NUMERICAL METHODS FOR THE DESCRIPTION OF "CERAMIC STEELS" (ZrO_2 CONTAINING CERAMICS)

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The stress-induced transformation toughening of ZrO_2 containing ceramics is examined by means of a quantitative model. Transformed ZrO_2 inclusions are represented by pressurized, spherical holes surrounding a Griffith crack in an infinite, two-dimensional matrix. The K_I-value for a given configuration will be calculated numerically using a pertubation procedure in order to solve generalized Isida equations (1). Knowing K_I and the K_{IC} of the matrix it is possible to predict the increase in fracture toughness of the compound. Numerical errors will be estimated and the final results are compared with experimental data.

INTRODUCTION

The increase in fracture toughness of ZrO_2 containing ceramics has been reported and discussed extensively (e.g.) by Claussen und Rühle (2), Evans (3) and Evans and Heuer (4). Two mechanisms are mainly responsible for the improvement: "Transformation toughening by microcracking" and "stress-induced transformation toughening". The latter (to which we restrict ourselves in this paper) is based upon the stabilization of tetragonal t-ZrO_2 far below room-temperature by means of a surrounding matrix (e.g. Al_2O_3). Under the influence of the enormous stress-concentration in the neighbourhood of an approaching crack tip the ZrO_2 inclusions will transform into their more voluminous, monoclinic m-version. Hence extra work is necessary to move the crack through the generated compressive process zone. The corresponding K_{IC}-increase was calculated approximatively in (2) using Irvin's energy concept. Müller showed in (5,6) that it is alternatively possible to apply a compounding technique. He studied the forces along a chord in a "sphere of action" around a transformed ZrO_2-particle and determined their influence on the K_I-value of a simple Griffith crack.

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Both methods were able to reproduce the experimental results fairly well, but nevertheless: they were not based upon exact solutions of a given boundary-value-problem. This shortcoming will be removed in the present paper.

MODEL

We consider a Griffith crack in an infinite two-dimensional plane under uniform stress σ at infinity: Figure 1. In order to modelize the toughening effect we shall assume that the crack is surrounded by several transformed ZrO_2-particles. These inclusions are represented by circular holes which are subject to an enormous pressure (cp.(5,6)) so that the t → m volume increase is taken into account.

Now, if for a given configuration of a crack and pressurized holes K_I would be known, one could calculate the "toughening ratio"

$$\alpha = \frac{K_I}{K_{IC}\ (Al_2O_3)} = \frac{K_I}{\sigma_c \sqrt{\pi a}} \tag{1}$$

where σ_c is the critical stress which must be overcome in order to propagate a crack of length a through an Al_2O_3-matrix. σ_c can be calculated for any given a, since

$$K_{IC}\ (Al_2O_3) \simeq 3\mathrm{MNm}^{-3/2} \tag{2}$$

One concludes that for $\alpha < 1$ the crack is stabilized and the smaller α, the more important is the increase in toughness. Obviously the final problem is the determination of K_I. We proceed to discuss this point.

ANALYSIS

Throughout the following we shall use the notation of the Isida paper (1). Proofs are kept to a minimum and rely decisively upon (1). The reader who is mainly interested in the results is recommended to glance through the presumptions and propositions and then should go directly to the next chapter.

Basic equations

Presumptions:
Consider (Figure 2) an infinite plate containing an arbitrary distribution of various elliptical holes (j = 1,..., N), some of which may be circles or cracks. The holes are subject to a pressure p_0^j* which can be zero or not. The plate is assumed to be subjected to the following stresses at infinity:

* In (1) only "free" holes were considered, i.e. $p_0^j = 0$.

$$t^{\infty}_{xx} = \sigma(\alpha + \delta \frac{Y}{d}) \ , \quad t^{\infty}_{yy} = \sigma(\beta + \mu \frac{X}{d}) \ , \quad t^{\infty}_{xy} = \sigma\gamma \qquad (3)$$

where σ is a reference stress, X, Y are cartesian coordinates and α, β, γ. δ, μ are constants as shown in Figure 2.

Proposition I:
The Airy stress function of the problem can be written (with respect to any hole j) as:

$$\chi = \sigma d^2 \, \mathrm{Re} \left\{ \bar{z}_j \, \Phi_j (z_j) + \Psi_j (z_j) \right\}$$

$$\Phi_j(z_j) = \sum_{n=o}^{\infty} \left\{(F^{\cdot}_{n,j} + i F'_{n,j}) \, z_j^{-(n+1)} + (M^{\cdot}_{n,j} + i M'_{n,j}) \, z_j^{n+1}\right\} \qquad (4)$$

$$\Psi_j(z_j) = -D^{\cdot}_{o,j} \ln z_j + \sum_{n=1}^{\infty} (D^{\cdot}_{n,j} + i D'_{n,j}) \, z_j^{-n} + \sum_{n=o}^{\infty} (K^{\cdot}_{n,j} + iK'_{n,j}) z_j^{n+2} ,$$

where dots and primes correspond to real and imaginary parts resp.; z_j is the complex, dimensionless variable with respect to the j-th coordinate system (cf. Figure 2):

$$z_j := \frac{1}{d} (X_j + i \, Y_j) \qquad (5)$$

with a reference length d.

