*J. G. Merkle**

An Application of the J integral to an Incremental Analysis of Blunting Crack Behaviour

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ABSTRACT This paper describes an analytical approach to estimating the elastic–plastic stresses and strains near the tip of a blunting crack with a finite root radius. Rice's original derivation of the path independent J-integral considered the possibility of a finite crack tip root radius. For this problem Creager's elastic analysis gives the relation between the stress intensity factor K_I and the near tip stresses. It can be shown that the relation $K_I^2 = E'J$ holds when the root radius is finite. Recognising that elastic–plastic behaviour is incrementally linear then allows a derivation to be performed for a bi-elastic specimen having a crack tip region of reduced modulus, and the result differentiated to estimate elastic–plastic behaviour. The result is the incremental form of Neuber's equation. This result does not require the assumption of any particular stress–strain relation. However by assuming a pure power law stress–strain relation and using Ilyushin's principle, the ordinary deformation theory form of Neuber's equation, $K_\sigma K_\varepsilon = K_t^2$, is obtained. Applications of the incremental form of Neuber's equation have already been made to fatigue and fracture analysis. This paper helps to provide a theoretical basis for these methods previously considered semi-empirical.

Notation

E	Modulus of elasticity, MPa
E'	Effective modulus of elasticity for plane stress or plane strain, as specified, MPa
E_T	Notch-tip region elastic modulus for a bi-elastic specimen, MPa
J	The J-integral, $MJ \cdot m^{-2}$
J_T	Value of the J-integral computed by integrating on a contour close to the notch tip, $MJ \cdot m^{-2}$
J_∞	Value of the J-integral computed by integrating on a contour remote from the notch tip, $MJ \cdot m^{-2}$
K_I	Linear-elastic crack-tip stress-intensity factor, $MPa \cdot \sqrt{m}$
K_T	Linear-elastic crack-tip stress-intensity factor corresponding to the stress magnitudes near a crack tip, $MPa \cdot \sqrt{m}$
K_∞	Linear-elastic stress-intensity factor calculated on the basis of the remote nominal stress, crack size, and geometry for a specimen assumed to be homogeneous, $MPa \cdot \sqrt{m}$
K_t	Theoretical elastic stress concentration factor, dimensionless
K_ε	Actual strain concentration factor, dimensionless
K_σ	Actual stress concentration factor, dimensionless
r	Radial distance from notch-tip focal point, m
S	Nominal stress, MPa

* Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831-8049, USA.

W	Strain energy density, MPa
x	Coordinate distance measured parallel to the notch centreline, m
Δx	Distance behind the notch tip measured along the notch centreline, m
y	Coordinate distance measured perpendicular to the notch centreline, m
Γ_t	Notch-tip contour (not an algebraic quantity)
ε	Strain near the notch tip, particularly peak strain where so stated, dimensionless
θ	Position angle measured at the notch-tip focal point, radians
λ	Nominal strain, dimensionless
ν	Poisson's ratio, dimensionless
ρ	Notch-tip radius of curvature, m
σ	Stress near the notch tip, particularly peak stress where so stated, MPa
σ_e	Stress in the notch region calculated for the assumption of homogeneous elastic behaviour, MPa
σ_t	Principal stress acting tangential to the notch-tip contour, MPa
σ_x, σ_y	Normal stresses acting in the x and y direction, respectively, in the notch-tip region, MPa

Introduction

The development of linear-elastic fracture mechanics (LEFM) has made it possible to measure fracture toughness and to estimate safety margins against fracture for structures constructed of high-yield-strength, low-toughness materials. However, the majority of structures are designed and built of materials that are selected with the objective of preventing fracture at stresses below the yield stress. Thus, estimating fracture loads and safety margins for most structures, based on actual measured or probable undetected flaw sizes, requires some form of elastic–plastic fracture mechanics.

