*B. A. Bilby**

Fundamentals of Ductile Fracture

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Introduction

In the fracture scene today we see a vast practical engineering effort on design, monitoring, and testing, coupled on the research side with very great refinements of experimental technique which enable us to see what is happening on the finest scale. At the same time, an explosion in computing power is occurring, so that there is now real hope of making realistic models relevant to the actual geometry of specimens and structures. Nevertheless, a steady flow of engineering failures, many of which have very serious human and economic consequences, is reported in the daily press. At times, this flow becomes such a spate as to prompt articles like that in the *Sunday Times* of 5 March 1989 about the aircraft industry, with the heading 'Courting disaster', in which the potential loss of public confidence is discussed. The major catastrophe often involves a combination of misfortunes, each of which in isolation might have been contained by the protective systems envisaged in the design. The chief factors causing failure include faulty design and materials; inadequate monitoring and maintenance; system failure; and last, but not least, human error. It is a measure of the success of our limited area of responsibility that many catastrophes are not due entirely to a failure of structural integrity, but to this combination of causes. However, even if our structures demonstrate sometimes a toughness and robustness far beyond expectations and so avert a disaster, there is still every reason for continued vigilance and investigation, for there are many aspects of failure which are still only imperfectly understood and which are controlled in a semi-empirical way. It is disturbing that under present economic pressures there is increasing difficulty in obtaining support for the fundamental studies without which we cannot trust to the 'solid ground of Nature' in designing our preventative measures. Too much time now seems to be spent in writing plausible accounts of what might be done in years one, two and three if money were provided, and not enough in grappling with the problems themselves. There are even courses(but not yet an Institute) on what one might call 'Advanced Propositioning Technology'.

* Department of Mechanical and Process Engineering, University of Sheffield, Mappin Street, Sheffield S1 3LD, UK.

Energy and work

In thinking about fracture today a cornerstone is still the work of Griffith **(54)** **(55)**, as modified by Orowan **(84)** and Irwin **(62)**, together with the latter's concept of the crack extension force **(62)**–**(64)**. About such fundamental ideas, as the recent issues of our Newsletter attest, there is still controversy. Here and elsewhere in fracture mechanics where competing methodologies with eager protagonists abound, we do well to remember the edict of Francis Bacon that 'Truth will come sooner out of error than confusion'. Stevens and Guiu **(91)** are right to draw attention to the difference in thermodynamic status between surface free energy and plastic work, though their comments do not invalidate the concept that one may balance the localized work done in fracturing against the energy release rate **(62)**. In theory the ideal elastic brittle solid in vacuo under a uniform remote stress is in unstable equilibrium, with energy stored firstly in the crack surfaces; secondly as elastic energy in the body; and thirdly as potential energy of the loading system. As many have discussed, the equilibrium is, in principle, reversible. The crack will close up (completely) if its length decreases slightly and extend (indefinitely) if its length increases slightly. As it closes by δa the change in the potential energy of the loading mechanism is $\delta E^{POT} = +w$, the change of elastic energy of the body is $\delta E^{EL} = -w/2$ and the change of free energy stored in the crack surfaces is $4\gamma\delta a = -w/2$. The surface free energy helps to lift the weight or store energy in the spring. Conversely, as the crack extends, the loss of potential energy of the loading mechanism is stored, half as elastic energy in the body and half as surface free energy.

