Asymptotic and Numerical Analysis of Crack Tip Fields of Mode I in Steady Crack Growth with Linear Hardening Materials

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In the present paper the asymptotic stress, strain fields of mode I crack extension steadily and quasi-statically in an elastic plastic material are investigated. The material is characterized by J_2 - flow theory with linear strain hardening. All stresses and strains of the asymptotic crack tip field are separable functional forms of r and θ which represent the polar coordinates centered at the actual crack tip. For comparison with the asymptotic solutions numerical simulations of stable crack growth by the finite element method were performed on plane strain models of a compact tension specimen (CT) taking linear hardening material with coefficient $\alpha = E_t/E = 0.03$ (E is elastic modulus and E_t tangent modulus) into account. The results of the asymptotic solution and the numerical simulation are in qualitative good agreement.

1. Introduction

Knowledge of the stress and strain fields near the crack tip in elastic plastic material is essential for continued development of fracture mechanics. The asymptotic solution for a stationary crack (HRR-theory)/1,2/ supply a theoretical basis for elastic plastic fracture mechanics. The asymptotic solution of the stress and strain fields for a growing crack is much more complicated due to difficulfies in field equation formulation and inconsistency between elastic and plastic strain increments. Under the assumption of quasi-static growth and small-strain plasticity some special solutions have been found. The main progress in understanding the stress and deformation field for a growing crack tip has been limited to elasic-perfectly plastic material. Chitaly and McClintock /3/ found an asymptotic solution under anti-plane shear condition. For the plane strain mode I case, a complete near tip solution has been generated by Drugan, Rice and Sham /4/. Recently, a more general solution throughout the range from small-scale yielding

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up to large yielding has been found by Drugan and Chen /5/.

However, the asymptotic fields at a growing crack tip in hardening materials are not completely understood. An asymptotic analysis for a linear hardening material under anti-plane shear and mode I plane stress and plane strain has been presented by Amazigo and Hutchinson /6/. Based on this analysis plastic reloading has been introduced by Castanede /7/.An investigation about the asymptotic field for a power hardening material was performed by Gao and Hwang /8/. The main feature of this solution was an interaction between the elastic and plastic strain increments.

Now, the existent solutions at a growing crack tip in hardening materials have been confined to the field singularities and related problems. The issues are the complete solution for stress and strain and the relation between the amplitude factor of the field equations and loading. Threfore, we will investigate the stress and strain fields at a steady growing crack tip for linear hardening material in this paper. We will also compare the asymptotic solutions with the results from a finite element simulation which has used the node shift and node release technique. The calculation assumes either plane stress or plane strain and mode I.

2. Formulation and Governing Equations

Let x_i (i=1,2,3) be a Cartesian coordinate system of fixed orientation traveling with the crack tip such that the x_3 - axis coincides with the straight crack front and the x_1 - axis is in the direction of crack advance. Similarly, let r and θ be polar coordinates corresponding to x_1 and x_2. In steady crack growth analysis the crack tip velocity is constant so that the material derivative is given by

$$\frac{d}{dt} = -w \frac{\partial}{\partial x_1} , \qquad (1)$$

where w is the crack growing velocity.

The stress components at the growing crack tip must satisfy the equilibrium equations. We introduce Airy's stress function ϕ which always satisfies the equilibrium equations. Then the stress components in polar coodinates derive from Airy's stress function by

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} , \\ \sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} , \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) . \end{aligned} \qquad (2)$$

Here is assumed that the crack tip field is divided into a plastic loading zone and an elastic unloading zone, Fig.1. The constitutive equations in the plastic loading zone, accounting for strain hardening, and characterized by J_2-flow theory and a bilinear stress-strain curve, Fig.2.

