture mechanics parameter is commonly adopted. In practical problems often a mixed mode loading situation occurs. To describe this mixed mode crack problem adequately by the *J* integral, a splitting of the complete value into parts according to the different fracture modes is necessary.

For two-dimensional problems Ishikawa *et al.* **(1)** proposed a method of separation of *J* into J_I and J_{II}, corresponding to the mode I and mode II part, respectively, of the *J* integral. The same separation formulae were deducted by Bui by means of complex functions for linear elastic material behaviour. In our paper linear elastic and elastic–plastic finite element calculations are presented of two-dimensional and three-dimensional models of the compact tension shear (CTS) specimen developed by Richard **(2)**. By rotation of the loading direction the CTS specimen permits the study of pure mode I and mode II as well as mixed mode (i.e. combined mode I and mode II) loading.

The calculations show the applicability of the *J* separation procedures. In the linear elastic two-dimensional analyses a very good coincidence with a reference solution is found.

Introduction

Cracks in real structures are often exposed to combined loadings. Each of these parts of the loading, consisting of tensile and shear stresses, influences the further growth of the crack. From the literature several procedures are known (e.g. **(3)**–**(5)**) for the prediction, of whether, and in what direction, a crack will grow.

Experiments (for instance, described in **(2)**) have shown that a reasonable prediction of the crack behaviour is possible in the case of brittle materials, where linear elastic fracture mechanics may be used. Mostly the stress intensity factor concept is adopted in this case, which allows a separate calculation of K_I, K_{II}, or K_{III} for the different fracture modes, shown in Fig. 1.

For ductile material the application of the *J* integral concept is well established in fracture mechanics, though, strictly speaking, the path independency of the *J* integral is only proved for linear and nonlinear elastic material behaviour. As in the brittle case, a separation of *J* into its parts J_I, J_{II}, J_{III} is necessary to predict crack growth for a mixed mode loading situation.

In this paper two- and three-dimensional elastic and elastic–plastic finite element calculations of a special mixed mode fracture mechanics specimen are presented.

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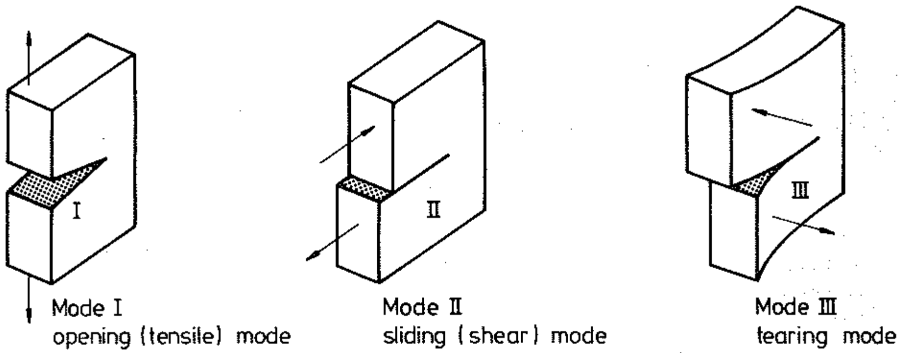

Fig 1 Principal fracture modes for tensile and shear loading cases

Calculation of separated *J* integral values

The original formulation of the J integral proposed by Rice **(6)** is a line integral. For finite element calculations it has been shown to be more convenient to use a proposal by DeLorenzi **(7)**, which makes use of the virtual crack extension technique according to Parks **(8)**, and is applicable for two- and three-dimensional problems.

For two-dimensional calculations the separation of J into the mode I and II parts J_{I} and J_{II} follows a rather straightforward formulation by Ishikawa *et al.* **(1)**.

Thereby the stresses, strains, and displacements in the integration region are combined according to the following equations

$$\begin{bmatrix}\sigma^{\mathrm{I}}_{11}\\ \sigma^{\mathrm{I}}_{22}\\ \sigma^{\mathrm{I}}_{12}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\sigma_{11}+\sigma'_{11}\\ \sigma_{22}+\sigma'_{22}\\ \sigma_{12}-\sigma'_{12}\end{bmatrix}\qquad \begin{bmatrix}\sigma^{\mathrm{II}}_{11}\\ \sigma^{\mathrm{II}}_{22}\\ \sigma^{\mathrm{II}}_{12}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\sigma_{11}-\sigma'_{11}\\ \sigma_{22}-\sigma'_{22}\\ \sigma_{12}+\sigma'_{12}\end{bmatrix}$$