The mechanics of a very long, fully ductile inclined shear crack in a large steel plate is similar, but the situation is not reversible and the crack will not close up. However, if long enough, it is unstable with respect to extension and propagates in a mode which, like that of the brittle crack, has been called cumulative **(35)(37)**. If the plate is very large, about half the loss of potential energy of the loading system as the crack advances is stored as an increase in the elastic energy of the plate, but the remainder is dissipated, largely as heat, since only a very small fraction of the plastic work is stored as the self and interaction energies of the microstructural defects and surfaces created. To discuss this process in the light of thermodynamics we have to appeal to the theory of irreversible processes. However, the point of instability can still be assessed in the Griffith manner by comparing the rate of change of the elastic energy release rate with that of the work rate required for fracturing, provided only that we have a situation (small scale yielding) where the fracture process zone is clearly definable, with dimensions much smaller than the crack length. Then the elastic parameter K can be used to estimate the overall crack extension force. It is when we cannot clearly associate nearly all the plasticity or other irreversible processes with a well defined fracture zone that our difficulties begin. Several classifications of fractures in this context have been made. One distinguishes two modes, the cumulative (referred to above) and the non cumulative **(35)(37)**; another considers three types, elastic, elastic–plastic, and plastic **(3)**.

The crack opening displacement

A series of important developments followed the explicit representation of cracks by crack dislocations **(32)(51)**, the first being the use by Cottrell **(33)** of the DBCS model **(18)(43)** to estimate the crack opening displacement, and his subsequent exploitation of it as a fracture criterion **(33)**–**(37)**. Wells **(93)(27)** also independently made a similar application, using a quite different approximate method to estimate the COD, relating it to the crack extension force and stress intensity factor in small scale yielding. It is interesting to note that when Wells first introduced it, the acronym COD meant for him crack opening *dislocation*, and not, as it does to most people today, crack opening *displacement*. Barenblatt **(6)(7)** also used procedures similar to the DBCS model to remove the crack tip singularity. The subsequent exploitation of these DBCS (or strip yield) models and their developments has been very extensive **(5)(4)(17)(19)(20)(23)(31)(87)(92)** and continues to this day. Cottrell made effective use of the DBCS model **(17)(33)(35)**–**(37)** to discuss notch sensitivity and size effects and to classify a broad range of fractures, by choosing appropriate values of the parameters δ_t, the crack tip displacement, and σ_o, the finite stress at the tip, with $\sigma_o \cdot \delta_t$ as the work of fracture. This uses the DBCS model **(37)** to provide a crude rectangular pulse representation of the law of force between the separating surfaces in the fracture process. It is interesting to compare this work with the crack band model of Bazant and its developments **(8)(9)**, largely concerned with the failure of materials of the concrete type with a strain softening law, where the crack band width plays a role very similar to the crack opening displacement. The analogy with strip yield models has been noted **(9)**.

The strip yield model has also contributed to the development of the R6 procedure. Cottrell **(36)(37)** explained fractures below general yield in large structures by taking $\sigma_o = \sigma_y$ and using a constant crack opening displacement criterion. Taking σ_o equal to σ_u, the ultimate stress, Heald, Spink, and Worthington **(59)** made it possible to use the DBCS line of constant crack opening displacement to interpolate between LEFM and failures near the ultimate stress. This procedure was used to interpret 'invalid' data obtained from small tough specimens. Dowling and Townley **(42)** then introduced a diagram, defined in terms of load ratios, so that the DBCS line became an interpolation between failure by LEFM and plastic collapse (instead of failure at the ultimate stress **(59)**). Finally, Harrison, Loosemore, and Milne derived the original R6 diagram **(58)** by interchanging the axes and taking the ordinate to be K/K_{Ic}, which gives a convenient linear movement of the representative point from the origin as the load increases. Again the assessment line was the DBCS interpolation based on a constant crack opening displacement. Since the

DBCS model has no wake, caution is needed in using such a diagram to discuss crack growth **(11)**, although procedures for doing this were at one time proposed **(29)(73)(74)**. In the most recent R6 procedure **(75)(76)**, the DBCS interpolation has been abandoned and the discussion of crack growth by the most sophisticated method (Option 3, Category 3) is stated to correspond to analysis by J_R curves, though it is not yet clear how complete this correspondence is.