The occuring complex coefficients $(F_{n,j}, D_{n,j})$ and $(M_{n,j}, K_{n,j})$ have to satisfy the following equations which relate them to the $(F_{n,k}, D_{n,k})$, $(M_{n,k}, K_{n,k})$ of all other holes and ensure the validity of (3):

$$M^{\cdot}_{n,j} = \frac{\Delta^n_o}{4} \left\{\beta+\alpha+r_j(\mu \cos \beta_j+\delta \sin \beta_j)\right\} + \frac{\Delta^n_1}{8} (\mu \cos \alpha_j+\delta \sin \alpha_j) + $$
$$+ \sum_{p=o}^{\infty} \sum_{k\neq j}^{N} (e^{p,k}_{n,j} F^{\cdot}_{p,k} + f^{p,k}_{n,j} F'_{p,k})$$

$$M'_{n,j} = \frac{\Delta^n_o}{4} r_j(\mu \sin \beta_j-\delta \cos \beta_j) + \frac{\Delta^n_1}{8} (\mu \sin \alpha_j-\delta \cos \alpha_j) + \qquad (6)_1$$
$$+ \sum_{p=o}^{\infty} \sum_{k\neq j}^{N} (-f^{p,k}_{n,j} F^{\cdot}_{p,k} + e^{p,k}_{n,j} F'_{p,k})$$

$$K^{\cdot}_{n,j} = \frac{\Delta^n_o}{4} \left\{(\beta-\alpha) \cos 2\alpha_j-2\gamma \sin 2\alpha_j+r_j \cos 2\alpha_j(\mu \cos \beta_j-\delta \sin \beta_j)\right\}+$$
$$+ \frac{\Delta^n_1}{24} (\mu \cos 3\alpha_j-\delta \sin 3\alpha_j) +$$
$$+ \sum_{p=o}^{\infty} \sum_{k\neq j}^{N} (a^{p,k}_{n,j} D^{\cdot}_{p,k} + b^{p,k}_{n,j} D'_{p,k} + c^{p,k}_{n,j} F^{\cdot}_{p,k} + d^{p,k}_{n,j} F'_{p,k})$$

$$K'_{n,j} = \frac{\Delta_o^n}{4}\Big\{(\beta-\alpha)\sin 2\alpha_j + 2\gamma\cos 2\alpha_j + r_j \sin 2\alpha_j(\mu\cos\beta_j - \delta\sin\beta_j) +$$

$$+\frac{\Delta_1^n}{24}(\mu\sin 3\alpha_j + \delta\cos 3\alpha_j) +$$

$$+\sum_{p=o}^{\infty}\sum_{k\neq j}^{N}\left(-b_{n,j}^{p,k}D^{\cdot}_{p,k} + a_{n,j}^{p,k}D'_{p,k} - d_{n,j}^{p,k}F^{\cdot}_{p,k} + c_{n,j}^{p,k}F'_{p,k}\right) \quad ,$$

where

$$a_{n,j}^{o,k} = \frac{\cos\left\{(n+2)(\beta_{jk}-\alpha_j)\right\}}{(n+2)(r_{jk})^{n+2}}, \quad b_{n,j}^{o,k} = \frac{\sin\left\{(n+2)(\beta_{jk}-\alpha_j)\right\}}{(n+2)(r_{jk})^{n+2}}, \quad r_{jk} = \frac{\rho_{jk}}{d}, \quad r_j = \frac{\rho_j}{d}$$

$$\begin{Bmatrix} a_{n,j}^{p,k} \\ b_{n,j}^{p,k} \end{Bmatrix} = (-1)^p \binom{n+p+1}{n+2} \frac{\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(p(\alpha_j-\alpha_k)+(n+p+2)(\beta_{jk}-\alpha_j)\right)}{(r_{jk})^{n+p+2}} \qquad (6)_2$$

$$\begin{Bmatrix} c_{n,j}^{p,k} \\ d_{n,j}^{p,k} \end{Bmatrix} = (-1)^p \binom{n+p+2}{n+2} \frac{\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left((p+2)(\alpha_j-\alpha_k)+(n+p+4)(\beta_{jk}-\alpha_j)\right)}{(r_{jk})^{n+p+2}}$$

$$\begin{Bmatrix} e_{n,j}^{p,k} \\ f_{n,j}^{p,k} \end{Bmatrix} = (-1)^{p+1} \binom{n+p+1}{n+1} \frac{\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left((p+2)(\alpha_j-\alpha_k)+(n+p+2)(\beta_{jk}-\alpha_j)\right)}{(r_{jk})^{n+p+2}} \quad .$$

(Δ_j^o refers to the Kronecker symbol!)

Furthermore all coefficients must obey the "pressure hole relations" in order to guarantee that all hole boundaries j are only subject to the pressure p_o^j and to no other forces:

$$D^{\cdot}_{2n,j} = \sum_{p=o}^{\infty}\lambda_j^{2n+2p+2}\left(P_{2p,j}^{2n}K^{\cdot}_{2p,j} + R_{2p,j}^{2n}\left(M^{\cdot}_{2p,j} + \underline{\frac{p_o^j}{2\sigma}\Delta_p^o}\right)\right)$$

$$F^{\cdot}_{2n,j} = -\sum_{p=o}^{\infty}\lambda_j^{2n+2p+2}\left(Q_{2p,j}^{2n}K^{\cdot}_{2p,j} + S_{2p,j}^{2n}\left(M^{\cdot}_{2p,j} + \underline{\frac{p_o^j}{2\sigma}\Delta_p^o}\right)\right)$$

$$D^{\cdot}_{2n+1,j} = \sum_{p=o}^{\infty}\lambda_j^{2n+2p+4}\left(P_{2p+1,j}^{2n+1}K^{\cdot}_{2p+1,j} + R_{2p+1,j}^{2n+1}M^{\cdot}_{2p+1,j}\right)$$

$$F^{\cdot}_{2n+1,j} = -\sum_{p=o}^{\infty}\lambda_j^{2n+2p+4}\left(Q_{2p+1,j}^{2n+1}K^{\cdot}_{2p+1,j} + S_{2p+1,j}^{2n+1}M^{\cdot}_{2p+1,j}\right) \qquad (7)_1$$

$$D'_{2n,j} = -\sum_{p=o}^{\infty}\lambda_j^{2n+2p+2}\left(T_{2p,j}^{2n}K'_{2p,j} + V_{2p,j}^{2n}M'_{2p,j}\right)$$

$$F'_{2n,j} = \sum_{p=o}^{\infty}\lambda_j^{2n+2p+2}\left(Q_{2p,j}^{2n}K'_{2p,j} + W_{2p,j}^{2n}M'_{2p,j}\right)$$

$$D'_{2n+1,j} = -\sum_{p=o}^{\infty} \lambda_j^{2n+2p+4} \left(T^{2n+1}_{2p+1,j} K'_{2p+1,j} + V^{2n+1}_{2p+1,j} M'_{2p+1,j}\right)$$