$$\begin{bmatrix}\varepsilon^{\mathrm{I}}_{11}\\ \varepsilon^{\mathrm{I}}_{22}\\ \varepsilon^{\mathrm{I}}_{12}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\varepsilon_{11}+\varepsilon'_{11}\\ \varepsilon_{22}+\varepsilon'_{22}\\ \varepsilon_{12}-\varepsilon'_{12}\end{bmatrix}\qquad \begin{bmatrix}\varepsilon^{\mathrm{II}}_{11}\\ \varepsilon^{\mathrm{II}}_{22}\\ \varepsilon^{\mathrm{II}}_{12}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\varepsilon_{11}-\varepsilon'_{11}\\ \varepsilon_{22}-\varepsilon'_{22}\\ \varepsilon_{12}+\varepsilon'_{12}\end{bmatrix}$$

$$\begin{bmatrix}u^{\mathrm{I}}_{1}\\ u^{\mathrm{I}}_{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}u_{1}+u'_{1}\\ u_{2}-u'_{2}\end{bmatrix}\qquad \begin{bmatrix}u^{\mathrm{II}}_{1}\\ u^{\mathrm{II}}_{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}u_{1}-u'_{1}\\ u_{2}+u'_{2}\end{bmatrix}$$

σ_{ij} and σ'_{ij}, ε_{ij} and ε'_{ij}, and u_i and u'_i are stresses, strains, and displacements at a point P of the integration area, and a corresponding point P', which lies symmetrically to P with respect to the crack plane.

The above-mentioned equations were also deduced by Bui **(13)** for linear elastic material behaviour, using complex functions.

An analogous separation procedure was given for three-dimensional problems by Atluri and Nishioka **(9)** and in similar form developed by one of the

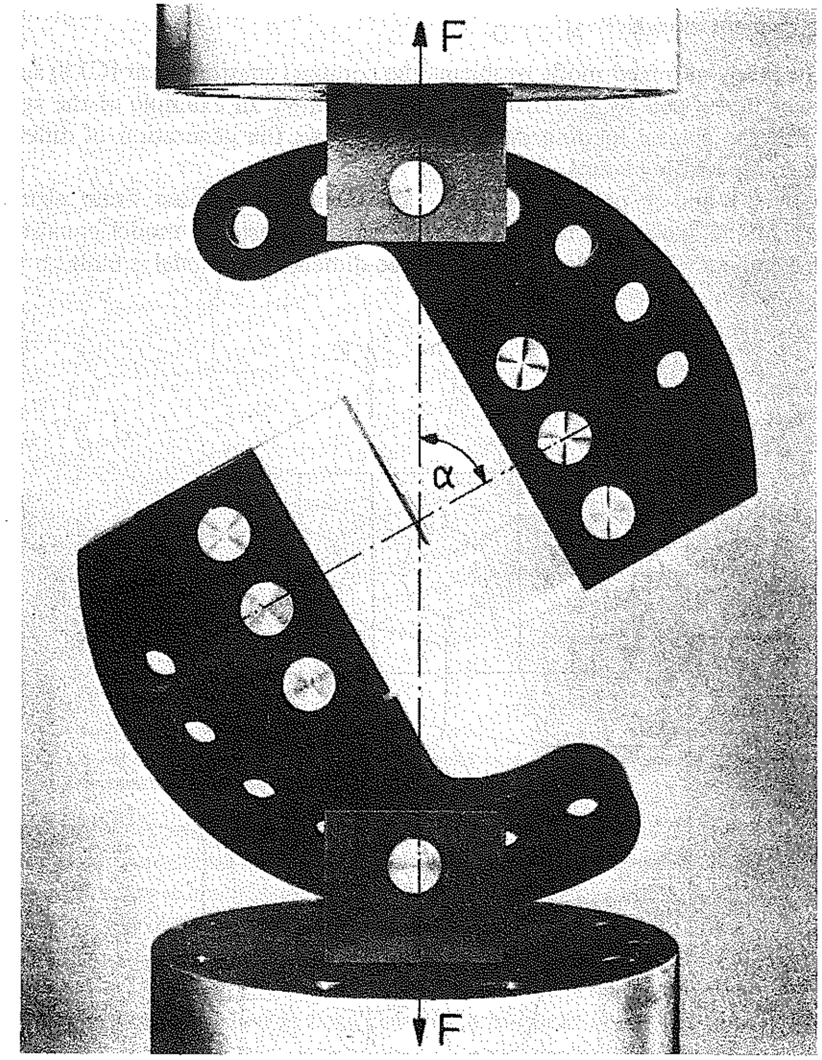

Fig 2 Compact tension shear (CTS) specimen and loading device for mixed-mode fracture experiments

present authors. Applications of the three-dimensional J separation will be published in the near future.

