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An Energetic Analysis of Elastic–Plastic Fracture

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ABSTRACT The objective of this communication is to present a discussion on the energetic aspect of elastic–plastic fracture. In the classical framework of continuum mechanics and thermodynamics, two separated but closely related problems on the propagation of a linear crack in an elastic–plastic solid are considered: the dissipation analysis of a propagating crack and the construction of an energetic criterion of propagation in relation with the expression of the asymptotic fields near crack: crack tip singularity and discontinuity surfaces, in particular in perfect plasticity.

Introduction

In fracture mechanics a basic problem associated with the description of crack propagation is the characterization of the local state of stress and strain. This problem has been successfully studied in the context of linear elasticity. Substantial results have been obtained and important notions such as stress intensity factors, energy release rate, and path independent integrals have been introduced and widely applied in the study of brittle fracture of usual materials under static or dynamic loading conditions.

However, the extension of this mechanical approach to the case of dissipative materials is not straightforward. For example, the important case of elastic–plastic materials leads to many difficulties because of the material nonlinearity. The determination of stress and strain singularity near the crack tip in this case is rather cumbersome. Up to the present day, the definition of a relevant and satisfactory fracture parameter in the description of the crack extension for ductile fracture is still an open problem although many discussions have been given in the literature on the subject. Since the pioneering works of **(1)**, **(2)**, many authors such as **(3)**, **(4)**, **(5)**, **(6)** have given a certain number of asymptotic 'solutions' of the propagating crack problem. These solutions are incomplete solutions in the sense that they are asymptotic, the connection with the far crack response is not established. Some of these 'solutions' also present discontinuities of stress and strain **(7)**.

The existence of such discontinuity is rather common in continuum mechanics. In quasi-static evolution, solutions presenting strong discontinuity are frequently obtained when there is softening i.e. loss of ellipticity (see, for example, **(8)**). In dynamic evolution, shock waves are observed even in linear elasticity, when the loading is fast enough. In plasticity, especially in perfect plasticity, most of the elementary solutions present strong discontinuity. From

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the experimental point of view, the presence of thin zones of strong gradient near the crack tip is also a physical reality.

The objective of this discussion is to present a thermodynamic description of the running crack problem in arbitrary dissipative continua. The energetic aspect in the modelling of ductile fracture is once again considered. Our attention will be focused on the assumption of the existence of surfaces of strong discontinuity propagating with the crack motion.

The consequences of first and second principles of thermodynamics are derived in the first part.

The expressions of the surface heat source and surface entropy production (per unit shock surface) are given.

The consideration of global entropy production in the cracked solid suggests a special expression of thermodynamic force associated with the crack propagation velocity.

The second part is devoted to an application of the preceding analysis in elastic–plastic fracture. In particular, the validity of some simple asymptotic 'solutions' of propagating crack in perfect plasticity is discussed.

Thermodynamic analysis of the running crack

Let us consider the propagation of a linear crack in an inelastic solid undergoing small, dynamic and bi-dimensional transformation (plane strain, plane stress, or antiplane shear).

Figure 1 represents schematically the cracked solid Ω. On the crack surface S, the unilateral contact is assumed to be without friction. Implied force and displacement F^d, u^d are prescribed, respectively, on the complementary portions $\partial\Omega_F$ and $\partial\Omega_u$ of the boundary $\partial\Omega$. The crack, of length $l(t)$, is assumed to propagate in its direction, i.e. the problem of crack kinking is not considered here.

The crack tip A is a singular point in the sense that the material is highly strained at its vicinity. If the displacement is always continuous, two common situations can be observed.

(i) A is an isolated singular point. This means that all local physical quantities (stress, strain...) vary with continuity near A, except at the point A.

(ii) Existence of first order surfaces of discontinuity Σ in the vicinity of the crack tip. In this case the fields of displacement velocity, strain, and perhaps stress are not continuous across Σ.

The case (i) is quite familiar in the static response of linear elastic solids and thus corresponds to the well-known framework of Linear Fracture Mechanics.