J integrals

A generalization of crack characterizing parameters, which has seen widespread development, occurred when, independently, Rice, Eshelby, and Cherepanov expressed the crack extension force of Irwin in the form of a surface or line integral, which, for a linear or non linear elastic material is independent of the path **(19)(30)(86)**. For Eshelby, this was a development of his work on forces on elastic singularities, which had led him to the introduction of the elastic energy momentum tensor **(46)–(48)**. The wide use in fracture mechanics of these integrals, and their generalizations related to the invariance of elastic fields under operations other than translation **(26)(49)(56)(66)**, has stimulated much research on conservation laws in many branches of continuum mechanics **(22)**. It is convenient to denote the integral of Eshelby by F; that of Rice by J; and a third, introduced by Bilby and Eshelby **(10)**, and independently by Miyamoto, and Kageyama **(77)**, by Q. In F, W is the strain energy density and u_i the elastic displacement, while in J, W is the density of stress working and u_i is the total (shape) displacement. Thus J can be evaluated for a path through plastic material while F cannot, unless u_i is taken to be the total displacement. When evaluated through plastic material J remains path independent if the loading is proportional, but not strictly otherwise. Nevertheless, finite element calculations of J have shown it to be very nearly path independent over much of the field in many situations, so that it has become widely used in two dimensional problems. The integral Q is derived from the continuum theory of dislocations and so may be evaluated through a plastic region if the plasticity is represented by a distribution of dislocations. It then gives the resultant force on the crack and all dislocations inside the path over which it is evaluated. Its value decreases to zero as the path shrinks on to a crack tip in an elastic–plastic material and thus confirms that there is no crack extension force when there is relaxation by quasistatic continuum plasticity **(10)**. Guided by the interpretations of the F and J integrals, the practical engineer has evolved very carefully controlled standard tests in which other quantities called J, with various suffixes, are deduced from measurements on the load displacement records of cracked specimens. These quantities would exactly correspond to the path independent integral giving the energy release rate if the material were non linear elastic. Moreover, they correlate well with elastic–plastic calculations of this integral. Methods for the approximate estimation of J for

various geometries have also been evolved. All this, and parallel developments with the COD parameter, have led to methodologies for the characterization of fracture initiation involving K, J, and COD. However, for three dimensional geometries and the discussion of R curves, and for the translation of results from test specimens to structures, there is still debate about the best way to proceed. Also, as the papers here show, much effort is still devoted towards the development of new and simpler characterisation procedures **(25)(90)(89)**.

Two parameter characterization

One might hope to improve the characterization of the crack tip field in different geometries by using more than one parameter. From the general theory of matched asymptotic expansions **(44)** the constant term T_{11} in the expansion of the 11 stress ahead of the crack tip **(87)** is a prime candidate for consideration (the crack is in the 13 plane with its front along the 3 axis); so too is the S_{33} stress, which acts along the crack edge **(57)**. More generally, as discussed at this meeting **(85)**, one may consider also other stress components not giving rise to singular fields in a linear elastic material. The influence of the T_{11} stress was first studied by Larsson and Carlsson **(68)**. Recently, finite deformation plane strain elastic–plastic fields have been calculated for a number of specimen types, including the compact tension (CTS), centre cracked tension (CCP), biaxially loaded centre cracked tension (CCB), and single edge cracked bend (SECB). The results have been compared with similar fields produced round semi-infinite cracks under K and T_{11} loading **(14)(16)(52)**. The latter fields, by analogy with fluid mechanics, are referred to as modified boundary layer fields (MBL fields) **(68)**, or when $T_{11} = 0$, of course, as small scale yielding fields (SSY). The recent calculations were carried out using the TOMECH finite element program, which has been under development at Sheffield since 1980. The updated Lagrangean method was used **(72)**, together with a modified variational principle to ensure that constant volume deformation does not impose artificial constraints **(79)**. For edge (or centre) cracked mode 1 specimens, $T_{11}/\sigma_y = BK_I/\sigma_y\sqrt{(\pi a)}$, where a is the crack length (or half of it) and σ_y the yield stress. The parameter B can be evaluated for different specimens by various methods **(69)(28)(65)**. Under load F per unit thickness, the single edge cracked bend specimen (SECB) has $T_{11}/\sigma_y = 3BYFH/W^2\sigma_y$, where $2H$ is the length and W the width, while the compact tension specimen (CTS) has $T_{11}/\sigma_y = BYF/\sqrt{(\pi)}W\sigma_y$. For the biaxially loaded centre cracked panel (CCB) with $\sigma_{11}/\sigma_{22} = \lambda$, $T_{11}/\sigma_y = B_\lambda Y\sigma_{22}/\sigma_y$, where $B_\lambda = (B + \lambda/Y)$. The centre cracked panel under pure tension (CCT) has $\lambda = 0$, while $B_\lambda = 0$ for $\lambda = 1.378$. Here Y is the dimensionless factor giving $K_I = Y\sigma_{22}\sqrt{(\pi a)}$ for the CCB and CCT specimens. The J parameters evaluated were the path integral of Rice **(86)** and J^i defined by $J^i = \eta^i U/tb$, where U is the area under the load versus load line displacement curve, t the thickness, and $b = W - a$ the remaining ligament. The parameter $\eta^i = \alpha + \beta(b/W)$, where W is the specimen width. For