$$F'_{2n+1,j} = \sum_{p=o}^{\infty} \lambda_j^{2n+2p+4} \left(Q^{2n+1}_{2p+1,j} K'_{2p+1,j} + W^{2n+1}_{2p+1,j} M'_{2p+1,j}\right)$$

where:

$$P^o_{2p,j} = (1-\varepsilon_j^2)^{p+1} \frac{(p+1)}{2^{2p}} \binom{2p+1}{p}, \qquad R^o_{2p,j} = P^o_{2p,j} \frac{(R_j^2+R_j^{-2})}{2}$$

$$\left\{\begin{matrix} P^{2n}_{2p,j} \\ T^{2n}_{2p,j} \end{matrix}\right\} = \frac{(1-\varepsilon_j^2)^{n+p+1}}{2^{2p+2}} \left[\left\{\begin{matrix} 4(p+1) \binom{2p+1}{p} \left(1+\frac{1}{n}\right) A_{n,1} \\ 0 \end{matrix}\right\} + \sum_{m=1}^{p+1,n} \binom{2p+2}{p-m+1} \left((2p+1)R_j^{4m} \mp 1\right) A_{n-m,2m}\right]$$

$$\left\{\begin{matrix} R^{2n}_{2p,j} \\ V^{2n}_{2p,j} \end{matrix}\right\} = \frac{(1-\varepsilon_j^2)^{n+p+1}}{2^{2p+1}} \left[\left\{\begin{matrix} (p+1) \binom{2p+1}{p} (R_j^2+R_j^{-2}) \left(1+\frac{1}{n}\right) A_{n,1} \\ 0 \end{matrix}\right\} + 2p \sum_{m=o}^{n,p} \binom{2p+1}{p-m} R_j^{4m+2} A_{n-m,2m+1}\right] \quad (n \geq 1) \tag{7$_2$}$$

$$Q^{2n}_{2p,j} = (1-\varepsilon_j^2)^{n+p+1} \frac{(p+1)}{2^{2p}} \sum_{m=o}^{n,p} \binom{2p+1}{p-m} R_j^{4m+2} A_{n-m,2m+1}$$

$$\left\{\begin{matrix} S^{2n}_{2p,j} \\ W^{2n}_{2p,j} \end{matrix}\right\} = \frac{(1-\varepsilon_j^2)^{n+p+1}}{2^{2p+1}} \left[(p+1)\binom{2p+1}{p} A_{n,1} \pm \sum_{m=o}^{n,p} \binom{2p+1}{p-m} A_{n-m.2m+1} + 2(2p+1) \sum_{m=1}^{n+1,p} \binom{2p}{p-m} R_j^{4m} A_{n-m+1,2m}\right]$$

$$\left\{\begin{matrix} P^{2n+1}_{2p+1,j} \\ T^{2n+1}_{2p+1,j} \end{matrix}\right\} = \frac{(1-\varepsilon_j^2)^{n+p+2}}{2^{2p+3}} \left[\pm(p+2)\binom{2p+3}{p+1} A_{n,1} + \sum_{m=o}^{n,p+1} \binom{2p+3}{p-m+1} \left((2p+2)R_j^{4m+2} \mp 1\right) A_{n-m,2m+1}\right]$$

$$\begin{Bmatrix} R^{2n+1}_{2p+1,j} \\ V^{2n+1}_{2p+1,j} \end{Bmatrix} = \frac{(1-\varepsilon_j^2)^{n+p+2}}{2^{2p+2}} \left[\pm(p+1)\begin{pmatrix}2p+1\\ p\end{pmatrix}(R_j \pm R_j^{-1})^2 A_{n,1} + \right.$$

$$\left. +(2p+1)\sum_{m=1}^{n+1,\,p+1}\begin{pmatrix}2p+2\\ p-m+1\end{pmatrix} R_j^{4m} A_{n-m+1,\,2m}\right]$$

$$Q^{2n+1}_{2p+1,j} = (1-\varepsilon_j^2)^{n+p+2}\,\frac{(2p+3)}{2^{2p+2}}\sum_{m=1}^{n+1,\,p+1}\begin{pmatrix}2p+2\\ p-m+1\end{pmatrix} R_j^{4m} A_{n-m+1,\,2m}$$

$$\begin{Bmatrix} S^{2n+1}_{2p+1,j} \\ W^{2n+1}_{2p+1,j} \end{Bmatrix} = \frac{(1-\varepsilon_j^2)^{n+p+2}}{2^{2p+2}} \left[\pm \sum_{m=1}^{n+1,\,p+1}\begin{pmatrix}2p+2\\ p-m+1\end{pmatrix} A_{n-m+1,\,2m} + \right.$$

$$\left. +2(2p+2)\sum_{m=0}^{n+1,\,p}\begin{pmatrix}2p+1\\ p-m\end{pmatrix} R_j^{4m+2} A_{n-m+1,\,2m+1}\right] \qquad (n \geqslant 0)$$

(Where for the bounds of summations smaller values must be taken!)
and:

$$\lambda_j := \frac{a_j}{d}\ , \quad \varepsilon_j := \frac{b_j}{a_j}\ , \quad R_j = \left(\frac{a_j + b_j}{a_j - b_j}\right)^{1/2}, \quad c_j = \frac{(a_j^2 - b_j^2)^{1/2}}{d}$$

$$A_{n-m,\,2m} = \frac{m}{2^{2n} n}\begin{pmatrix}2n\\ n-m\end{pmatrix}, \quad A_{n-m,\,2m+1} = \frac{2m+1}{2^{2n+1}(2n+1)}\begin{pmatrix}2n+1\\ n-m\end{pmatrix}$$

<u>Proof</u>:
The only difference between (7) and the free hole conditions of (1) are the underlined terms which we proceed to derive.