Furthermore, for linear elastic cases the energy release rates G_{I}, G_{II}, and G_{III} (which are equal to the corresponding J values in the linear elastic application) may be calculated by means of the virtual crack closure method, as presented by Buchholz *et al.* **(10)**. The method is used here in a formulation for 20-node solid finite elements, introduced by Shivakumar *et al.* **(11)**.

Finite element models

For the study of mixed-mode problems the compact tension shear (CTS) specimen was developed by Richard (**2**). Figure 2 shows an example of the specimen and the special loading device, which allows the generation of different mixed-mode loading states.

Figures 3 and 4 present the two- and three-dimensional finite element models of the specimens considered. The two-dimensional model consists of plane elements with 8 nodes, while the three-dimensional model is built up with

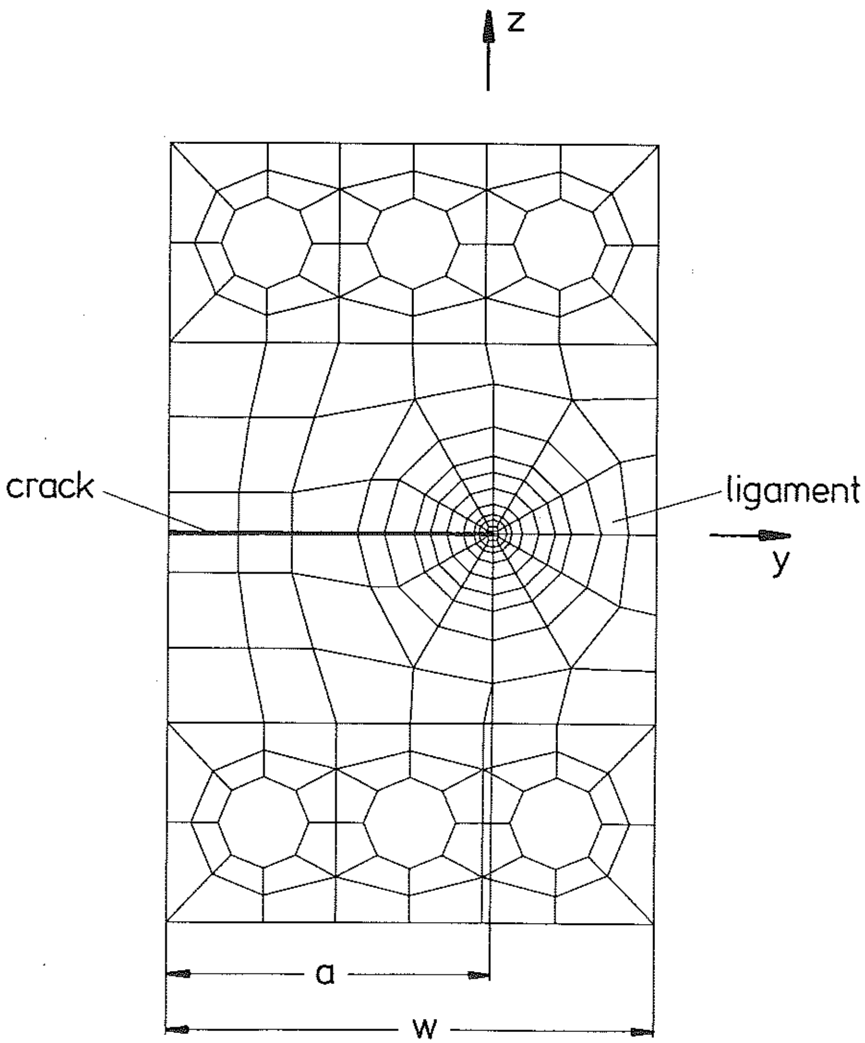

Fig 3 Finite element model of the CTS specimen used for two-dimensional calculations