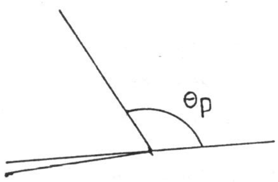

Fig.1 Crack tip fields

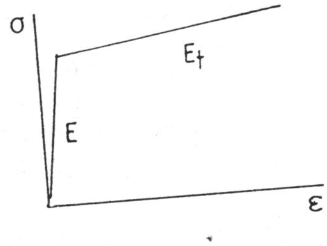

Fig.2 Bilinear material

$$\begin{aligned} E_t\dot{\varepsilon}_{ij} &= \alpha[(1+\nu)\dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}] + \frac{3}{2}(1-\alpha)\frac{\dot{\sigma}_e}{\sigma_e}s_{ij} \quad (\dot{\sigma}_e \geq 0) \\ E_t\dot{\varepsilon}_{ij} &= \alpha[(1+\nu)\dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}] \quad (\dot{\sigma}_e < 0) \end{aligned} \qquad (3)$$

The strain increments must satisfy the compatibility condition

$$\frac{1}{r}\frac{\partial^2\dot{\gamma}_{r\theta}}{\partial r\partial\theta} - \frac{\partial^2\dot{\varepsilon}_{\theta\theta}}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\dot{\varepsilon}_{rr}}{\partial\theta^2} + \frac{1}{r}\frac{\partial\dot{\varepsilon}_{rr}}{\partial r} + \frac{1}{r^2}\frac{\partial\dot{\gamma}_{r\theta}}{\partial\theta} - \frac{2}{r}\frac{\partial\dot{\varepsilon}_{\theta\theta}}{\partial r} = 0 \qquad (4)$$

which is the conditional equation for the stress function. For asymptotic analysis we introduce separating power-law functions for the stress function, stresses and strains

$$\phi(r,\theta) = A\sigma_o r^{-s+2}\tilde{\phi}(\theta)$$
$$\sigma_{ij}(r,\theta) = A\sigma_o r^{-s}\tilde{\sigma}_{ij}(\theta) \qquad (5)$$
$$\varepsilon_{ij}(r,\theta) = A\varepsilon_o r^{-s}\tilde{\varepsilon}_{ij}(\theta)$$

Introducing the equations (5) into (2),(3) and (4) considering zero stress components $\sigma_{xz},\sigma_{yz},\sigma_{zz}$ for plane stress and zero strain components $\varepsilon_{xz},\varepsilon_{yz},\varepsilon_{zz}$ for the plane strain and eliminating the amplitude A and the radius r, we obtain ordinary differential equations for the angle θ of fifth order

$$\tilde{\phi}^{(5)}(\theta) = F(\ \tilde{\phi}^{(4)},\tilde{\phi}^{(3)},\tilde{\phi}'',\tilde{\phi}',\tilde{\phi},\alpha,s,\theta_p,\theta\ ) \qquad \textit{plane stress} \qquad (6)$$
$$\tilde{\phi}^{(5)}(\theta) = F(\ \tilde{\phi}^{(4)},\tilde{\phi}^{(3)},\tilde{\phi}'',\tilde{\phi}',\tilde{\phi},\alpha,s,\theta_p,\theta,\tilde{\sigma}_{zz}\ ) \qquad \textit{plane strain}$$

We have the boundary conditions and symmetry conditions for mode I

$$\tilde{\phi}(\pi) = \tilde{\phi}'(\pi) = \tilde{\phi}'(0) = \tilde{\phi}(0)''' = 0$$
$$\tilde{\phi}^{(4)}(0) = F[\tilde{\phi}(0),\tilde{\phi}''(0),s,\alpha] \qquad \textit{plane stress} \qquad (7)$$
$$\tilde{\phi}^{(4)}(0) = F[\tilde{\phi}(0),\tilde{\phi}''(0),s,\alpha,\tilde{\sigma}_{zz}] \qquad \textit{plane strain}$$
$$\tilde{\sigma}_{zz}(0) = F_1[\tilde{\phi}(0),\tilde{\phi}''(0),s,\alpha] \qquad \textit{plane strain}$$

The differential equations (6) with boundary conditions (7) constitute a homogeneous problem so that we can set

$$\tilde{\sigma}_e(0) = 1 \qquad (8)$$

Now we must introduce continuity conditions between the loading zone and unloading zone. They are formulated at $\theta=\theta_p$

$$[\ \sigma_{ij}\ ] = [\ \dot{\sigma}_{ij}\ ] = [\ u_i\ ] = [\ \dot{u}_i\ ] = 0 \qquad (9)$$

The system (6),(7),(8) and (9) is solved by numerical integration. Thus we obtain the singularity s,unloading angle θ_p and the stress components. By integrating the constitutive equations (3), considering the initial value (8), the strain components for plane stress and plane strain are obtained. For control the obtained results are introduced into the compatibility condition

$$\frac{1}{r}\frac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} - \frac{\partial^2\varepsilon_{\theta\theta}}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r} + \frac{1}{r^2}\frac{\partial\gamma_{r\theta}}{\partial\theta} - \frac{2}{r}\frac{\partial\varepsilon_{\theta\theta}}{\partial r} = 0 \qquad (10)$$

which was satisfied by all obtained results.