Consider a single elliptical hole under the influence of an internal pressure p_o. Introduce the mapping function

$$z = \Omega(\zeta) = \frac{c}{2}\left(R\zeta + \frac{1}{R\zeta}\right), \tag{8}$$

which transforms the hole boundary and its external region into a unit circle and its external region (Figure 3). The Kolosov equations allow to calculate all stresses along the hole edge:

$$t_{nn} + t_{ss} = 2\sigma\left\{\Phi'(z) + \overline{\Phi}'(\overline{z})\right\}_{\zeta = e^{i\beta}}$$

$$t_{nn} - t_{ss} - 2it_{ns} = -2\sigma\left[\frac{\zeta}{\overline{\zeta}\,\overline{\Omega}'(\overline{\zeta})}\left\{\overline{\Omega}(\overline{\zeta})\frac{d\Phi'(z)}{d\zeta} + \frac{d\psi'(z)}{d\zeta}\right\}\right]_{\zeta = e^{i\beta}}, \tag{9}$$

where the Goursat functions can be expanded into Laurent series:

$$\Phi'(\Omega(\zeta)) = \alpha_o + \sum_{m=1}^{\infty} (\alpha_m \zeta^m + \beta_m \zeta^{-m}) \tag{10}$$

$$\frac{d\Psi'(\Omega(\zeta))}{d\zeta} = \frac{c}{2}\left\{\gamma_o + \sum_{m=1}^{\infty} (\gamma_m \zeta^m + \delta_m \zeta^{-m})\right\}$$

Since t_{nn} is no longer zero but equal to $-p_o$ the resulting relations between the expansion coefficients ((2.74) in (1)) have to be changed:

$$\beta_{2m+2} = \beta_{2m+2}\,(\text{in}(1)) + \frac{p_o}{\sigma} R^{-(2m+2)} \tag{11}$$

$$\delta_{2m+2} = \delta_{2m+2}\,(\text{in}(1)) + \frac{p_o}{\sigma}(2m+1)\left[R^{-(2m-1)} + R^{-(2m+3)}\right]$$

Since:

$$(R\zeta)^{-2m} = \sum_{n=m}^{\infty} A_{n-m,\,2m} \left(\frac{c}{z}\right)^{2n}, \quad (R\zeta)^{-(2m+1)} = \sum_{n=m}^{\infty} A_{n-m,\,2m+1} \left(\frac{c}{z}\right)^{2n+1} \tag{12}$$

the $\Phi(z)$ and $\Psi(z)$ of (1) have to be extended as follows:

$$\Phi(z) = \Phi(z)[\text{in}(1)] - c\,\frac{p_o}{\sigma} \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{1}{2n-1} A_{n-m,\,2m} \left(\frac{z}{c}\right)^{-(2n-1)} =$$

$$= \sum_{n=o}^{\infty} \left\{(F_n^{\cdot} + i\,F_n') \, z^{-(n+1)} + (M_n^{\cdot} + i\,M_n')\, z^{n+1}\right\}$$

$$\Psi'(z) = \Psi'(z)\;[\text{in}(1)] - \frac{c}{2}\frac{p_o}{\sigma}(R^2 + R^{-2}) \sum_{m=o}^{\infty} \sum_{n=m}^{\infty} A_{n-m,\,2m+1} \left(\frac{z}{c}\right)^{-(2n+1)} =$$

$$= - D_o^{\cdot}\,\frac{1}{z} - \sum_{n=1}^{\infty} n(D_n^{\cdot} + i\,D_n')\, z^{-(n+1)} + \sum_{n=o}^{\infty} (n+2)(K_n^{\cdot} + i\,K_n')\, z^{n+1} \tag{13}$$

where

$$D_{2n}^{\cdot} = \sum_{p=o}^{\infty} \lambda^{2n+2p+2} \left(P_{2p}^{2n} K_{2p}^{\cdot} + R_{2p}^{2n} \left[M_{2p}^{\cdot} + \frac{p_o}{2\sigma} \Delta_p^o\right]\right) \tag{14}$$

$$F_{2n}^{\cdot} = - \sum_{p=o}^{\infty} \lambda^{2n+2p+2} \left(Q_{2p}^{2n} K_{2p}^{\cdot} + S_{2p}^{2n} \left[M_{2p}^{\cdot} + \frac{p_o}{2\sigma} \Delta_p^o\right]\right) .$$

All other coefficients satisfy the original conditions in (2.77) of (1). Now, in the case of N holes (13) and (14) can be used analogously as in (1) and one finally arrives at (4) – (7), q.e.d..

Proposition II:
The stress intensity factors K_I and K_{II} for modes I and II resp. can be calculated via:

$$(K_I - i\,K_{II}) = 2\sigma(2\pi d)^{1/2} \lim_{z_j \to \lambda_j} \left\{(z_j - \lambda_j)^{1/2}\, \Phi_j'(z_j)\right\} \tag{15}$$

provided that $\Phi_j(z_j)$ for the crack j is known.

Proof:
Cf. (1) for a motivation.

Numerical procedure

In order to enable an explicit calculation of K_I and K_{II} for various hole configurations one has to determine the unknown coefficients $D_{n,j}$, $F_{n,j}$, $M_{n,j}$, $K_{n,j}$ ($n = 0,1,2,\ldots$; $j = 1,2,\ldots,N$) from the relations (6) and (7). Use is made of the Isida perturbation technique:

Presumptions:

Let $$\lambda_j = s_j\lambda \tag{16}$$

with a new perturbation parameter λ which has to be determined suitably and constants s_j representing the ratio of hole lengths. All unknowns are assumed as the following power series is λ (perturbation ansatz):

$$D_{2n,j} = \sum_{p=n+1}^{\infty} D_{2n,j}^{(2p)} \lambda^{2p}, \qquad F_{2n,j} = \sum_{p=n+1}^{\infty} F_{2n,j}^{(2p)} \lambda^{2p}$$

$$D_{2n+1,j} = \sum_{p=n+2}^{\infty} D_{2n+1,j}^{(2p)} \lambda^{2p} \qquad F_{2n+1,j} = \sum_{p=n+2}^{\infty} F_{2n+1,j}^{(2p)} \lambda^{2p} \tag{17}$$

$$M_{n,j} = M_{n,j}^{(o)} \Delta_n^o + \sum_{p=1}^{\infty} M_{n,j}^{(2p)} \lambda^{2p}, \quad K_{n,j} = K_{n,j}^{(o)} \Delta_n^o + \sum_{p=1}^{\infty} K_{n,j}^{(2p)} \lambda^{2p}$$