(2), $i = A$, and $(\alpha, \beta) = (1.97, 0.815)$; for **(83)**, $i = C$, and $(\alpha, \beta) = (2, 0.522)$. The calculations show that the stresses and strains, plotted as functions of X/δ_t or $X\sigma_y/J$, where X is the original distance ahead of the tip and δ_t the CTOD, correlate well with the T_{11}/σ_y parameter, a striking result being that for the CCB specimen with $\lambda = 1.378$, for which $B_\lambda = 0$, the data coincide with the SSY curves down to very small values of $b\sigma_y/J$; see **(12)(14)(16)** and Fig. 1, from recent work by Goldthorpe **(52)** on an austenitic steel, which shows the variation of the ratio of the mean normal stress σ_m to the equivalent stress σ_e. Thus this CCB specimen can be used to study behaviour under small scale yielding.

The growth of voids ahead of the crack tip depends on the whole history of the elastic–plastic field there. This growth may be calculated approximately using a simple void growth theory. In this, the history of the elastic–plastic field at a point ahead of the tip is used as the remote field history in a theory of the growth of an isolated void in an infinite medium in a perfectly plastic material **(88)**. In the simple theory, it is assumed that the void grows under

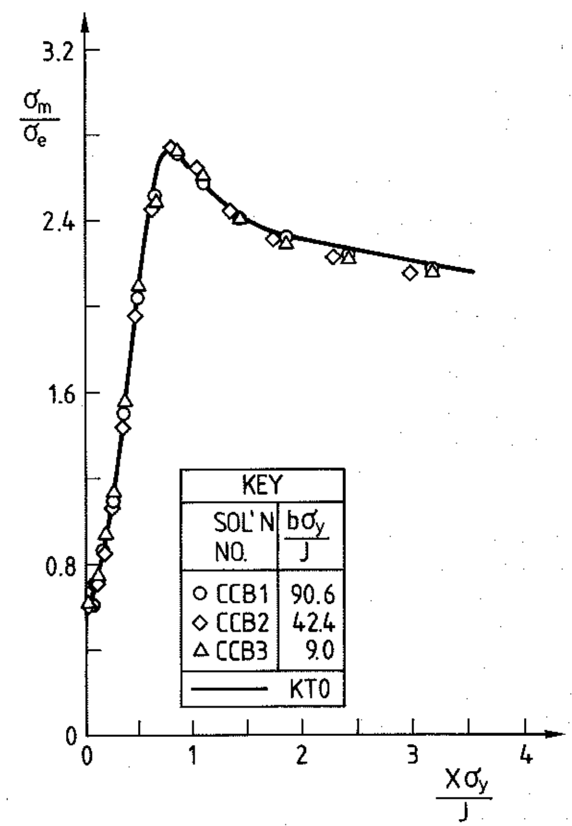

Fig 1

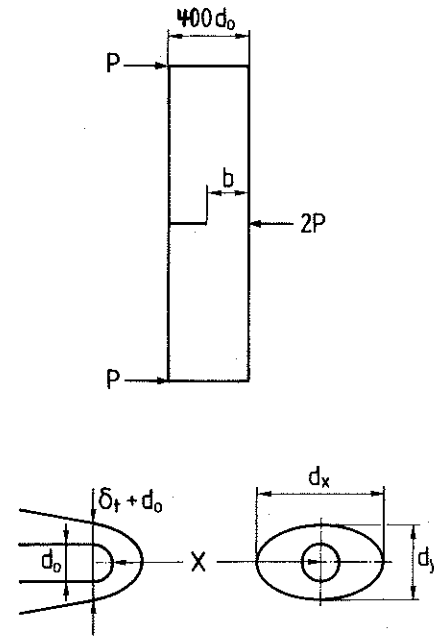

Fig 2

conditions where the hydrostatic part of the stress is dominant, so that the void remains nearly spherical. This simple void growth theory has been used by a number of workers **(16)(40)(45)(70)(78)**, and forms one component of the 'local approach' to fracture developed particularly in France. It was noted that the void growth calculated by the simple