The case (ii) is also familiar in dynamic problems when there is propagation of shock waves and in static problems when there is strain softening. In particular, such surfaces of strong discontinuity are frequently observed in perfect

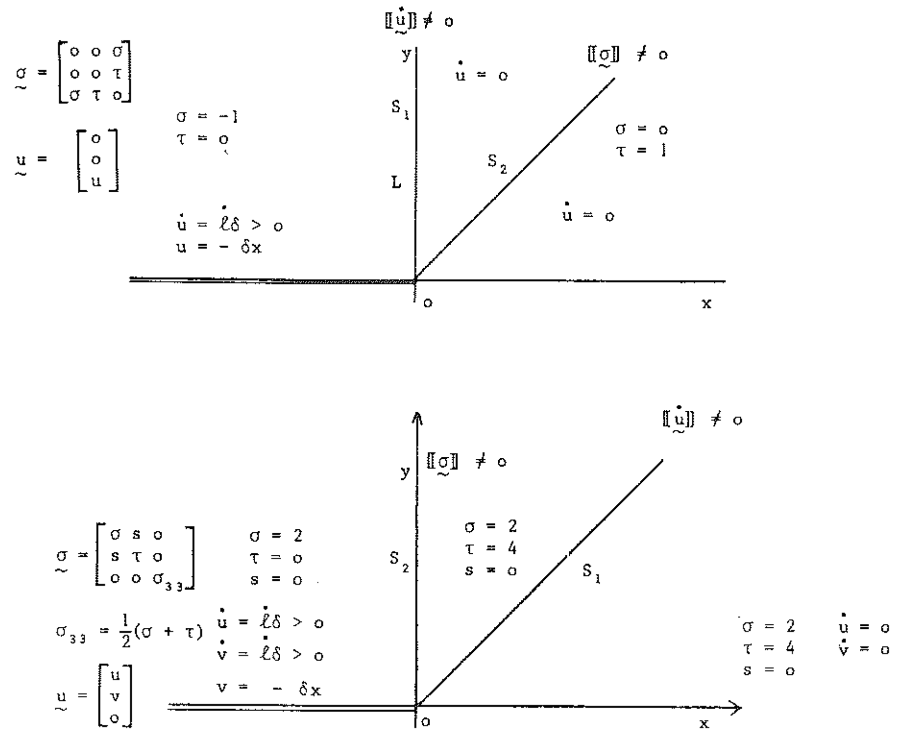

Fig 1 Asymptotic 'solutions' with stress and strain discontinuity

plasticity. Because of the hyperbolic nature of the governing equations of the problem, these surfaces are also moving surfaces in translation with the crack motion. In particular, the existence and the location of these surfaces have been discussed by **(8)** for the static response of cracked solids in nonlinear elasticity with strain softening. Such surfaces will be denoted indifferently here as shock, strong, or first order surfaces of discontinuity.

In both cases, because of the motion of singular points or of discontinuity surfaces, the energetical analysis is not classical and must be handled with caution **(9)**.

The consequences of the two principles of thermodynamics are now studied for the system of a solid with propagating crack. To simplify the presentation, it will be assumed that the whole system is isolated thermically (experiment in vacuum for example).

Let us recall first that for any system of material points occupying a volume M of boundary ∂M, the first and second principles of thermodynamics correspond to the energy balance

$$\dot{E} + \dot{C} = P_{cal} + P_{ext} \tag{1}$$

and the condition of non negative internal entropy production

$$\dot{S} + \int_{\partial M} \frac{q \cdot n}{T} \, \mathrm{d}a \geqslant 0 \tag{2}$$

where E, S, C, q, T, P_{cal}, P_{ext} denote respectively the internal energy, the entropy, the kinematic energy, the heat flux, the temperature, the received calorific power, and the external power of the system. If ρ, e, s represent respectively the volumic mass, the specific internal energy and the specific entropy, E and S are given by

$$E = \int_M \rho e \, \mathrm{d}V, \qquad S = \int_M \rho s \, \mathrm{d}V \tag{3}$$

while the others terms are

$$C = \int_M \tfrac{1}{2}\rho v^2 \, \mathrm{d}V, \qquad P_{\mathrm{cal}} = -\int_{\partial M} q \cdot n \, \mathrm{d}a \tag{4}$$

$$P_{\mathrm{ext}} = \int_{\partial M} v \cdot n \, \mathrm{d}a$$

if no body force and no external heat source are assumed in order to simplify the presentation.

The two principles can be applied to the system of solid Ω with propagating crack by taking $M = \Omega$. However, the result is not straightforward because of the existence of possible moving singularity (isolated singular point A or discontinuity surfaces Σ). For example, in the case of an isolated singular point, the expression of $\mathrm{d}C/\mathrm{d}t$ cannot always be written as the integral in Ω of the function $\rho v \, \dot{v}$ since this function can be strongly singular at A and is not necessarily an integrable function. The same remark can also be made concerning the computation of E and S.