Proposition III:
All expansion coefficients can be calculated successively via ($j = 1,\ldots,N$):

$$M'^{(o)}_{o,j} = \frac{1}{4}(\beta+\alpha), \quad M''^{(o)}_{o,j} = 0$$

$$M'^{(o)}_{n,j} = M''^{(o)}_{n,j} = 0 \quad (n \geq 1)$$

$$K'^{(o)}_{o,j} = \frac{1}{4}\left\{(\beta-\alpha)\cos 2\alpha_j - 2\gamma \sin 2\alpha_j\right\}, \quad K''^{(o)}_{o,j} = \frac{1}{4}\left\{(\beta-\alpha)\sin 2\alpha_j + 2\gamma\cos 2\alpha_j\right\}$$

$$K'^{(o)}_{n,j} = K''^{(o)}_{n,j} = 0 \quad (n \geq 1)$$

$$D'^{(2n+2)}_{2n,j} = s_j^{2n+2}\left\{(P_o^{2n})_j K'^{(o)}_{o,j} + (R_o^{2n})_j \left[M'^{(o)}_{o,j} + \frac{P_o^j}{2\sigma}\right]\right\}$$

$$D''^{(2n+2)}_{2n,j} = -s_j^{2n+2}\left\{(T_o^{2n})_j K''^{(o)}_{o,j} + (V_o^{2n})_j M''^{(o)}_{o,j}\right\}$$

$$F^{(2n+2)}_{2n,j} = -s_j^{2n+2}\left\{(Q_o^{2n})_j\, K^{(o)}_{o,j} + (S_o^{2n})_j\left[M^{(o)}_{o,j} + \frac{p_o^j}{2\sigma}\right]\right\}$$

$$F^{\{2n+2)}_{2n,j} = s_j^{2n+2}\left\{(Q_o^{2n})_j\, K^{\{o)}_{o,j} + (W_o^{2n})_j\, M^{\{o)}_{o,j}\right\}$$

$$M^{(2)}_{n,j} = \sum_{k\neq j}^{N}\left(e^{o,k}_{n,j}\, F^{(2)}_{o,k} + f^{o,k}_{n,j}\, F^{\{2)}_{o,k}\right)$$

$$M^{\{2)}_{n,j} = \sum_{k\neq j}^{N}\left(-f^{o,k}_{n,j}\, F^{(2)}_{o,k} + e^{o,k}_{n,j}\, F^{\{2)}_{o,k}\right)$$

$$K^{(2)}_{n,j} = \sum_{k\neq j}^{N}\left[a^{o,k}_{n,j}\, D^{(2)}_{o,k} + b^{o,k}_{n,j}\, D^{\{2)}_{o,k} + c^{o,k}_{n,j}\, F^{(2)}_{o,k} + d^{o,k}_{n,j}\, F^{\{2)}_{o,k}\right]$$

$$K^{\{2)}_{n,j} = \sum_{k\neq j}^{N}\left[-b^{o,k}_{n,j}\, D^{(2)}_{o,k} + a^{o,k}_{n,j}\, D^{\{2)}_{o,k} - d^{o,k}_{n,j}\, F^{(2)}_{o,k} + c^{o,k}_{n,j}\, F^{\{2)}_{o,k}\right] \qquad (18)$$

$$D^{(2n+2q)}_{2n,j} = \sum_{p=o}^{q-1} s_j^{2n+2p+2}\left\{(P^{2n}_{2p})_j\, K^{(2q-2p-2)}_{2p,j} + (R^{2n}_{2p})_j\, M^{(2q-2p-2)}_{2p,j}\right\}$$

$$D^{\{2n+2q)}_{2n,j} = -\sum_{p=o}^{q-1} s_j^{2n+2p+2}\left\{(T^{2n}_{2p})_j\, K^{\{2q-2p-2)}_{2p,j} + (V^{2n}_{2p})_j\, M^{\{2q-2p-2)}_{2p,j}\right\}^{*}$$

$$F^{(2n+2q)}_{2n,j} = -\sum_{p=o}^{q-1} s_j^{2n+2p+2}\left\{(Q^{2n}_{2p})_j\, K^{(2q-2p-2)}_{2p,j} + (S^{2n}_{2p})_j\, M^{(2q-2p-2)}_{2p,j}\right\}$$

$$F^{\{2n+2q)}_{2n,j} = \sum_{p=o}^{q-1} s_j^{2n+2p+2}\left\{(Q^{2n}_{2p})_j\, K^{\{2q-2p-2)}_{2p,j} + (W^{2n}_{2p})_j\, M^{\{2q-2p-2)}_{2p,j}\right\}$$

$$D^{(2n+2q)}_{2n+1,j} = \sum_{p=o}^{q-2} s_j^{2n+2p+4}\left\{(P^{2n+1}_{2p+1})_j\, K^{(2q-2p-4)}_{2p+1,j} + (R^{2n+1}_{2p+1})_j\, M^{(2q-2p-4)}_{2p+1,j}\right\}$$

$$D^{\{2n+2q)}_{2n+1,j} = -\sum_{p=o}^{q-2} s_j^{2n+2p+4}\left\{(T^{2n+1}_{2p+1})_j\, K^{\{2q-2p-4)}_{2p+1,j} + (V^{2n+1}_{2p+1})_j\, M^{\{2q-2p-4)}_{2p+1,j}\right\}$$

$$F^{(2n+2q)}_{2n+1,j} = -\sum_{p=o}^{q-2} s_j^{2n+2p+4}\left\{(Q^{2n+1}_{2p+1})_j\, K^{(2q-2p-4)}_{2p+1,j} + (S^{2n+1}_{2p+1})_j\, M^{(2q-2p-4)}_{2p+1,j}\right\}$$