theory showed a transition in the SECB specimen **(16)**. The void growth ratio for a small specimen (small $b\sigma_y/J$) was greater than that for SSY for small values of X/δ_t, but smaller than that for SSY for large values, the transition being at about $X/\delta_t = 1.9$. Thus if the critical fracture process distance is small the specimen would give conservative results, but might not do so if it is large **(16)**. The simple void growth theory, being based on the behaviour of an isolated void in an infinite medium under a uniform stress, does not take account of the fact that the many voids grow in the large stress and strain gradients near the crack tip, so that their growth itself modifies the crack tip field; thus, it ignores crack–void and void–void interactions. It also neglects the influence of hardening (except through changes in the applied field) and, of course, any modifications of constitutive relations due to the damage which is occurring.

Accordingly it was decided to assess the simple void growth theory, particularly in the SECB specimen, by studying the simultaneous large deformation of crack tip and void together in two and three dimensions allowing for full crack–void interaction. In March 1988, when the work was started, no 3D calculations had been published, although the problem had been treated in

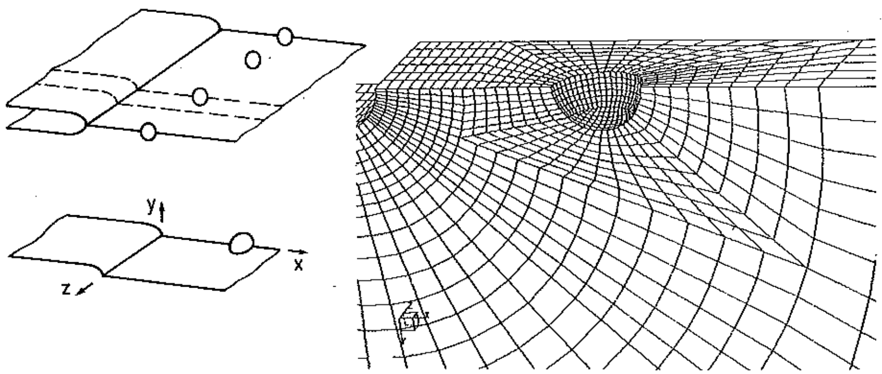

Fig 3

SSY in 2D **(1)**. Preliminary accounts of the SECB work, which has been carried out in collaboration with Mr G. E. Cardew and Dr M. R. Goldthorpe, with occasional assistance from Dr Z. H. Li, have been given **(50)(13)(12)**. In these, comparisons have been made as far as is possible **(12)** with the studies of

