CONSERVATION LAW OF J INTEGRAL TYPE FOR MULTILAYERED SHELLS

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In this paper we examine the balance laws in multilayered shells. Using the Euclidean group of transformation, the equivalence between the balance law and the Euclidean invariance is demonstrated. An example is then consider and extension one of these balance laws to simple problems for plates theory is carried out.

INTRODUCTION

Conservation laws (or balance laws) have been the subject of considerable research in recent years. One of these laws, the J-integral, has been applied extensively to the fracture mechanics problems with much success. In this paper, we examine similar type of integrals for multilayered shells in the context of mutidirector surfaces theory, based on the assumption of piece-wise linearity of the displacements field across the thickness. Such a treatment identifies each layer with a 2-D array of material vectors so that the shell is in factor regarded as a surface endowed with n-director fields, n being the number of layers. Such a picture strongly suggests the concept of a multidirector Cosserat surface.

Conservation laws for classical shells have been considered by Bergez and Radenkovic' (1), and Bergez (2). Lo (3) examined path-independent integrals for cylindrical shells and shells of revolution. Studies made by Kienzler and Golebievska-Herrmann (4) show that conservation laws are derived from variational principle in the context of higher-order shells theories. Based on the Naghdi's theory of thin shells, Sedmak, Berković and Jarić (5) have derived path independent integral for generally shaped shells.

In this paper the intention is to derive conservation laws (or balance laws) using invariant characteristic of variational principle in relation to the Euclidean group of transformation. Using Euclidean group of transformation, the equivalence between the conservation law and the Euclidean invariance is demonstrated. As a consequence a nowel result for the conservation law (or balance law) for multilayered shells has ben obtained. Finally, one of the laws is an example is used to illustrate its application.

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EQUATIONS OF VARIATIONAL INVARIANCE

Let $\xi=(\xi_i)\in R_i$, $i=0,1,2$, be the independent and $\phi=(\phi_\alpha)\in R_\alpha$, $\alpha=1,m$ dependent vector variables, describing the behaviour of material system under consideration.

The following action integral can be defined now:

$$A(\phi) = \int_T \int_B L \, dSdt = \int_R L(Y) \, dx \tag{1}$$

where L represents real scalar function of ξ, ϕ, $\phi,_\alpha$, defined and differentiable for all values of its arguments and $Y = Y(\xi,\phi,\phi,_\alpha)$.

For the action integral (1), the small transformations of dependent and independent variables are introduced as follows:

$$\xi^{*i} = \xi^i + \delta\xi^i = \xi^i + \alpha^i\eta + O(\eta^2)$$
$$\phi^* = \phi + \delta\phi = \phi + b\,\eta + O(\eta^2) \qquad (2)_{1,2}$$

where the quantities $\delta\xi^i = \alpha^i$, $\delta\phi = b$ etc. are taken to be of infinitesimal order and η is a small parameter.

Now a special form of Noether's theorem can be defined, which is used here to derive the conservation laws (the proof of this theorem can be found in (6)):

THEOREM: If the fields ϕ satisfy the corresponding Euler-Lagrange equations $E(L)_\phi = Q$, then the functional (1) remains infinitesimally invariant at ϕ under the small transformations (2), if and only if ϕ satisfies

$$\frac{\partial}{\partial\xi^\alpha}\left[\{L,_{\phi,_i}, m\} + L\,\alpha^i\right] - \{m, Q\} = 0 \tag{3}$$

where the vector m is defined as

$$m = b - \phi,_i\alpha^i \tag{4}$$

It was convenient (in eqs (3) and (4)) to use abbreviated notation suggested by Ericksen (7): $\{\phi_1, \phi_2\} = \{(a_1, b_1)(a_2, b_2)\} = a_1 a_2 + b_1 b_2$.

MULTILAYERED SHELLS

The starting point is the elastic multilayered shell theory by Epstein and Glockner (8), and Ericksen and Truesdell (9). Only the basic elements of the theory are given here and details can be found in (8,9)

Let $R=R(X^\alpha)$ be the position vector of a generic point of the reference surface S of a shell in the reference configuration, with curvilinear Gaussian coordinates X^α ($\alpha=1,2$). Associated with it is a complete description of the shell and the supplementary director fields $D_I = D_I(X^\alpha)$, $I=1,2,\ldots..n$.