Several methods of elastic–plastic fracture analysis have come into use in recent years. The methods potentially easiest to apply to structures are approximate methods based on some particular simplified yet hopefully still general description of cracked structural behaviour. The only theoretically based method of elastic–plastic fracture analysis currently in use is the J-integral **(1)(2)**. This method has been used to calculate fracture toughness values for small two-dimensional test specimens and, more recently, by means of three-dimensional elastic–plastic finite element stress analysis computer programs, to calculate fracture strengths for large structures containing three-dimensional flaws. Nevertheless, such programs are still expensive to use, not universally available and not necessarily straightforward to use for parametric analyses, especially, for example, when the object is to find the critical flaw size for a given load under conditions of stable crack growth. Some approximate procedures currently in use require the assumption of a power law (Ramberg–Osgood) stress–strain curve, despite the fact that not all stress–strain data are

well fitted by such curves. What are still needed, at least for initial estimating purposes, are approximate methods of elastic–plastic fracture analysis that can be used with any reasonably accurate representation of measured stress–strain data and that have good theoretical justifications.

Analysis development

Theoretical considerations

The J-integral **(1)(2)** is the most widely accepted basis for elastic–plastic fracture analysis because its definition is physically based and mathematically exact; and, by virtue of path independence, its value can be determined analytically or experimentally by several different methods. However, its definition is based on the deformation theory of plasticity, which is known to be physically less accurate than incremental theory **(3)**. The incremental theory of plasticity is accepted as a more accurate method of elastic–plastic stress analysis because it recognises that the principal stress ratios at a point can change during loading **(3)**. Nevertheless, the majority of methods currently in use for elastic–plastic fracture analysis are based, explicitly or implicitly, on deformation theory, because they calculate the crack-driving force directly as a function of the currently applied load.

The J-integral is commonly justified as a fracture criterion because it serves as the loading parameter for the deformation theory stress distribution near the tip of an infinitely sharp crack in a material with a pure power-law stress–strain curve **(4)**. For this type of stress–strain curve and proportional applied loading only, deformation theory and incremental theory agree exactly, and stress analysis solutions are scalable according to Ilyushin's principle **(5)(6)**. Goldman, Hutchinson, Shih, and Kumar **(7)(8)** have taken advantage of this fact to develop a J-integral estimating procedure based on scaling. However, as stated earlier, deformation theory is in general an approximation; cracks do not remain infinitely sharp during loading, especially in the elastic–plastic range; and most materials do not deform exactly according to a pure power-law stress–strain curve. Furthermore, the estimating procedure of Shih and Kumar **(8)** still requires non-linear numerical analysis, and for this reason extensions to three-dimensional problems are still difficult and expensive.

Another analytical approach potentially useful for modelling the elastic–plastic aspects of crack-tip behaviour was developed by Neuber **(9)** in the form of the equation **(10)**

$$K_\sigma K_\varepsilon = K_t^2 \tag{1}$$

where K_σ and K_ε are the ratios of the peak to the nominal stresses and strains, respectively, and K_t is the theoretical elastic stress concentration factor. Neuber's equation (equation (1)) is by far the simplest equation available for the elastic–plastic stress analysis of notches. However, its application to cracks is hampered by the need to specify a value of the crack-tip root radius and by the lack of an analytical relationship connecting it to the more widely accepted equations of fracture mechanics.