It is necessary to isolate the singular region by a closed curve Γ delimiting a volume V_Γ in translation with the crack tip, the volume V_Γ contains in its interior all the singularity: crack tip A or surface Σ. If Ω_Γ denotes the geometric volume $\Omega - V_\Gamma$, by definition all fields are regular in Ω_Γ. The energy balance can now be obtained in the following way. Since

$$E + C = \int_{\Omega_\Gamma} \rho(e + \tfrac{1}{2}v^2) \, \mathrm{d}V + \int_{V_\Gamma} \rho(e + \tfrac{1}{2}v^2) \, \mathrm{d}V$$

one obtains in the case of discontinuity surface

$$\dot{E} + \dot{C} = \lim_{\Gamma \Rightarrow \Sigma} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\Gamma} \rho(e + \tfrac{1}{2}v^2) \, \mathrm{d}V \tag{5}$$

because the contribution of the second term is zero when the curve Γ reduces in the limit to the two sides of Σ. The same remark is true for an isolated singular point A, one has to take the limit $\Gamma \Rightarrow 0$, when the curve Γ shrinks to

the point A. This result is quite general and follows from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_\Gamma} \rho(e + \tfrac{1}{2}v^2) \, \mathrm{d}V = \int_{V_\Gamma} \rho(e + \tfrac{1}{2}v^2) \mathring{g} \, \mathrm{d}V$$

where $\mathring{g}$ denotes the time derivative in the moving axes AXY of the quantity g. The *transport of singularity* property implies that the derivative $\mathring{g}$ of any integrable physical quantity g such as energy or entropy has the necessary regularity to be an integrable function and thus its contribution vanishes when the domain of integration vanishes. Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\Gamma} \rho(e + \tfrac{1}{2}v^2) \, \mathrm{d}V = \int_{\Omega_\Gamma} \rho(e + \tfrac{1}{2}v^2) \cdot \mathrm{d}V - \int_\Gamma \rho(e + \tfrac{1}{2}v^2) \dot{l} n_1 \, \mathrm{d}\Gamma$$

the energy balance of the thermally isolated solid with propagating crack then leads to

$$\lim_{\Gamma \Rightarrow \Sigma} \left\{ P_{\mathrm{ext}} - \int_{\Omega_\Gamma} \rho(e + \tfrac{1}{2}v^2) \cdot \mathrm{d}V + \int_\Gamma \rho(e + \tfrac{1}{2}v^2) \dot{l} n_1 \, \mathrm{d}\Gamma \right\} = 0 \tag{6}$$

The energy exchange near the crack tip can be then derived from the energy balance of the system of material points occupying the geometric volume Ω_Γ at time t. For this system one obtains by taking $M = \Omega_\Gamma$

$$\int_{\Omega_\Gamma} \rho(e + \tfrac{1}{2}v^2) \cdot \mathrm{d}V = \int_\Gamma q \cdot n \, \mathrm{d}\Gamma + P_{\mathrm{ext}} - \int_\Gamma v \cdot \sigma \cdot n \, \mathrm{d}\Gamma \tag{7}$$

The combination of (6) and (7) leads finally, in the case of an isolated singular point, to

$$\lim_{\Gamma \Rightarrow 0} \left\{ \int_\Gamma q \cdot n \, \mathrm{d}\Gamma - \int_\Gamma (\rho(e + \tfrac{1}{2}v^2) \dot{l} n_1 + v \cdot \sigma \cdot n) \, \mathrm{d}\Gamma \right\} = 0 \tag{8}$$

and in the case of discontinuity surfaces to the classical jump equation

$$[q \cdot n] = \rho \dot{l} n_1 [e + \tfrac{1}{2}v^2] + [v \cdot \sigma \cdot n] \tag{9}$$

defined on discontinuity surfaces Σ. Formulae (9) gives the expression of the distributed surface heat source defined on the shock. The quantity

$$H = \lim_{\Gamma \Rightarrow 0} \int_\Gamma q \cdot n \, \mathrm{d}\Gamma = \lim_{\Gamma \Rightarrow 0} \int_\Gamma (\rho(e + \tfrac{1}{2}v^2) \dot{l} n_1 + v \cdot \sigma \cdot n) \, \mathrm{d}\Gamma \tag{10}$$

is the crack tip heat source by definition. Formulae (10) gives in the case of an isolated singular point the expression of the crack tip heat source resulting from energy balance.