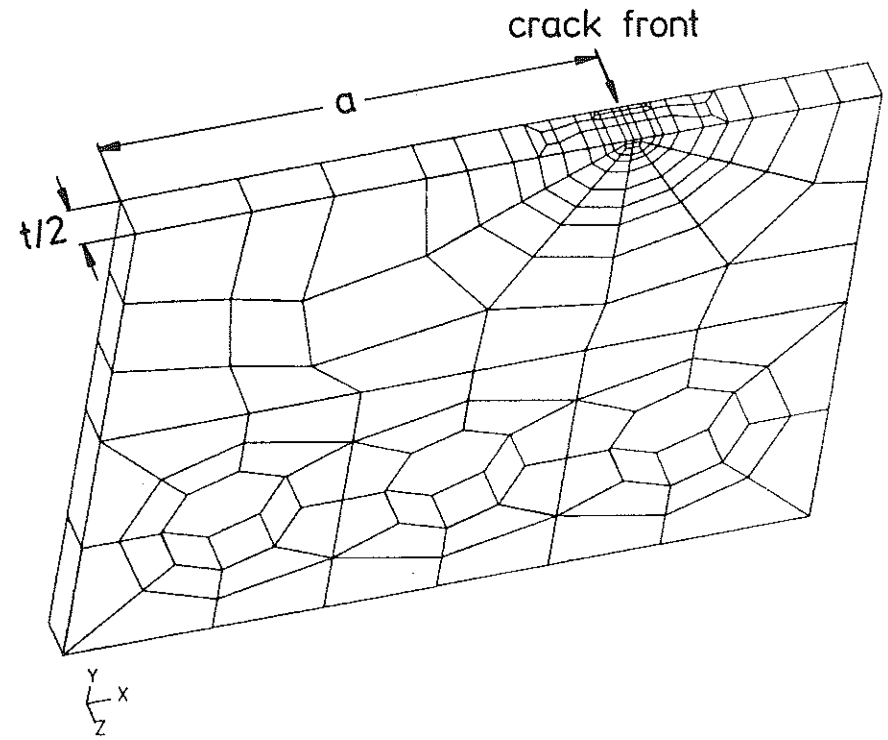

Fig 4 Lower half part of the finite element model used for three-dimensional calculations with refinement in thickness direction

solid elements with 20 nodes each. For symmetry reasons only the half of the specimen is modelled in the thickness direction in the three-dimensional case.

For the two-dimensional calculations the material data of a steel were used ($E = 210\,000$ MPa, $\nu = 0.3$, $E_t = 4\,200$ MPa, $\sigma_y = 400$ MPa), while in the three-dimensional analyses the data of aluminium were chosen ($E = 70\,000$ MPa, $\nu = 0.3$, $E_T = 1\,045$ MPa, $\sigma_y = 545$ MPa). The elastic–plastic material behaviour was modelled in both cases by a bi-linear approach.

Slightly differing crack lenghts were modelled in the different calculations. In the two-dimensional case linear elastic analyses were carried out with a ratio of a/w of 0.6, and elastic–plastic calculations with $a/w = 0.65$. In the three-dimensional case $a/w = 0.67$ was used.

The loading (nodal forces at the pinholes) and the displacement constraints were chosen according to Richard (**12**) for an appropriate modelling of different load angles.