Neuber's equation was originally developed implicitly for deformation theory conditions. However, it has since been applied successfully, in an incremental form, for cyclic fatigue analysis **(11)(12)** although without a completely theoretical basis. In the incremental form of equation (1), the total stress and strain terms on the left-hand side of the equation are simply replaced by the corresponding increments. Intuitively, an extension of this approach to an incremental analysis of elastic–plastic fracture under monotonic loading seems possible. Nevertheless, in all likelihood, such an approach would only be considered acceptable for use if a good theoretical basis could be found and agreement with experimental data demonstrated. The primary potential advantage of Neuber's equation is that the effects of geometry are estimated from a linear-elastic analysis and do not have to be re-determined in the plastic range, thus avoiding the need for a difficult and expensive non-linear numerical analysis. Neuber concluded that equation (1) was independent of the shape of the stress–strain curve. However, noting Neuber's **(9)** result that the solution for a Mode III-loaded crack also provides a solution for a notch with a finite root radius, Rice **(13)** pointed out that the elastic–perfectly-plastic solution described by McClintock and Irwin **(14)** agrees with equation (1) only for the case of small-scale yielding. Therefore, equation (1) apparently cannot be completely general. Nevertheless, if an incremental form of Neuber's equation can be found that is independent of the stress–strain curve, then it may still provide a valid basis for a practical method of elastic–plastic fracture analysis that does not involve unnecessary assumptions about the elastic–plastic behaviour of cracked structures. In addition, the principles of the theory of plasticity will probably have to be applied explicitly to clarify the applicability and limitations of Neuber's equation with respect to elastic–plastic fracture analysis.

Analytical starting points

In searching for an analytical basis for an incremental form of Neuber's equation, it was noted that Hutchinson **(15)** has obtained a solution for the stresses near the tip of a sharp crack in elastic-linear strain-hardening material and that the result is the classical elastic solution, multiplied by the factor $\sqrt{(E_T/E)}$, where E_T and E are the crack region and the remote region tangent moduli, respectively. Furthermore, Wang **(16)** obtained the same result for the case of a bi-elastic adhesive joint specimen. These results suggest that an analytical basis for an incremental form of Neuber's equation might be obtainable by using a bi-elastic specimen as an analytical starting point and converting the bi-elastic solution to an incremental solution by differentiation. Such an analytical model is shown in Fig. 1, which depicts a blunted crack in an elastic body the near-tip region of which has an elastic modulus of E_T and the remainder an elastic modulus of E. Because of path independence, the value of

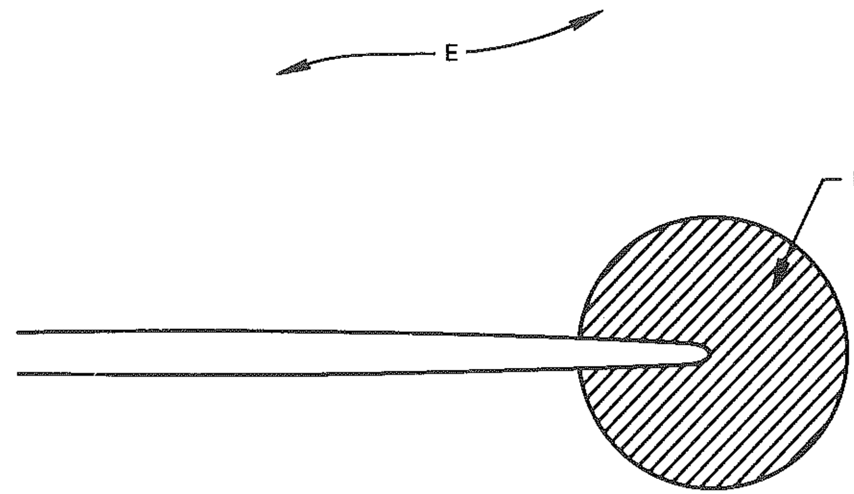

Fig 1 Schematic drawing of a blunted crack in a bi-elastic specimen with notch-tip region elastic modulus E_T and elastic modulus E elsewhere

J should be the same for all contours surrounding the notch tip that begin and end on parallel segments of the notch faces. Thus,

$$J_T = J_\infty \tag{2}$$

where J_T is the value of J computed for a contour lying close to the notch tip entirely within the region of modulus E_T and J_∞ is the value of J computed for a contour located remote from the notch tip entirely within the region of modulus E.