$$F^{\{2n+2q)}_{2n+1,j} = \sum_{p=o}^{q-2} s_j^{2n+2p+4}\left\{(Q^{2n+1}_{2p+1})_j\, K^{\{2q-2p-4)}_{2p+1,j} + (W^{2n+1}_{2p+1})_j\, M^{\{2q-2p-4)}_{2p+1,j}\right\}$$

$$M^{(2q)}_{n,j} = \sum_{k\neq j}^{N}\sum_{p=o}^{2q-2}\left\{e^{p,k}_{n,j}\, F^{(2q)}_{p,k} + f^{p,k}_{n,j}\, F^{\{2q)}_{p,k}\right\}$$

$$M^{\{2q)}_{n,j} = \sum_{k\neq j}^{N}\sum_{p=o}^{2q-2}\left\{-f^{p,k}_{n,j}\, F^{(2q)}_{p,k} + e^{p,k}_{n,j}\, F^{\{2q)}_{p,k}\right\}$$

$$* \quad D^{\{2)}_{o,j} = 0, \qquad D^{\{2q)}_{o,j} = 0\ .$$

$$K^{\cdot(2q)}_{n,j} = \sum_{k\neq j}^{N} \sum_{p=o}^{2q-2} \left\{ a^{p,k}_{n,j} D^{\cdot(2q)}_{p,k} + b^{p,k}_{n,j} D'^{(2q)}_{p,k} + c^{p,k}_{n,j} F^{\cdot(2q)}_{p,k} + d^{p,k}_{n,j} F'^{(2q)}_{p,k} \right\}$$

$$K'^{(2q)}_{n,j} = \sum_{k\neq j}^{N} \sum_{p=o}^{2q-2} \left\{ -b^{p,k}_{n,j} D^{\cdot(2q)}_{p,k} + a^{p,k}_{n,j} D'^{(2q)}_{p,k} - d^{p,k}_{n,j} F^{\cdot(2q)}_{p,k} + c^{p,k}_{n,j} F'^{(2q)}_{p,k} \right\}$$

$$j = 1,2,\ldots,N, \qquad q = 2,3,\ldots,M, \quad M \to \infty.$$

Proof:
Insertion of the perturbation ansatz into (6), (7) and inspection of equal powers in λ yields immediately equations (18).

Proposition IV:
Insertion of (18) into (15) gives for the j-th crack:

$$K_I = \sigma\,(\pi a_j)^{1/2} \cdot$$

$$\cdot \left\{ \sum_{n=1}^{M} \lambda^{2(n-1)} \sum_{p=o}^{n-1} \frac{(p+1)^2}{2^{2p-1}} \binom{2p+1}{p} s_j^{2p} \left[K^{\cdot(2n-2p-2)}_{2p,j} + M^{\cdot(2n-2p-2)}_{2p,j} + \frac{p_o^j}{2\sigma} \Delta^o_p \Delta^1_n \right] + \right.$$

$$\left. + \sum_{n=1}^{M} \lambda^{2n-1} \sum_{p=o}^{n-1} \frac{(p+1)(2p+3)}{2^{2p}} \binom{2p+1}{p} s_j^{2p+1} \left(K^{\cdot(2n-2p-2)}_{2p+1,j} + M^{\cdot(2n-2p-2)}_{2p+1,j} \right) \right\} \tag{19}$$

A computer program can now be prepared acc.to (18). It automatically computes the coefficients of all holes and the stress intensity factors of all cracks for various geometric and loading parameters.

RESULTS

The following configurations have been studied in detail: i) one and two pressurized 2μm-hole(s) resp., circling at a fixed distance d around the centre of a crack (Figures 4-8); ii) one and two pressurized 2μm-hole(s) resp., located at various distances x above a crack tip (Figures 9,10). In all situations the critical load σ_c of the Al_2O_3-matrix was applied at infinity (1,2). The internal pressure was assumed to be

$$p_o = 2{,}5 \text{ GPa} \tag{20}$$

which is a typical value for a transformed 2μm ZrO_2 particle as was shown by Müller and Müller in (8) and in (5,6).

One observes (Figures 4-8) that at a small angle of inclination ß the toughening ratio α is greater than one, so that the crack becomes unstable. With increasing ß the toughening ratio of crack tip A decreases distinctively, assumes a minimum and meets finally at ß = 90° the α of crack tip B (symmetrical point); the crack is stabilized. The existence of stability and instability zones was already observed in (5,6) where a compounding method was used: Figure 11 shows a normalized distribution of forces along a chord in a sphere of action around a transformed ZrO_2 particle. If a crack enters from the left, it becomes unstable under the influence of a traction zone. Thus it advances towards the interior of the sphere, where it will be stabilized by pressure. And in fact, the smaller the chord distance x, the greater the pressure becomes and the more effective is the stabilization. This is shown quantitatively in Figures 9, 10 (crack tip A).

DISCUSSION AND OUTLOOK

The truncation of the series (17) and the resulting closure of the iteration scheme (18) leads inevitably to numerical errors in (19). As in (1) it was observed that for

$$\lambda = \frac{a}{d} \rightarrow 1 \tag{21}$$

the convergence is rather weak so that even very high order terms have to be taken into account in order to guarantee an accurracy in α of at least 1%. It shall be noted that in Figures 9, 10 for $x \varepsilon [1,5;3]$ μm M = 60 was used whereas in all other cases it was sufficient to choose $12 \leq M \leq 24$.

The calculated values of α show that it is possible to achieve an increase of 50% and more if the particle is placed suitably*. But if not, the toughness decreases. Therefore one should consider random distributions of pressurized holes around a crack and calculate a mean value of α. This will be done in a subsequent paper.