It is important to note that on Σ, the following Hadamard's compatibility condition is obtained from the continuity of displacement

$$[v] = -\dot{l} n_1 [u_{,1}] \tag{11}$$

because of the translation motion of Σ. In the same way, transport of singularity implies in the case of isolate singular point

$$v = -\dot{l}n_1 u_{,1} + \text{more regular function} \tag{12}$$

It follows that formulae (9) and (10) can also be written as

$$[q \cdot n] = \dot{l}[\rho(e + \tfrac{1}{2}v^2)n_{,1} - n \cdot \sigma \cdot u_{,1}] \tag{13}$$

and as

$$H = \lim_{\Gamma \to 0} \int_\Gamma (\rho(e + \tfrac{1}{2}v^2)n_{,1} - n \cdot \sigma \cdot u_{,1})\, \mathrm{d}\Gamma \cdot \dot{l} \tag{14}$$

The consequence of the second principle can be derived in the same way. The internal entropy production of the whole system is

$$E_{\mathrm{py}} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho s\, \mathrm{d}V \geqslant 0 \tag{15}$$

from the assumption of thermal isolation. One obtains as before

$$E_{\mathrm{py}} = \lim_{\Gamma \to \Sigma} \left\{ \int_{\Omega_\Gamma} \rho \dot{s}\, \mathrm{d}V - \int_\Gamma \rho s \dot{l} n_1\, \mathrm{d}\Gamma \right\}$$

This quantity is also the sum of two non-negative terms representing the internal entropy production of the systems of material points occupying the volume Ω_Γ and V_Γ.

The internal entropy production of the system of material points occupying the volume Ω_Γ at time t is

$$E_{\mathrm{py}}(\Omega_\Gamma) = \int_{\Omega_\Gamma} \rho \dot{s}\, \mathrm{d}V - \int_\Gamma \frac{q \cdot n}{T}\, \mathrm{d}\Gamma \geqslant 0 \tag{16}$$

The entropy production on Σ is thus

$$E_{\mathrm{py}}(\Sigma) = E_{\mathrm{py}} - \lim_{\Gamma \to \Sigma} E_{\mathrm{py}}(\Omega_\Gamma) \geqslant 0 \tag{17}$$

$$= \lim_{\Gamma \to \Sigma} \int_\Gamma \left(\frac{q \cdot n}{T} - \rho s \dot{l} n_1 \right) \mathrm{d}\Gamma \geqslant 0$$

and due to the contribution of the surface entropy production e_{py}

$$e_{\mathrm{py}} = \left[\frac{q \cdot n}{T} \right] - \rho \dot{l} n_1 [s] \geqslant 0 \tag{18}$$

In solid mechanics, the assumption of continuous temperature on Σ is physically significant and will be accepted in our analysis. Formulae (13) and (18) then lead to

$$e_{\mathrm{py}} = \frac{1}{T} \dot{l}[\rho(w + \tfrac{1}{2}v^2)n_1 - n \cdot \sigma \cdot u_{,1}] \geqslant 0 \tag{19}$$

where $w = e - Ts$ denotes specific free energy of the material. The case of one singular point A can be obtained in the same way. One simply obtains when the crack tip temperature is finite

$$E_{\mathrm{py}}(\mathrm{A}) = \frac{1}{T_\mathrm{A}} \lim_{\Gamma \to 0} \int_\Gamma (w + \tfrac{1}{2}v^2)n_1 - n \cdot \sigma \cdot u_{,1})\, \mathrm{d}\Gamma \cdot \dot{l} \geqslant 0 \tag{20}$$

Remark 1

Thermodynamic restrictions are given by the non-negative condition of the entropy production at the crack tip or on the accompanying discontinuity surfaces Σ (formulae (20), (18) or (19)).

However, these conditions represent an overall estimate of the entropy production and do not take into account the constitutive equations.

In fact, a moving surface of strong discontinuity must be physically considered as the limit of a narrowing zone of transition of fast variations of the displacement velocity. In this zone, the material undergoes an arbitrary strain path from the initial state (+) to the final state (−) in order to ensure the jump $[\varepsilon]$.

It is then established **(10)** that for a standard elasto–plastic material, a stronger general restriction holds

$$[\rho c(w + \tfrac{1}{2}v^2) - n \cdot \sigma \cdot v] \geqslant cD(-[\varepsilon_\mathrm{p}]) \tag{21}$$

where D denotes the dissipative potential.