Since the purpose of the analysis to be developed is to examine the basis of equation (1), consideration of the existence of a finite notch root radius is necessary to prevent the value of K_t from becoming infinite. Rice's derivation of the J-integral **(1)** recognised the possible existence of a finite notch root radius, as demonstrated by the equation

$$J = \int_{\Gamma_t} W \, dy \tag{3}$$

where W is strain energy density, y is distance perpendicular to the notch centreline, and Γ_t is the notch-tip contour. Consequently, the existence of a finite notch root radius should not affect the path independence of J. Furthermore, since J is derived for non-linear elastic conditions, its path independence should not be affected by a difference in the elastic modulus values between the notch-tip region and other remote regions, as assumed in writing equation (2).

To relate J to the notch-tip stress field, a relation between these quantities must be used that is valid for notches having finite root radii as well as for infinitely sharp cracks. The relation

$$J = K_I^2/E' \tag{4}$$

where $E' = E/(1-\nu^2)$ for plane strain and E for plane stress, was derived by Irwin **(17)** for sharp cracks. However, implicitly, the same relation was derived by Rice **(2)** for notches with finite root radii. The latter derivation consisted of developing the equation for J by integrating along a contour surrounding the notch tip having a radius that was large compared with the notch root radius but small compared with other specimen dimensions, so that the only terms entering the expression for J were the singular inverse square root sharp crack terms. In Rice's derivation, the radius of the contour of integration was taken to be effectively infinite; thus, the problem actually analysed was a semi-infinite notch in an infinite body. Because any terms involving the notch root radius were eliminated at the input stage of Rice's analysis by the choice of an integration path, their null effect was only implicit in the results. However, explicit consideration of the effects of a notch root radius on the relation between J and K_I should be made possible by applying equation (3) to Creager's elastic notch-region stress equations **(18)(19)**. For each stress component, Creager's equations consist of the sharp-crack-stress term plus an additional term proportional to the product of K_I and the notch root radius, ρ.

For a narrow elliptic or hyperbolic notch, the near-tip region of which is shown in Fig. 2, the near-tip normal stresses are given by **(18)(19)**

$$\sigma_x = \frac{K_I}{\sqrt{(2\pi r)}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_I}{\sqrt{(2\pi r)}}\left(\frac{\rho}{2r}\right)\cos\frac{3\theta}{2}, \tag{5}$$

and

$$\sigma_y = \frac{K_I}{\sqrt{(2\pi r)}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_I}{\sqrt{(2\pi r)}}\left(\frac{\rho}{2r}\right)\cos\frac{3\theta}{2}, \tag{6}$$

where the coordinate axes x and y are aparallel and perpendicular, respectively, to the centreline of the notch. In equations (5) and (6), the origin of the cylindrical coordinates is the focal point of the notch profile, which is located

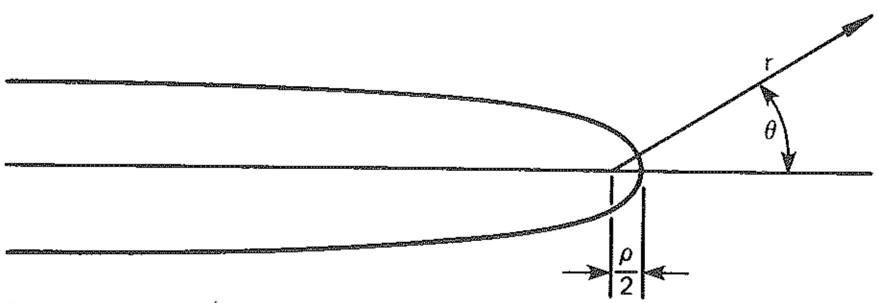

Fig 2 Profile of the near-tip region of a narrow elliptic or hyperbolic notch showing coordinate system used with Creager's elastic stress equations (18)

on the x axis at a distance $\rho/2$ behind the notch tip. In addition, the stress-intensity factor K_I has exactly the same definition in terms of nominal stress and notch depth as it would have for a sharp crack of the same depth.