SYMBOLS USED

a, a_j = crack length (m)

$D_{n,j}$, $F_{n,j}$, $M_{n,j}$, $K_{n,j}$ = dimensionless expansion coefficients

K_I, K_{II}, K_{IC} = stress intensity factors ($Nm^{-3/2}$)

p_o, p_o^j = pressure (Pa)

t_{ij} = stress component (Pa)

X, Y, X_j, Y_j = cartesian coordinates (m)

z, z_j = dimensionless complex variable

α = dimensionless toughening ratio

σ_c = critical stress (Pa)

Φ_j, Φ, Ψ_j, Ψ = dimensionless Goursat functions

χ = Airy stress function (N)

* In agreement with the experiments where K_{IC} ($Al_2O_3 + ZrO_2$) = 6 – 10 MN $m^{-3/2}$

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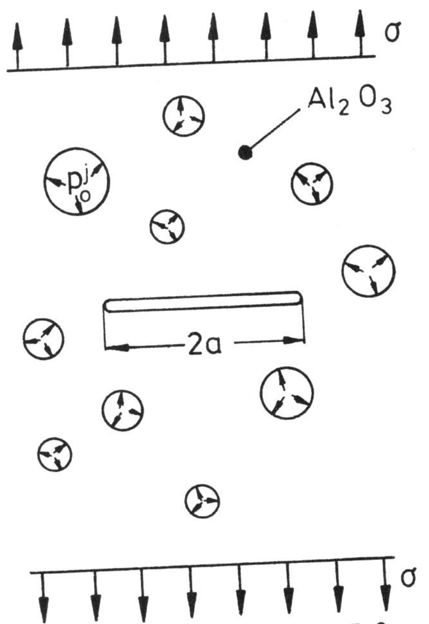