Results of the calculations

Two-dimensional analyses

In the linear elastic case the J values may be simply converted to stress inten-

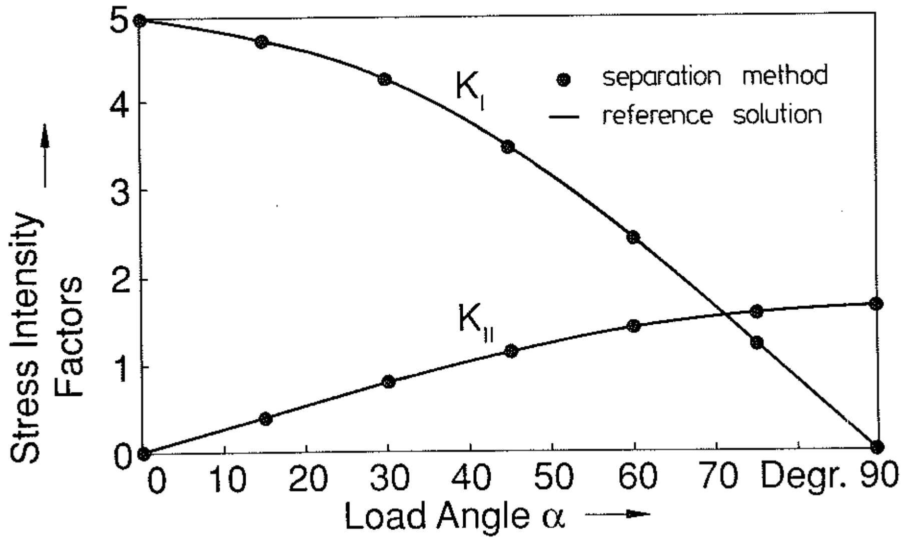

Fig 5 Normalized stress intensity factors K_I and K_{II} compared with reference values according to Richard (12) as functions of load angle

sity factors, which are shown in Fig. 5 together with a reference solution by Richard **(12)**.

While the mode I value K_I decreases with increasing load angle, K_{II} increases from 0 for pure mode I (load angle 0 degrees) to its maximum value for pure mode II (load angle 90 degrees).

Although the applicability of the separation procedure is not fully ensured in this case, the variation of J_I and J_{II} with the load for the elastic–plastic calculations is presented in Fig. 6 for the example of a load angle of 60 degrees. Further studies will be carried out, to clarify the assumptions for applying the separation method.

Three-dimensional analyses

Figure 7 presents the distribution of the total J value along the crack front. While for prevailing mode I the J integral-values in the middle of the specimen are higher than those at the surface, an opposite behaviour is obtained for mode II loading.

For the example of a load angle of 60 degrees, the separated values J_I, J_{II}, J_{III} are shown in Fig. 8 as a function of the position on the crack front. The values are calculated as energy release rates using the virtual crack closure method as described in **(10)** and **(11)**. While the mean value of the J_{III}-part over the specimen thickness is equal to 0, maximum values of J_{III} are found at the surface.