For either plane stress or plane strain, the only stress that contributes to the strain energy density on a traction-free notch surface is σ_t, the principal stress acting tangential to the notch surface. Mohr's circle can be used to show that, on the notch surface

$$\sigma_t = \sigma_x + \sigma_y \tag{7}$$

Thus, substituting equations (5) and (6) into equation (7) gives

$$\sigma_t = \frac{2K_I}{\sqrt{(2\pi r)}}\cos\frac{\theta}{2} \tag{8}$$

in which terms proportional to ρ do not appear because of their opposite signs in equations (5) and (6).

Blunted notch shape and relation between J and K_I

To apply equation (3) to determining the relation betweem J and K_I by integrating along the notch contour, an expression for the notch contour that allows the determination of J by closed-form integration is desirable. Considering a narrow elliptical notch, the tip region of which is shown in Fig. 3, it is straightforward to show **(18)** that for small values of r/a, the ellipse can be closely approximated by a parabola, for which

$$y = r\sin\theta \tag{9}$$

and

$$r = \frac{\rho}{1+\cos\theta} \tag{10}$$

The elastic strain energy density on the surface of a blunt notch is given by

$$W = \frac{\sigma_t^2}{2E'} \tag{11}$$

Combining equations (8) and (11) thus gives

$$W = \frac{K_I^2}{E'}\frac{\cos^2(\theta/2)}{\pi r} \tag{12}$$

Substituting equation (10) and the identity

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \tag{13}$$

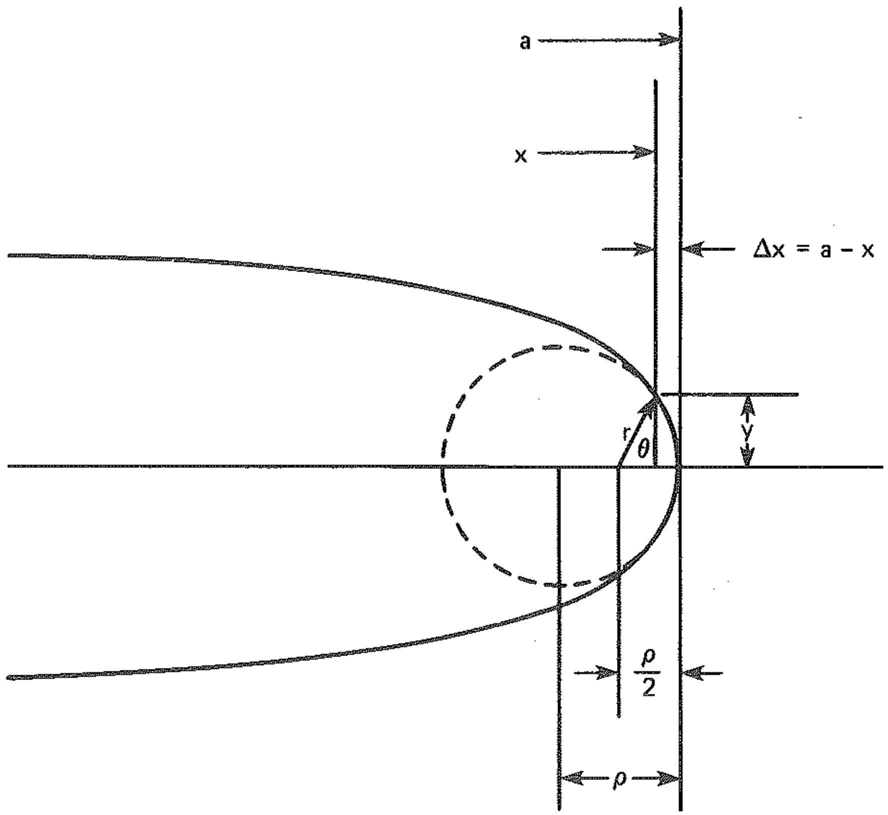

Fig 3 Near-tip region of an elliptical notch showing coordinates and dimensions used in the parabolic approximation of the notch profile