Figure 1 Idealized Al_2O_3-ZrO_2 texture

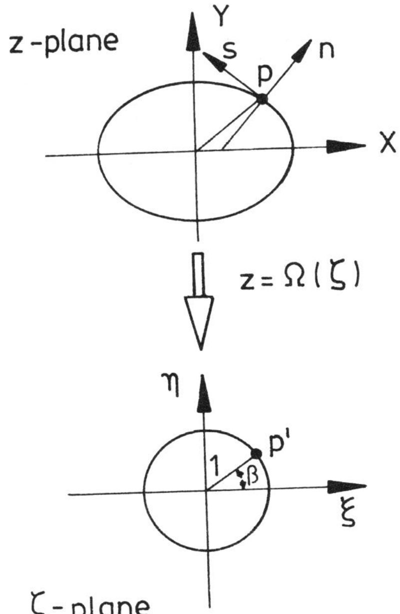

Figure 3 Conformal mapping: ellipse → unit circle

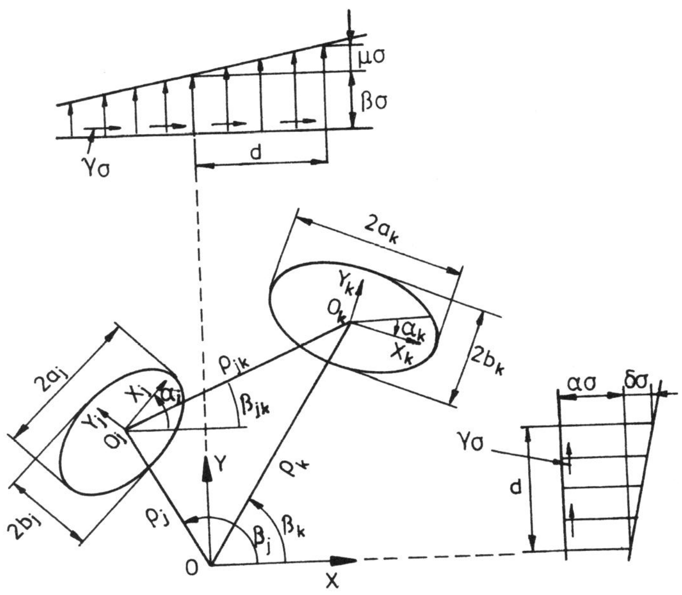

Figure 2 Elliptical holes in infinite plane under stress at infinity

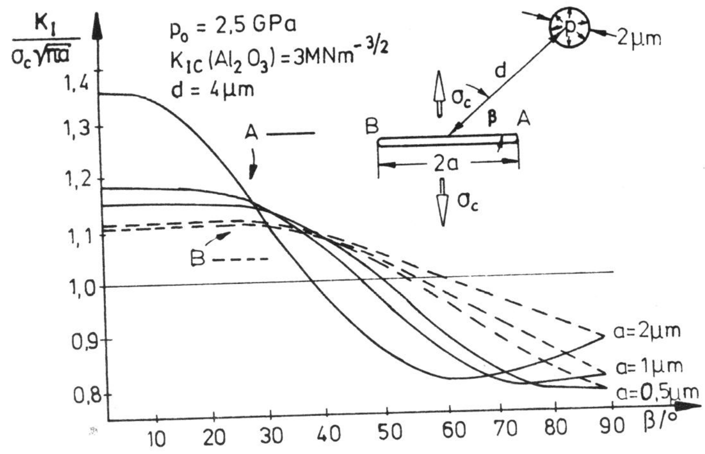

Figure 4 Pressurized hole at a fixed distance d = 4 μm from a crack

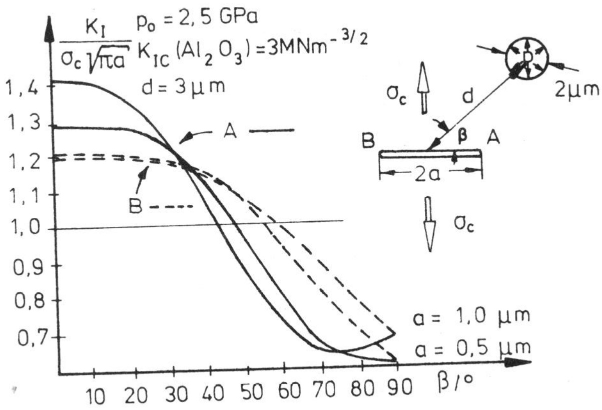

Figure 5 Pressurized hole at a fixed distance d = 3 μm from a crack

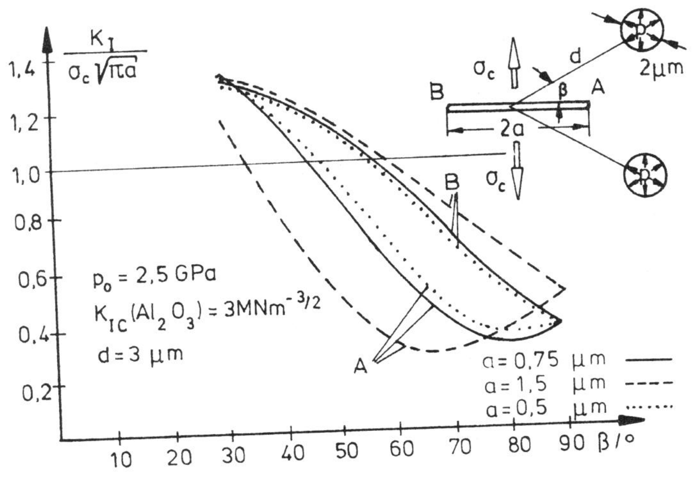

Figure 6 Pressurized holes at fixed distances d = 3 μm from a crack

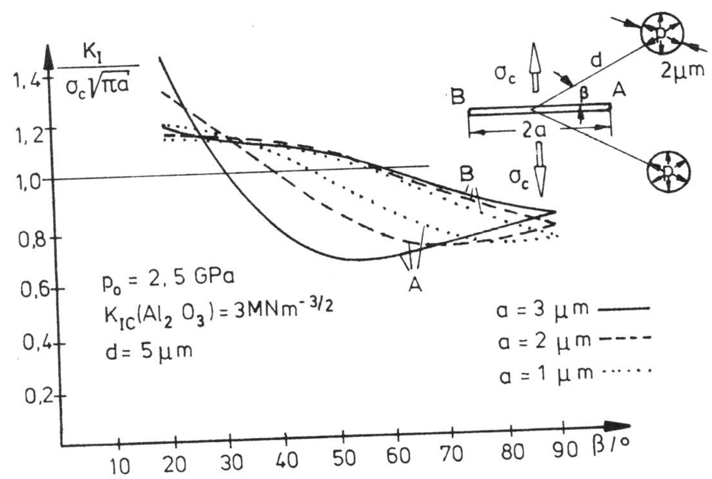

Figure 7 Pressurized holes at fixed distances d = 5 μm from a crack

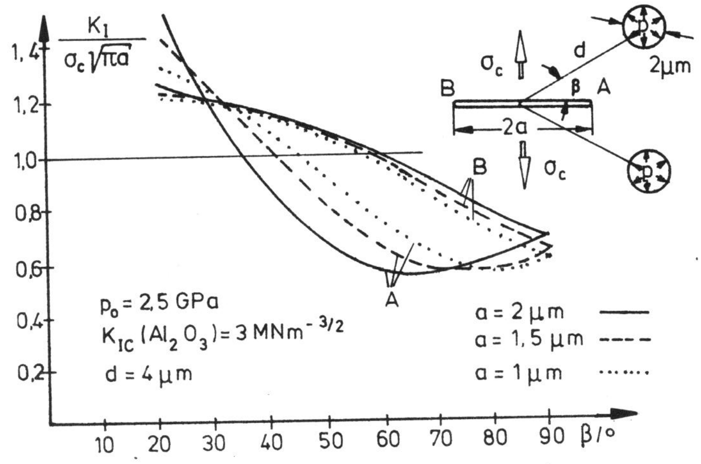

Figure 8 Pressurized holes at fixed distances d = 4 μm from a crack

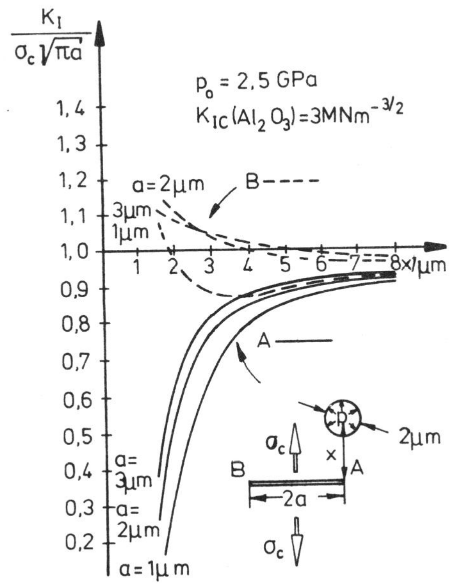

Figure 9 Pressurized hole above a crack tip

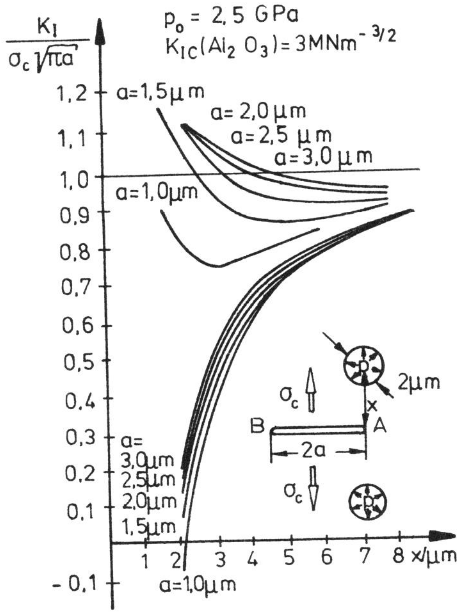

Figure 10 Pressurized holes above a crack tip

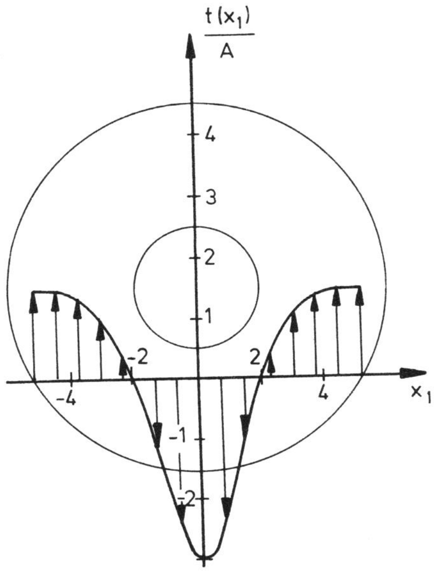

Figure 11 Forces along a chord in a sphere of action