Thus the analytical results have been compared with the numerical results from a finite element analysis simulating crack growth by a node shift and node release technique. Details of this computation will be found in /9/.

3. Numerical Results

Fig.3 shows the singularities for plane stress and plane strain. They are not identical for both cases as in the HRR-solution for a stationary crack. It is shown in Fig.4 that the unloading angles are dependent on the hardening coefficient α and run contrary for plane stress and plane strain cases. In Fig.5 and 6 we can show that the stress components σ_{xx} and σ_e for plane stress and σ_{xx},σ_{zz} and σ_e for plane strain are singular, and the effective stress σ_e for plane strain at θ=0^o is very small in relation to the other components. Fig.7 and 8 supply the strain results for both cases. The strain components for plane stress are nearly constant and only the components ε_{yy} and ε_{zz} on the crack flank are singular. On the other hand, the strains for plane strain depend on the angle θ, and ε_{yy} and γ_{xy} on the crack flank are singular likewise. In comparison with the results from finite element computation,Fig.9 and 10,we can show that both the stress distribution and the strain distribution around the growing crack tip for small strain hardening α=0.03 are in qualitative good agreement.

4. Conclusion

In this paper we have presented the complete solutions for the asymptotic crack tip stress and strain fields of steady state crack growth with linear hardening material. In our investigation power-law singularity functions are used in the crack tip field. The field singularities and their associated angular variations of the stress and strain fields are determined, leaving undetermined their amplitude factor which should be dependent on loadings and geomerties. We have first presented the strain distributions of steady crack growth for both cases whereas the existent solutions with this material have presented mostly the velocity fields round the growing crack tip. Finally, the asymptotic solutions have been compared with the numerical results from finite element computation.

Acknowlegements

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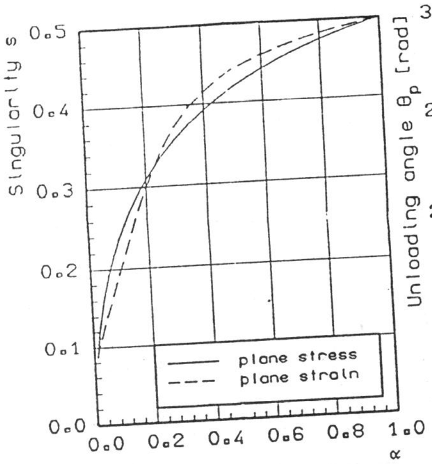

Fig. 3: Singularity

Bilinear material: $\nu = 0.3$

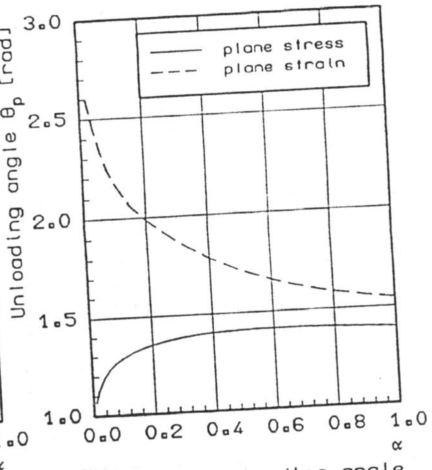

Fig. 4: Unloading angle

Bilinear material: $\nu = 0.3$

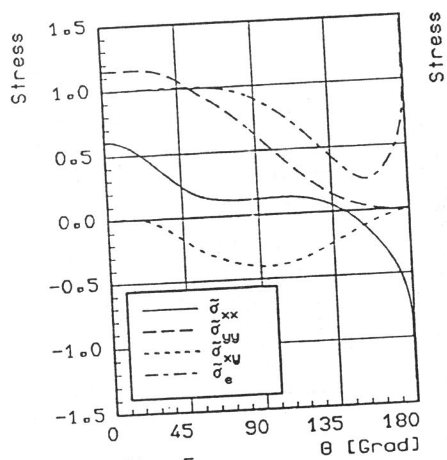

Fig. 5:

Stress for pl. stress

$\alpha=0.01$, $\nu=0.3$

Normalized such that $\tilde{\sigma}_e(\theta_p) = 1$

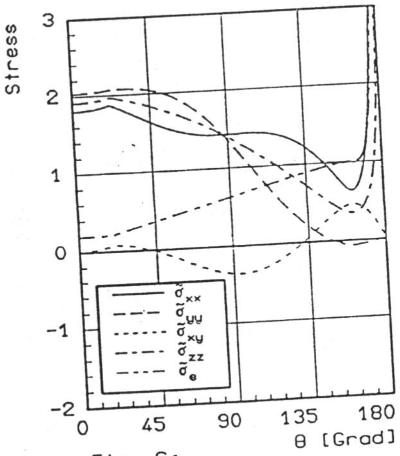

Fig. 6:

Stress for pl. strain

$\alpha=0.01$, $\nu=0.3$

Normalized such that $\tilde{\sigma}_e(\theta_p) = 1$

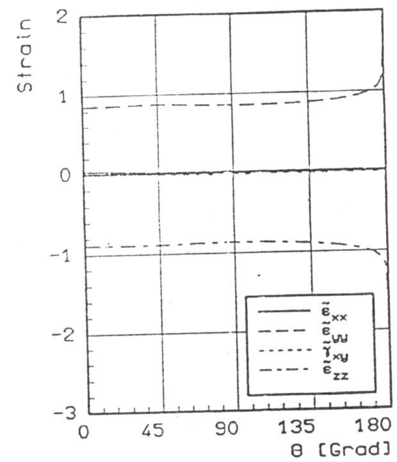

Fig. 7:

Strain for pl. stress
$\alpha=0.01$, $\nu=0.3$
Normalized such that $\tilde{\varepsilon}_e^p(\theta_p)=1$

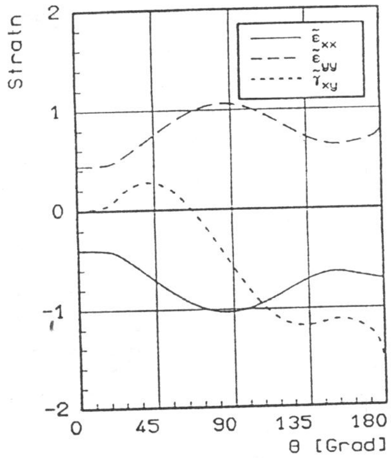

Fig. 8:

Strain for pl. strain
$\alpha=0.01$, $\nu=0.3$
Normalized such that $\tilde{\varepsilon}_e^p(\theta_p)=1$

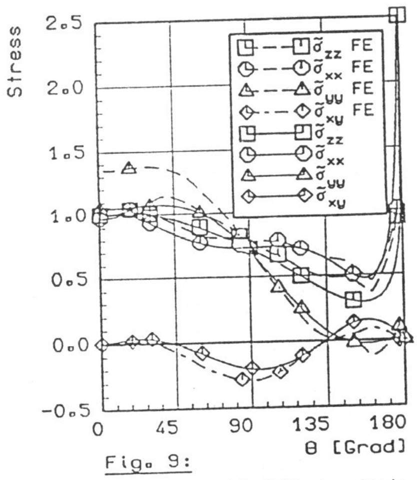

Fig. 9:

Comparison with FEM pl. strain
Normalized such that $\tilde{\sigma}_{xx}(0^\circ)=1$
$\alpha=0.03$, $\nu=0.3$

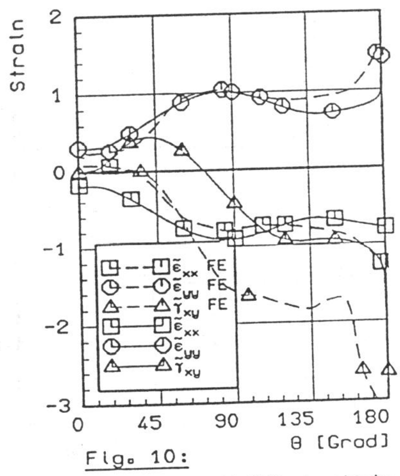

Fig. 10:

Comparison with FEM pl. strain
Normalized such that $\tilde{\varepsilon}_{yy}(90^\circ)=1$
$\alpha=0.03$, $\nu=0.3$