into equation (12) thus leads to

$$W = \frac{K_I^2}{E'}\frac{(1+\cos\theta)^2}{2\pi\rho} \tag{14}$$

Combining equations (9) and (10) gives

$$y = \frac{\rho \sin\theta}{(1+\cos\theta)} \tag{15}$$

so that

$$dy = \frac{\rho\, d\theta}{(1+\cos\theta)} \tag{16}$$

Thus, substituting equations (14) and (16) into equation (3) gives

$$J = \frac{K_I^2}{E'}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\cos\theta)\, d\theta \tag{17}$$

By direct integration, equation (17) gives

$$J = \frac{K_I^2}{E'} \tag{18}$$

Thus, the relation between J and K_I is unaffected by the existence of a finite root radius.

The problem analysed to derive equation (18) is the crack surface counterpart of the problem analysed by Rice **(2)**. It uses the near-notch-tip stress equations (as did Rice) and locates the end points of the notch profile at an effectively infinite distance from the notch tip (as did Rice) to ensure reaching a condition of parallel notch faces at the end points of the integration. The equations used are valid as long as the distance from the notch tip is small compared with the notch depth.

Stress concentration in a bi-elastic specimen

Returning to the blunted crack in the bi-elastic specimen shown in Fig. 1, applying equation (18) to both sides of equation (2) gives

$$J = K_T^2/E'_T = K_\infty^2/E' \tag{19}$$

where K_T is the stress-intensity factor of the actual near-tip stress field and K_∞ is the stress-intensity factor calculated on the basis of the remote nominal stress and crack size. The null effect of a small zone of reduced modulus near the notch tip on the value of K_∞ was pointed out by Rice **(2)**. Since there is no dependence of the near-tip stress variation on the numerical value of the elastic modulus, it follows that each of the actual near-tip stress components is given by

$$\sigma = \frac{K_T}{\sqrt{(2\pi r)}} f\left(\frac{\rho}{r}, \theta\right) \tag{20}$$

and that the near-tip stresses that would exist in a homogeneous elastic specimen are given by

$$\sigma_e = \frac{K_\infty}{\sqrt{(2\pi r)}} f\left(\frac{\rho}{r}, \theta\right) \tag{21}$$

Rearranging equation (19) and assuming no variation in Poisson's ratio within the specimen gives

$$K_T = K_\infty\sqrt{(E_T/E)} \tag{22}$$

which is identical to the result obtained by Wang **(16)** for a sharp crack in an elastic adhesive joint specimen. Then, by substituting the expressions for K_T and K_∞ obtained from equations (20) and (21) into equation (22), it follows that

$$\sigma = \sigma_e\sqrt{(E_T/E)} \tag{23}$$

that is, the near-tip stresses for a notch with a finite root radius in a bi-elastic specimen are the same as the stresses calculated for a homogeneous elastic specimen but then multiplied by the factor $\sqrt{(E_T/E)}$. This is exactly the same as the result obtained by Hutchinson **(15)** for a sharp crack in a material described as an elastic-linear strain-hardening material.

Incremental elastic–plastic analysis of notch-tip stresses and strains

Since equation (23) is valid for a notch with a finite root radius, it provides a basis for examining the incremental changes in stress and strain at the root of a notch. At the root of a notch

$$\sigma_e = K_t S \tag{24}$$

where K_t is the theoretical elastic stress concentration factor and S is the nominal stress. Combining equations (23) and (24) gives

$$\sigma = K_t S \sqrt{(E_T/E)} \tag{25}$$

Equation (25) still applies only for elastic conditions, but it can be used to develop an estimate for elastic–plastic conditions by recognising that the incremental response of an elastic–plastic body to a load increment is still linear – that is, although the tangent moduli at points throughout a body loaded into the plastic range may change from one load increment to the next, they do not change during a given load increment. Thus, for considering elastic–plastic behaviour, equation (25) can be differentiated, holding E_T and E constant as the tangent moduli governing the notch tip and the nominal strain increments, respectively. Therefore, for elastic–plastic conditions