Let us illustrate (21) in the simple case of Prager–Ziegler's linear kinematic hardening model of elasto-plasticity. In this case $\rho w = \frac{1}{2}(\varepsilon - \varepsilon_\mathrm{p})L(\varepsilon - \varepsilon_\mathrm{p}) + \frac{1}{2}\varepsilon_\mathrm{p} H \varepsilon_\mathrm{p} + \kappa T \cdot \mathrm{tr}\,(\varepsilon - \varepsilon_\mathrm{p}) + \rho w_0(T)$, ε_p is the plastic strain tensor, H denotes the kinematic hardening modulus, $H = 0$ corresponds to the special case of perfect plasticity. Inequality (21) can be explicitly written as

$$[\rho c(w + \tfrac{1}{2}v^2) - v \cdot \sigma \cdot n] \geqslant ck\| -[\varepsilon_\mathrm{p}]\| \tag{22}$$

Remark 2

It is also interesting to note that the conservation of momentum and the continuity of the displacement field imply some restrictions on the jumps of stress and strain across the shock surface. For example, in quasi-static transformation, many authors **(4)** or **(7)**, or **(6)** have shown that some components of strain must be continuous across the shock.

They have also introduced an *additional assumption* concerning their variations inside the shock which states that the strain path inside the transition layer must be radial, at least concerning these components, and concluded finally that stress discontinuity must be ruled out on quasi-static moving surface of discontinuity in perfect plasticity.

Elastic–plastic fracture

Energetic parameter and crack propagation

This paragraph is devoted to the problem of crack propagation in elastic–plastic fracture. Principal difficulties of an energetic approach based upon the classical framework of continuum mechanics and thermodynamics are presented. The introduction of discontinuity surfaces leads also to an interesting direction to be explored.

The first possibility derives from the thermodynamic analysis of the running crack. It is recalled that classical methods of thermodynamics can be applied to construct useful laws of crack propagation from the obtained expression of the entropy production.

Indeed, it follows from the previous discussion that, in the case of one singular point, the entropy production of the considered system is

$$E_{py}(\Omega) = \int_\Omega e_v \, \mathrm{d}V + \frac{G}{T_A} \dot{l} \tag{23}$$

where G denotes the familiar expression

$$G = \lim_{\Gamma \Rightarrow 0} J_\Gamma \tag{24}$$

with

$$J_\Gamma = \int_\Gamma (\rho(w + \tfrac{1}{2}v^2) - n \cdot \sigma \cdot u_{,1}) \, \mathrm{d}\Gamma$$

and e_v the volumic entropy production

$$e_v = \rho \dot{s} + \mathrm{div}\left(\frac{q}{T}\right)$$

$$= \frac{1}{T}\left(\sigma \cdot \dot{\varepsilon} - \rho(\dot{w} + s\dot{T}) - \frac{q}{T} \cdot \mathrm{grad}\ T\right)$$

representing the contribution of the regular response in Ω.

For example, if the material is elastic $e_v = 0$ by definition and the crack tip entropy production $\dot{l}$. G/T_A is the only source of irreversibility of the system.

Expression (23) shows that G/T_A is a thermodynamic force associated with the velocity of crack propagation $\dot{l}$. The classical method of thermodynamics of irreversible processes (T.I.P.) can be applied (see for example **(11)**) and suggests a phenomenological relation between force and velocity

$$\frac{G}{T_A} = \frac{\partial D}{\partial l}(\dot{l}) \tag{25}$$

via the expression of the dissipative potential $D(\dot{l})$.

For example, Griffith's law in brittle fracture $G \leqslant G_c = 2\gamma$ corresponds to the special case

$$D(\dot{l}) = \frac{2\gamma}{T_A} \cdot \dot{l}$$

where the surface energy 2γ appears to be a critical value of surface dissipation admissible for the material. In the presence of surfaces of discontinuity accompanying the crack motion, the total entropy production of the system is

$$E_{py}(\Sigma) = \int_\Sigma \frac{1}{T} [\rho(w + \tfrac{1}{2}v^2) - n \cdot \sigma \cdot u_{,1}] \, \mathrm{d}\Sigma \cdot \dot{l} + \int_\Omega e_v \, \mathrm{d}V \tag{26}$$

In (26), the expression of the thermodynamic force is clear. The preceding method is again applicable and suggests a criterion of crack propagation based upon the quantity

$$F = \int_\Sigma \frac{1}{T} [\rho(w + \tfrac{1}{2}v^2) - n \cdot \sigma \cdot u_{,1}] \, \mathrm{d}\Sigma \tag{27}$$

which, in the special case of isothermal transformation, reduces also to the following expression

$$F = \frac{1}{T} \lim_{\Gamma \Rightarrow \Sigma} J_\Gamma .$$

A criterion of propagation $F = F_c$ can be thus formally introduced in the same way as in the construction of Griffith's criterion. It is also clear that F is not a local quantity while G is. The computation of F is a difficult problem because knowledge of the solution in the vicinity of the crack is required.