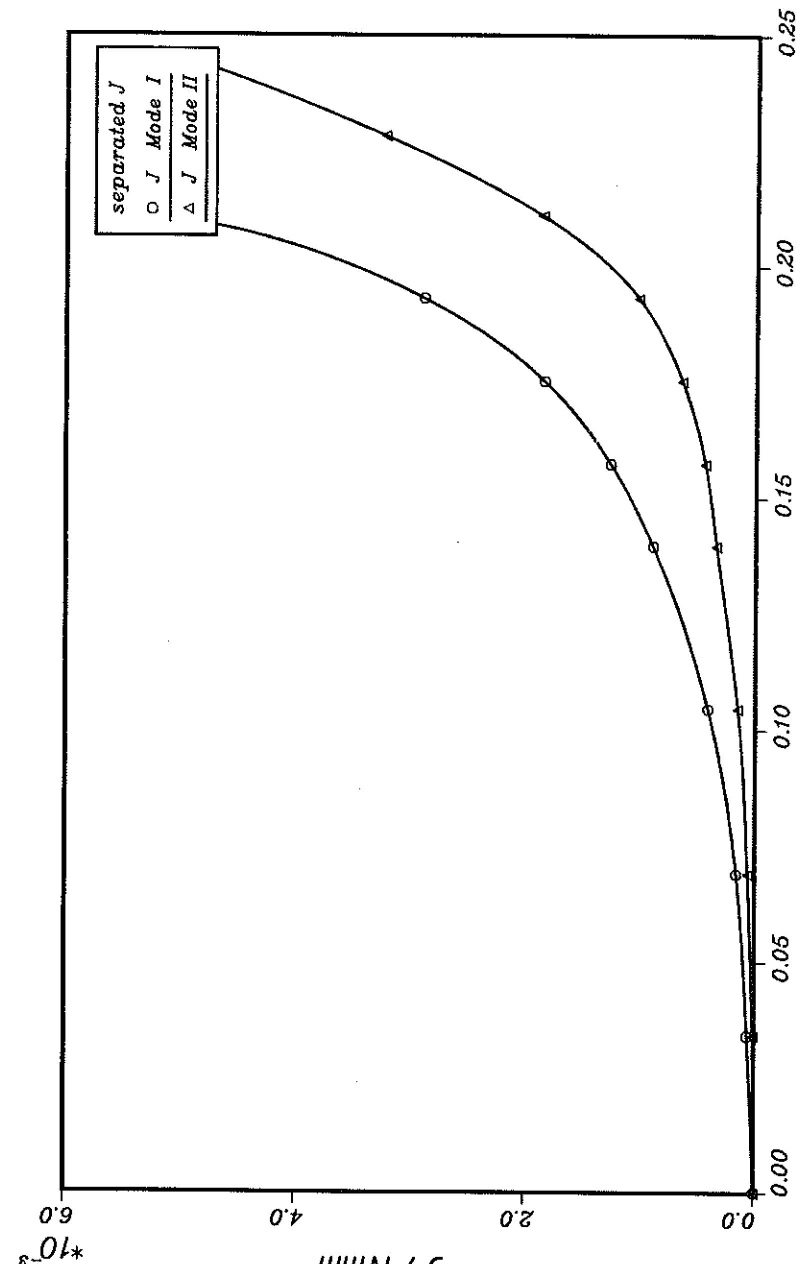

Fig 6 Variation of J_I and J_{II} with normalized external load ($F/wt\sigma_y$) for a load angle of 60 degrees

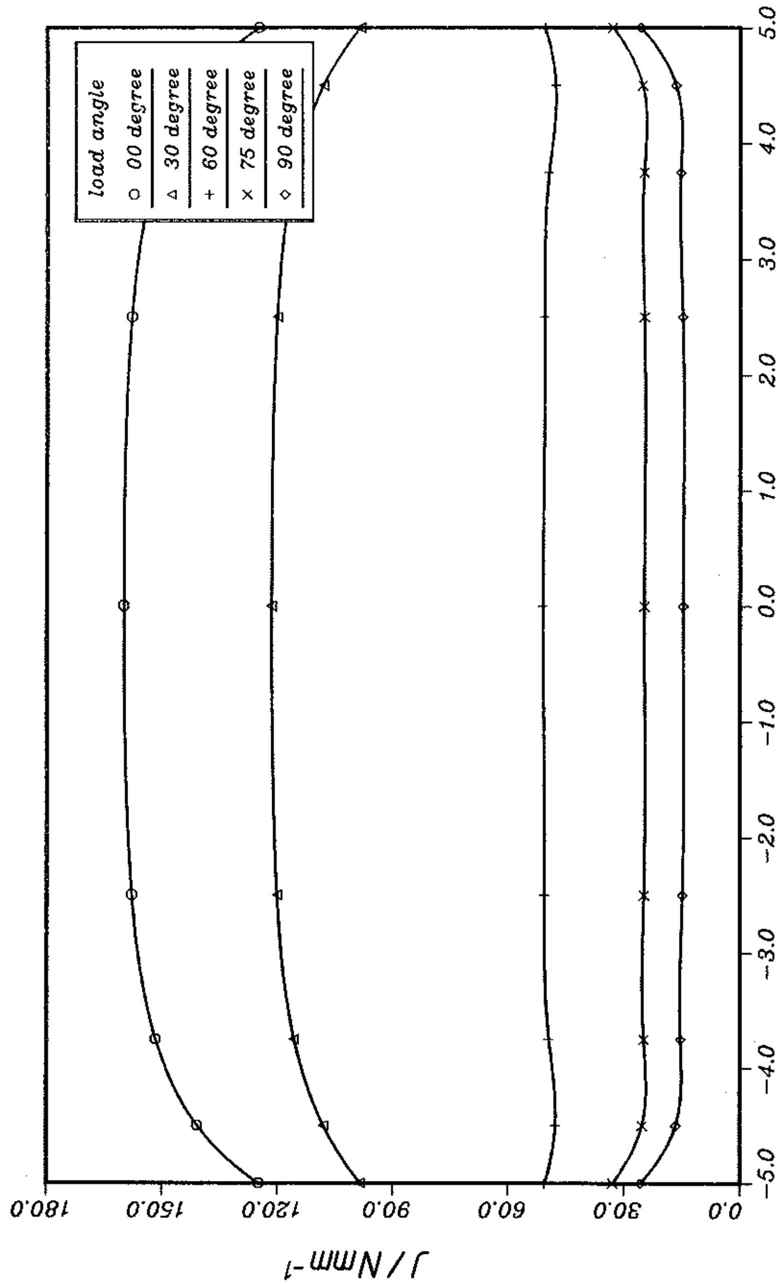

Fig 7 **Variation of total J along the crack front for different load angles (linear elastic calculations with $F = 20$ kN)**