$$d\sigma = K_t \, dS \sqrt{(E_T/E)} \tag{26}$$

Sqiaring both sides of equation (26) and rearranging gives

$$d\sigma(d\sigma/E_T) = dS(dS/E)(K_t^2) \tag{27}$$

Since

$$E_T = \frac{d\sigma}{d\varepsilon} \tag{28}$$

and

$$E = \frac{dS}{d\lambda} \tag{29}$$

where ε and λ are the peak and the nominal strains, respectively, substituting equations (28) and (29) into equation (27) and rearranging gives

$$(d\sigma/dS)(d\varepsilon/d\lambda) = K_t^2 \tag{30}$$

which is the incremental form of Neuber's equation being sought. It is important to note that the derivation of equation (30) involves no assumption about the shape of the stress–strain curve. However, equation (30) should be particularly easy to apply when the stress–strain curve is represented by a piecewise linear curve.

Total stress and strain analysis for a power-law stress–strain curve

As mentioned previously, Ilyushin's principle **(5)(6)** states that stress analysis solutions for proportional boundary loading are scalable for power-law stress–strain curves. This implies that for a pure power-law stress–strain curve, the incremental ratios in equation (30) can be replaced by the corresponding ratios of total stress and strain. Furthermore, a reverse proof of Ilyushin's principle was developed by Chang and Witt **(20)**, showing that if scaling is assumed to hold, then the stress–strain relation must be a pure power law. Therefore, for proportional boundary loading and a power-law stress–strain curve

$$\frac{d\sigma}{dS} = \frac{\sigma}{S} \tag{31}$$

and

$$\frac{d\varepsilon}{d\lambda} = \frac{\varepsilon}{\lambda} \tag{32}$$

Substituting equations (31) and (32) into equation (30) then gives

$$(\sigma/S)(\varepsilon/\lambda) = K_t^2 \tag{33}$$

Since by the usual definitions of peak stress and strain

$$\sigma/S = K_\sigma \tag{34}$$

and

$$\varepsilon/\lambda = K_\varepsilon \tag{35}$$

the result of substituting equations (34) and (35) into equation (33) is

$$K_\sigma K_\varepsilon = K_t^2 \tag{36}$$

which is the original form of Neuber's equations **(9)(10)** based on total stress and strain.

Discussions

The foregoing analysis clarifies and improves the relationship between Neuber's equation for estimating the stress and strain concentration factors of notches in the elastic–plastic range and the other equations of fracture mechanics. It also helps to explain the previous cases of good agreement between calculation and experimental data that have been obtained with methods of fatigue and fracture analysis based on the incremental form of Neuber's equation. One such method is the tangent modulus method of elastic–plastic fracture analysis **(21)(22)**, which has been used to analyse fracture test data from a series of large part-through surface-cracked tensile specimens, with the results