A motion of the shell is defined by specifying the position vector, **r**, of the deformed surface and the deformed directors, d_I, as function of the curvilinear coordinates, X^α and of time t, i.e.

$$\mathbf{r} = \mathbf{r}(X^\alpha, t) = \mathbf{r}(\xi^i) \tag{5}$$

$$\mathbf{d}_I = \mathbf{d}_I(X^\alpha, t) = \mathbf{d}_I(\xi^i)$$

Assume that m constraints are imposed on the deformation in

$$\Psi_i(\mathbf{r},_\alpha; \mathbf{d}_I, \mathbf{d}_{I,\alpha}) = 0 \qquad i=1,\dots m \tag{6}$$

which must satisfy frame indifference.

The Lagrangian density H associated with the multilayered shell is given by

$$H = L(Y) - \lambda^i \psi_i , \qquad Y = Y(\mathbf{r},_i; \mathbf{d}_I, \mathbf{d}_{I,i}) \tag{7}$$

and $\lambda^i = \lambda^i(X^\alpha, t)$ is theLagrange multiplier associated with the i-th constraint, eqn (6). The laws of motion, given by eqs (15a,b) in (8), are equivalent to the Euler-Lagrange equations

$$\frac{\partial}{\partial \xi^\alpha} \frac{\partial H}{\partial \phi,_\alpha} - \frac{\partial H}{\partial \phi} - Q = 0 \tag{8}$$

Then the Noether's theorem can be applied to our case. To confirm this statement we choose

$$L = H , \quad \phi = (r, d_I) , \quad Q = (F, F^I) , \quad m = (p, q_I) , \quad b = (\beta, \gamma_I) ,$$

$$P = \frac{\partial H}{\partial r} \quad P^I = \frac{\partial H}{\partial d_I} , \quad T^a = \frac{\partial H}{\partial r,_a} , \quad T^{Ia} = \frac{\partial H}{\partial d_{I,a}} \tag{9}$$

Before proceeding further, the integral form of the conservation laws is given, applying the Gauss theorem to (3):

$$\frac{d}{dt}\int_S (Pp + P^I q_I + L\alpha_0) ds + \int_C (T^\alpha p + T^{I\alpha} q_I + L\alpha^\alpha) n_\alpha dl + \int_S (Fp + F^I q_I) = 0 \tag{10}$$

where C is the smooth closed curve, bounding S and n is the unit normal (in S) to C.

THE CONSERVATION LAW

Following Toupin (10), one can postulate that the action density L is invariant under the group of Euclidean displacements. Since the group of Euclidean displacements is a connected Lie group, it is sufficient to require that the action density is invariant under infinitesimal transformations of the group of Euclidean displacements in order that it is invariant under arbitrary, finite transformations of the group. An infinitesimal transformation of the group has the form

$$X^\bullet = X + C\eta; \quad \phi^\bullet = \phi + (\Omega\phi + D)\eta; \quad \phi = (r, d_I); \quad t^\bullet = t + C_0\eta \tag{11}$$

where Ω is an antisymmetric tensor, $\Omega = \Omega^T$, and Ω, C, C_0 and D are arbitrary constants. By taking all of the arbitrary constants in (11) to be equal to zero, except the one in turn, we obtain the corresponding conservation laws:

(I) $D = 0, \quad \alpha_0 = 0, \quad \alpha^a = 0, \quad \beta = D, \quad \gamma_I = 0, \quad p = D$

The corresponding conservation law (3.6) now reads

$$\frac{d}{dt}\int P \, ds + \int T^{\alpha} n_{\alpha} dl - \int F \, ds = 0 \qquad (12)$$

(II) $\Omega = 0, \ \alpha_0 = 0, \ \alpha^{\alpha} = 0, \ \beta = \Omega\, r, \ \gamma_I = \Omega\, d_I, \ p = \Omega\, r, \ q_I = \Omega\, d_I$

This transformation represents rigid body rotation, and the corresponding conservation law reads

$$\frac{d}{dt}\int A^{*} ds - \int_{c} (r \times T^{\alpha} + d_I \times T^{I\alpha}) n_{\alpha} dl - \int_{s} (r \times F + d_I \times F^{I}) ds = 0 \qquad (13)$$

where

$$A^{*} = r \times P + d_I \times P^{I}$$

(III) $\alpha_0 = C_0 = 0, \ \alpha = 0, \ \beta = 0, \ \gamma_I = 0, \ p = -C_0 \dot{r}, \ q_I = -C_0 \dot{d}_I$

This transformation represent a shift of time, and the corresponding law reads

$$\frac{d}{dt}\int_{s} E \, ds - \int_{c} (T^{\alpha} \dot{r} + T^{I\alpha} \dot{d}_I) n_{\alpha} dl - \int_{s} (F \dot{r} + F^{I} d_I) ds = 0 \qquad (14)$$

where $E = P \dot{r} + P^{I} d_I - L$.