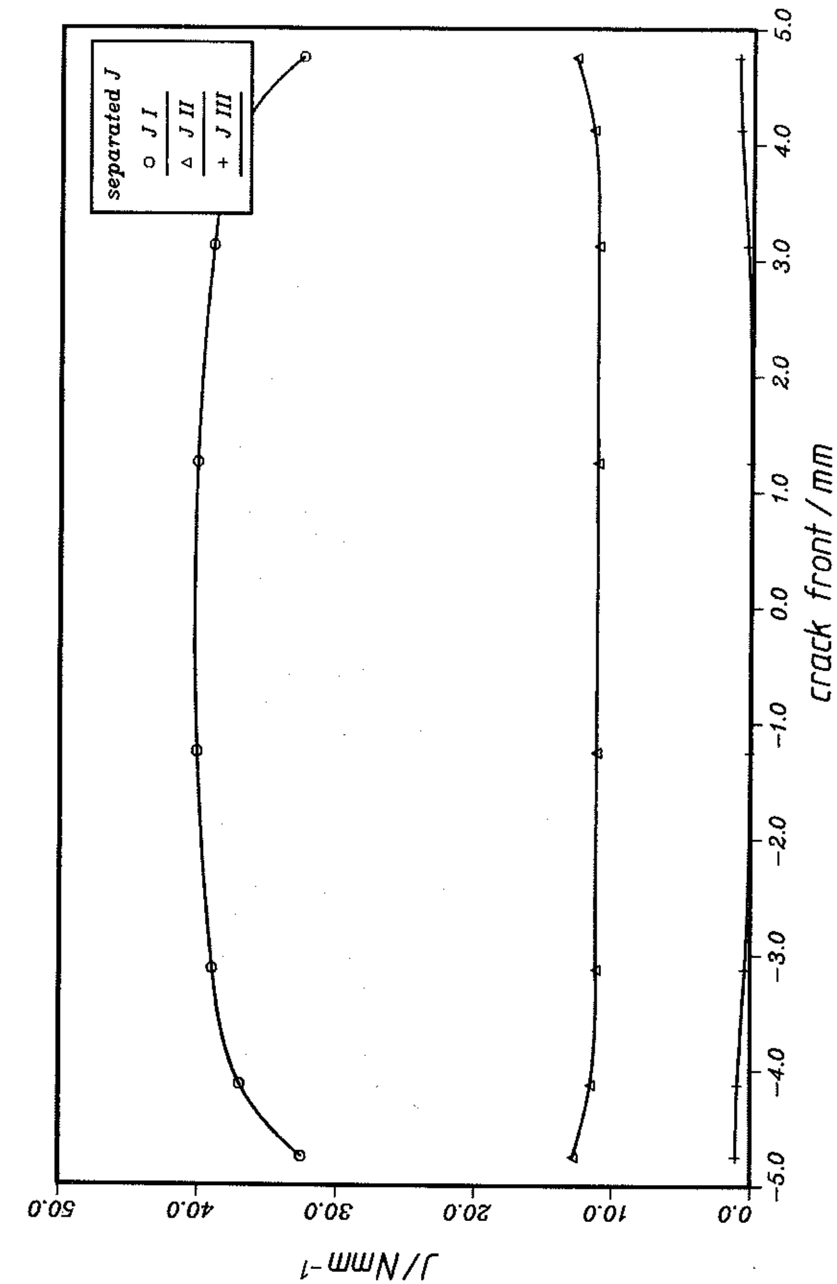

Fig 8 **Separated values J_I, J_{II}, and J_{III} along the crack front for a load angle of 60 degrees (linear elastic calculation with $F = 20$ kN)**

Discussion of the results and summary

With the two-dimensional linear elastic calculations, results were obtained which show a very good agreement with a reference solution given by Richard (**12**). In terms of stress intensity factors the maximum deviation is less than 1 percent, so that the numerical procedure of the J_{I} and J_{II} evaluation seems to be verified. Also, the elastic–plastic calculations should give a basis for an evaluation of the angle of the kinked crack, if a suitable criterion can be found and supported by experiments.

While the three-dimensional calculations show some initial interesting aspects, it is also evident, that much further work is necessary. Our next work in this area will be the completion and verification of the three-dimensional J separation procedure.

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