Asymptotic analysis in elastic–plastic fracture

Let us apply these results to the elastic–plastic modelling of ductile fracture. If the material is elastic–plastic with positive strain hardening the characterization of the local state of stress and strain has been first discussed by **(12)** for a propagating crack in quasi-static transformation. In this case, the crack tip is an isolated singular point since no first order discontinuity can be admitted. An energetic crack tip parameter is necessary, G given by formula (23). However, Amazigo and Hutchinson's results showed that the obtained crack tip singularity is not strong enough and leads always to the estimate $G = 0$ except in elasticity. This conclusion is also confirmed by other more refined discussions on the asymptotic singularity.

Thus, in this case no local energetic parameter can be found and in most analyses of the literature, a characteristic distance has to be introduced in the formulation of various criteria of propagation. For example **(13)** proposed the G_Δ criterion which is based upon the expression of J_Γ when Γ corresponds to a circle of ray Δ and centred at the crack tip. An alternating possibility which has often been adopted in the literature consists of accepting a criterion of crack propagation based upon a critical value of strain at a certain distance from the crack tip.

The case of *elastic perfectly plastic* material must be considered with more attention because of the eventual existence of discontinuity surfaces of stress and strain.

In this case, the determination of the asymptotic response for a propagating crack is an important problem. Many discussions have been devoted to it since the studies of **(1)** and **(2)**. Only some principal features of the obtained results in the literature are reported here in relation with strain and stress discontinuity and with the modelling of brittle fracture.

If the assumption of plastic state of stress all around the crack tip is accepted, many 'solutions' can be constructed satisfying all the mechanical equations and the plastic criterion. However, the jumps of stress and strain across surfaces of discontinuity must satisfy not only the mechanical conditions of conservation of mass, momentum, and energy but also the thermodynamic restriction of non-negative entropy production (20) or the general restriction (21) for standard materials.

For example, in mode III and in quasi-static transformation, Fig. 1 presents a possible solution with discontinuity of both stress and velocity fields (discussed in **(14)**) satisfying all conservation laws and thermodynamic restriction (20). However, condition (21) is violated and thus this 'solution' must be excluded.

In fact, the assumption of all around plastic state is not necessarily satisfied and most authors prefer the possibility of elastic unloading since **(2)**.

In mode I and in quasi-static transformation, a synthetic transformation, a synthetic review of the results has been given by **(15)**. Figure 2 presents the proposed asymptotic distribution of stress and the location of the velocity discontinuity.

As it is well known, the mathematical determination of the asymptotic response is rather cumbersome. Although this discussion is not already closed

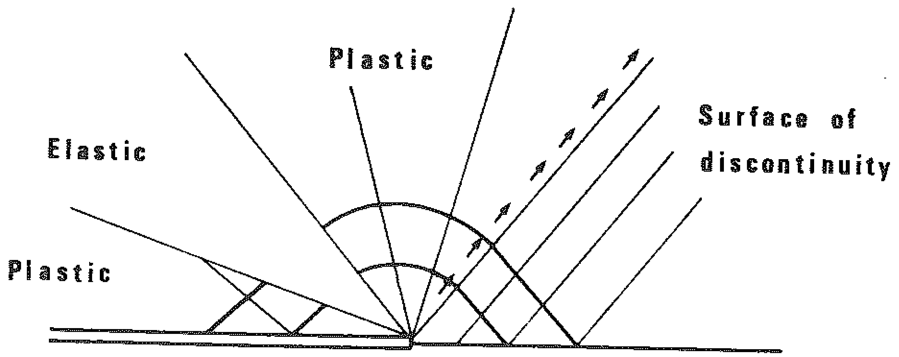

Fig 2 Asymptotic solution in mode I with strain discontinuity and unloading zone

in the literature, it seems interesting to underline here the role of strain discontinuity. Indeed, the crack opening displacement for a propagating crack can be obtained in these discussions as

$$u = Ar \log r + Br + \cdots \tag{28}$$

where the first term is due to the existence of elastic strain and the second term is principally due to the velocity discontinuity.

If the extension of the surface of discontinuity is large enough, the second term will be dominant at a certain distance from the crack tip and can be perhaps obtained by experimental measure as the crack tip opening angle CTOA.

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