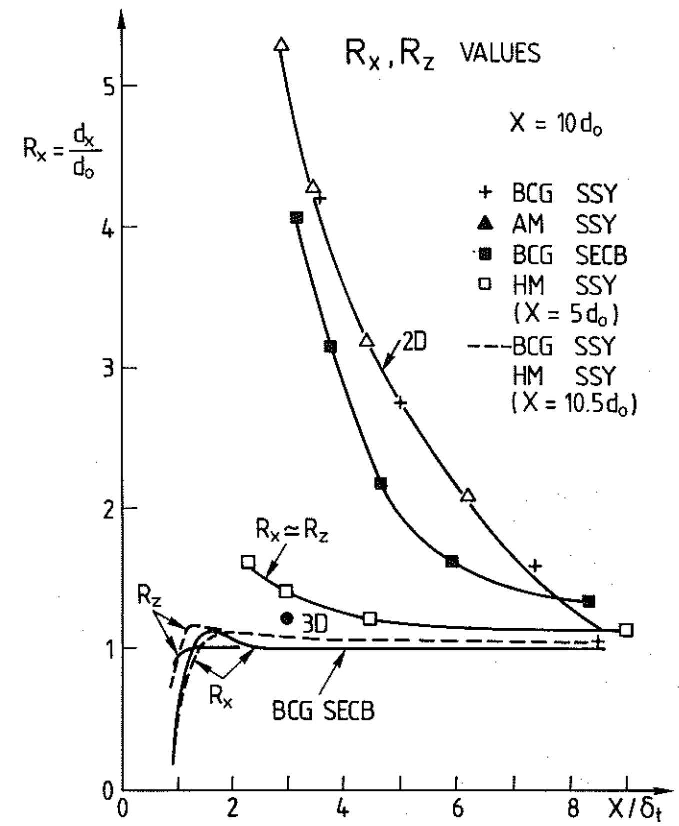

Fig 4

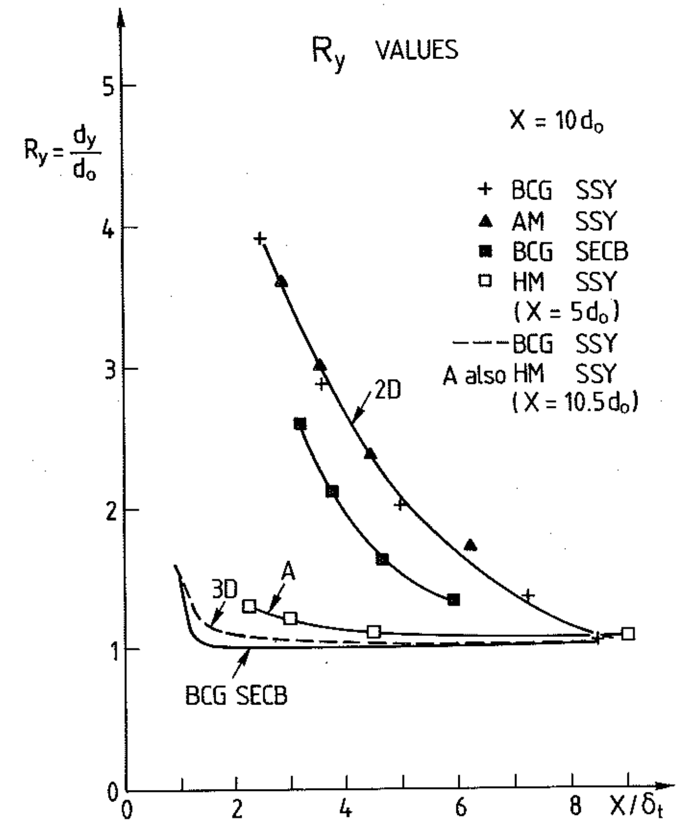

Fig 5

Hom and McMeeking **(60)(61)(71)**, who have now reported a detailed examination of the problem for small scale yielding. Our calculations for the SECB geometry **(12)** used isoparametric 8-noded elements, with fine and coarse meshes having respectively: 7920 and 1080 elements; 10 178 and 1720 nodes; ratios of maximum to minimum mesh size of 4092 and 2459; and 26 366 and 3923 degrees of freedom (DOF). On the CRAY X-MP/48, the fine mesh took 20 hours for 571 load increments. A coarse mesh SSY calculation (with 5448 DOF) has also been carried out. This took 60 hours on a HITECH 10. The geometry modelled was a long single row of initially spherical voids (initial diameter d_o) with centres spaced X apart parallel to z (Figs 2 and 3). The line of voids was initially at the same distance X ahead of the semicircular notch, also of diameter d_o, representing the crack tip. Because of the periodicity, it was necessary to model only a quadrant of the spherical void on one face of a thin slice of material of thickness X, whose faces were constrained in the z direction so as to model a plane strain behaviour remote from the tip. The meshes (produced with the TOMECH mesh generator) had 54 cells on the void quadrant for the coarse mesh and 216 cells on the void quadrant for the fine mesh. The fine inner mesh used for the final runs was similar but not identical to that shown in Fig. 3. The ratios $R_i = d_i/d_o$, $i = x, y, z$, indicating the change of initial void diameter d_o (Fig. 2) in the x, y, z directions are shown as functions of X/δ_t in Figs 4–7. The results from the simple theory are shown as R_s. All results are for perfect plasticity and the acronyms have the following meaning: BCG refer to **(12)**; AM to **(1)**; HM to **(60)** and M to **(70)**. The results confirm that the voids do not remain spherical and that the 3D voids deform much less than the two-dimensional cylindrical voids **(1)(60)**. Figure 6 suggests that the transition in R_y predicted by the simple theory is confirmed, but that it occurs at a smaller value of X/δ_t. However, there may be a mesh effect here and further calculations are needed. There is a transition in R_z of opposite type, and perhaps two in R_x. Comparisons of these transitions with the work of Hom and McMeeking **(60)(61)(71)** are not possible, since void distances differ and they did not continue their calculations below a value of X/δ_t less than about 2.3 due to numerical instability. The smallest X/δ_t result