The above conservation laws (12-14) represent the conservation of linear momentum, moment of momentum and energy, respectively. Thus, we have established the basic theorem of equivalence between conservation and invariance (10).

As a special case we consider

(IV) $\alpha^{\alpha} = C^{\alpha} = 0, \ \alpha_0 = 0, \ \beta = 0, \ \gamma_I = 0, \ p = -r_{,\alpha} C^{\alpha}, \ q_I = -d_{I\,,\alpha} C^{\alpha}$

This transformation represents the family of coordinate translations, and leads us to the conservation laws which are of a special interest for us.

$$\frac{d}{dt}\int_{s} (P r_{,a} + P^{I} d_{I,a}) ds - \int_{c} (L \delta^{b}_{a} - T^{b} r_{,a} - T^{Ib} d_{I,a}) n_b dl - \int_{s} (F r_{,a} + F^{I} d_{I,a}) ds = 0 \qquad (15)$$

The expressions (12-15) represent novel conservation laws for multilayered shells. Of special interest in fracture mechanics is the expression (15) which represent the conservation law of J integral type.

APPLICATION

A. SPECIAL CASE

Bearing in mind the application to elastic multilayered shells, it is convenient to assume a Lagrangian density decompossable as

$$L = W - K \tag{16}$$

with

$$2K = R^{00}\dot{r}\,\dot{r} + 2R^{0I}\dot{r}\,\dot{d}_I + R^{IJ}\dot{d}_I\dot{d}_J \tag{17}$$

and

$$W = W(r,_{\alpha}, d_I, d_{I,\alpha}, X^{\alpha}) \tag{18}$$

where the inertia coefficients R^{00}, R^{0I}, R^{IJ} are time independent. Under these circumstances K satisfies Euclidean invariance, as is easily verified, and since L has been assumed to be Euclidean invariant, it follows from (16) that the same is true for W.

For the case n=1, eqn (3.4) reduces to the theory of Cosserat surface with one variable director (7,8), which has been used for the theory of sendwich shell (9). Indeed, for n=0 and no constraints, eqn (3.4) and (16-18) reduce to (7):

$$\left(\frac{\partial W}{\partial \phi,_{\alpha}}\right),_{\alpha} + \frac{\partial W}{\partial \phi} - G = \kappa\phi \tag{19}$$

$$\phi = (r,d), \quad G = (f,g)$$

where

$$\kappa(a,b) = (R^{00}a+R^{01}b,\ R^{01}a+R^{11}b).$$

B. ELASTIC PLATES

Reformulation. We are interested in homogeneous flat plates, for which the reference configurations is of the form

$$r = r_R(X^1,X^2,0)$$

$$d = d_R = \text{const.} \tag{20}$$

X^1 and X^2 being rectangular Cartesian coordinates. Then, from (16), W and K will represent energies/unit undeformed or reference area. To describe its homogeneity, we restrict W by the condition that

$$W = W(r,_{\alpha},\ d,\ d,_b;X^c) = W(r,_{\alpha},\ d,\ d,_b;0) \tag{21}$$

where the coordinates are chosen so that the original lines are within the plate.

Recalling that W does not depend explicitely on r and has to be Euclidean invariant (7), one has

$$W(\phi,\phi,_{\alpha}) = W^*(U,U,_{\alpha}) \tag{22}$$

The equation (5.4) then becomes

$$\left(\frac{\partial W^*}{\partial U,_\alpha}\right),_\alpha - \frac{\partial W^*}{\partial U} + G^* = \kappa U \tag{23}$$

and the corresponding conservation law (10) in this case becomes

$$\frac{d}{dt}\int_S \{\kappa\dot{U}, U,_\alpha\}ds - \int_C (L\delta^b_\alpha - \{T^b, U,_\alpha\}n_b dl - \int_S \{G, U,_\alpha\}ds = 0 \tag{24}$$

where the curly brackets denote the iner product.

Another simple case can be obtained if (23) reduces to a system of ordinary differential equations. An obvious possibility is to try

$$U = U(x), \qquad G^* = 0, \qquad x = n_a X^a - Vt$$

where n_a and V are constant. Then (24) gives a integral:

$$\int_\Gamma (W^* + V^2\{\kappa U, U''\})dX^2 - \{T^a n_a, U'\}dl - \int_S V^2\{\kappa U, U'''\}ds = 0 \tag{25}$$

which is path-independent for any path $\Gamma(t)$ around the crack tip and $t > t_0 > 0$.

ACKNOWLEDGMENT

This work was supported by a grant from the National Science Foundation of Yugoslavia, Grant No 0402.

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