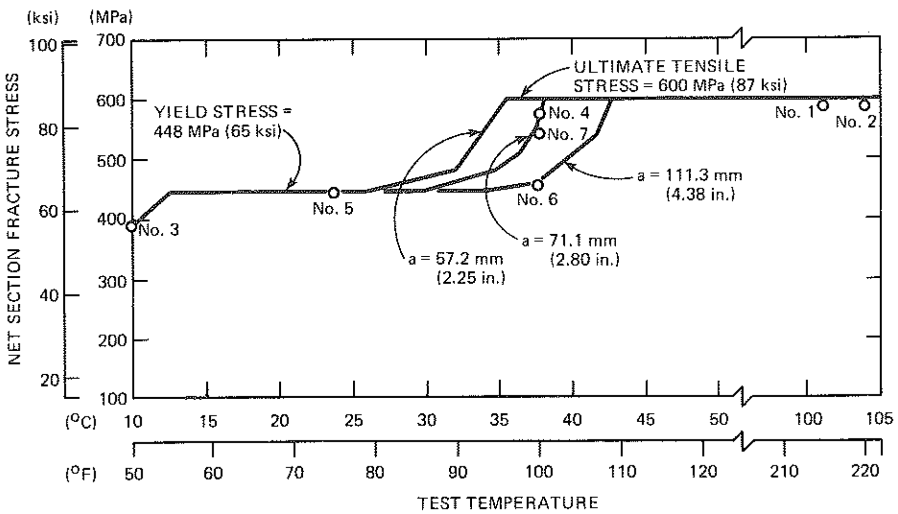

Fig 4 **Comparison of experimentally determined net section fracture–stress values with estimates made by the tangent modulus method for longitudinally oriented 152.4 mm-thick (6 in.) part-through surface-cracked intermediate tensile specimens of A533 grade B class 1 steel** (22)

shown in Fig. 4. Reference **(22)** contains the details of this analysis. As shown in Fig. 4, the analysis described in reference **(22)** is applicable up to but not beyond the average net section stress that causes necking and plastic instability to occur in the region of the flaw. This is because the derivation of equation (39) is based on small strain theory and includes only an approximate representation of the effects of stress redistribution due to yielding, by means of the variation of the nominal tangent modulus with strain. The tangent modulus method has also been used to analyse fracture test data from two HSST Program intermediate test vessels with inside nozzle-corner cracks **(23)**. This method of analysis is particularly well suited to the estimation of fracture strengths in the elastic–plastic range at temperatures at which rapid fracture can still take place. Stable crack growth can also be considered.

The incremental form of Neuber's equation (equation (30)) has been used for the cyclic fatigue analysis of notches **(11)(12)** and also to develop a relationship between fracture toughness, triaxial ductility, and other material properties **(24)**. The derivations presented therein help to provide a theoretical foundation for these methods of analysis and are thus responsive to expressions of uncertainty **(25)(26)** about their analytical basis and limitations.

Conclusions

(1) By using Creager's equations for the elastic stresses near the tip of a blunted notch, it is possible to show that $K_I^2 = E'J$, irrespective of the existence of a finite notch root radius.

(2) Considering the path independence of J, the stresses near the tip of a notch in a bi-elastic specimen with notch-tip region modulus E_T and remote elastic modulus E are demonstrated to be the same as for a homogeneous elastic specimen, except that the stresses are multiplied by the factor $\sqrt{(E_T/E)}$.

(3) The previous result for bi-elastic conditions can be transformed by differentiation and substitution into an incremental form of Neuber's equation for the elastic–plastic analysis of notch tips. The result, which is independent of the stress–strain curve, is the equation

$$(\mathrm{d}\sigma/\mathrm{d}S)(\mathrm{d}\varepsilon/\mathrm{d}\lambda) = K_t^2$$

where σ, S, ε, and λ are the peak and the nominal stresses and strains, respectively, and K_t is the theoretical elastic stress concentration factor.

(4) By using Ilyushin's principle, which states that the stress and strain solutions for proportionally loaded bodies obeying a pure power-law stress–strain curve can be scaled, the incremental form of Neuber's equation can be transformed into the familiar deformation theory form

$$K_\sigma K_\varepsilon = K_t^2$$

where K_σ and K_ε are the actual stress and strain concentration factors, respectively.

(5) Several methods of fatigue and fracture analysis based on the incremental form of Neuber's equation have been previously developed. The analyses presented here provide an analytical basis for these methods and also an explanation of their usefulness in analysing experimental data.

(6) The method of analysis suggested herein is applicable up to but not beyond the loads causing necking and plastic instability of the region near the flaw, because the effects of stress redistribution due to yielding are only approximated by decreasing the tangent modulus corresponding to the nominal strain at the flaw location.

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