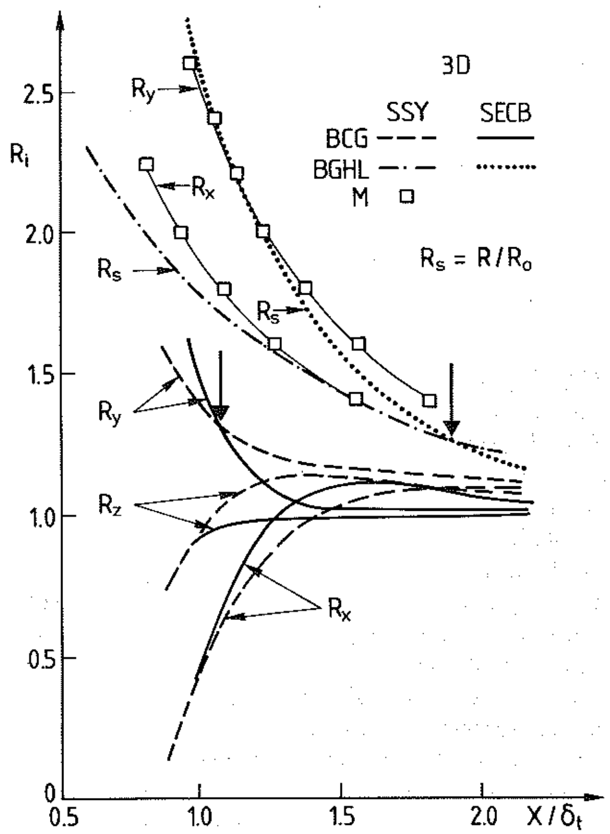

Fig 6

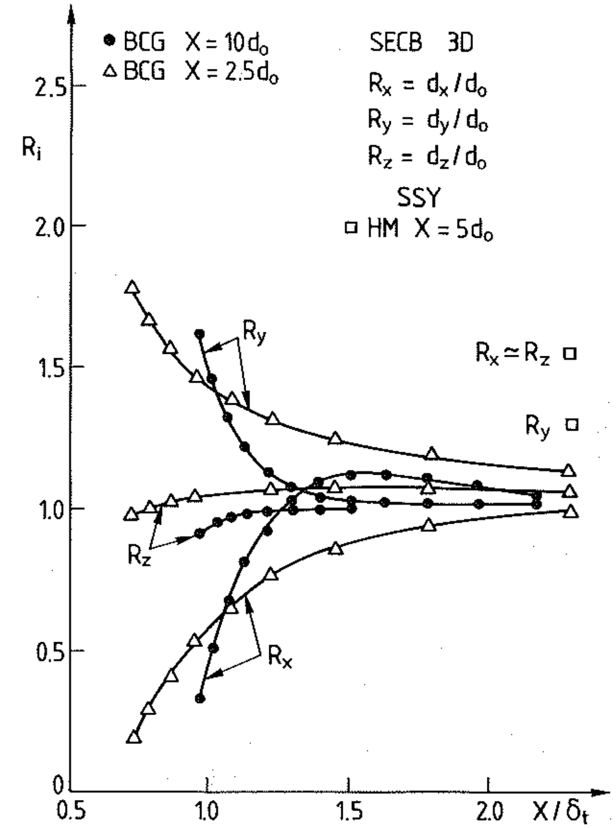

Fig 7

of **(60)**, is shown in Fig. 7. This figure also shows the effect of placing the void initially at $X = 2.5d_o$ (instead of at $10d_o$), which enhances void crack interaction. Again there may be mesh effects here, since the $X = 2.5d_o$ result was obtained with only 3923 DOF. Figure 6 shows that the simple theory (R_s), like the approximate ellipsoidal estimates of **(70)**, overestimates the growth (note particularly R_x). For perfect plasticity, the opposite result has been found **(60)**, although for a hardening material, R_s is found to overestimate the growth **(60)**. The figures show that there are some differences between the various results. However, no definite statements can be yet be made, since there are not only variations of mesh, but also of initial void distance. It is clear that there is a need for further work.

Damage theory and the local approach

It is not possible here to review the large effort now being devoted to continuum damage mechanics and the local approach to fracture. In the former, the constitutive relations are modified to model the progressive damage which develops as the material deforms; in the latter, the stress and strain histories calculated from conventional constitutive relations are used in models of brittle or ductile failure related to the microstructure, in order to discuss crack advance. In both, the objective is to present an alternative to the use of characterizing parameters, in which the theory predicts the advance of cracking, either because of the deterioration of the material itself, or through a criterion closely associated with the microstructure. Both approaches, as indeed do others using criteria calculated from the local crack tip field **(24)(53)**, seek to produce a more satisfactory transferability between behaviour in laboratory testing on the one hand, and in structures on the other, particularly in the complex three dimensional stress and strain histories experienced in engineering practice. The subject deals with a wide range of conditions and materials and the literature is already vast **(9)(67)**. It is clear that interest in this subject is likely to increase. On the one hand, we are becoming better able to categorize the microstructure and to recognize particularly the importance of the statistical variations which occur in it; and on the other, our computers are enabling us to deal with ever more complex problems. Perhaps one of the more urgent questions which should receive attention is the proper treatment of the various scalings involved. It has been argued **(9)** that problems involving softening theories are often not mathematically well posed and that this difficulty could be met by the use of an imbricate continuum having overlaying structures on different scales. An alternative is to make a deliberate choice of finite element parameters to represent an appropriate averaging over the microstructure **(41)(40)(78)**. With either approach, some choice of parameters reflecting the scale of the microstructure seems to be required. There does not yet appear to be a general consensus on the objectivity of these different methods. Perhaps the matter will not be entirely resolved until it is possible to incorporate in the region of the crack tip in the overall analysis, some specific modelling on the scale of the microstructure. In the field of short cracks in fatigue, a start has been made on this process by an ingenious use of the DBCS model **(80)–(82)**.

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