THE USE OF HENCKY'S EQUATIONS FOR ESTIMATING MULTIAXIAL ELASTIC-PLASTIC NOTCH STRESSES AND STRAINS

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A procedure for estimating notch stresses and strains of structures under proportional loading is presented. The procedure is based on a simplified handling of kinematic hardening, known local strain approximation formulas, Hencky's flow rule, and boundary conditions of the notch element.

Application is illustrated by the example of a notched round bar under combined loading.

1. INTRODUCTION

In the last two decades a concept for crack initiation life predictions has gained importance which is based on the assessment of the stress strain path at the mostly stressed volume element of the structure under consideration (see e.g. Dowling et al (1), Landgraf et al (2), Nowack et al (3) and Heuler (4)). This concept - called local strain approach - distinguishes itself by requiring only a small amount of experimental data, the cyclic stress-strain curve and the strain versus life curve both obtained from smooth specimen tests.

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The fatigue life analysis is divided in two principal steps. First, the local notch stress and strain histories must be predicted, and, second, the life resulting from this stress-strain versus time history must be estimated. The main concern of this paper is the prediction of the local stress-strain path for multiaxially stressed notches under proportional loading.

Sophisticated elastic-plastic analysis techniques such as Finite Elements could in theory be employed for this task. However, this would be extremely expensive and time consuming for load histories containing more than a few load reversals.

In chapter 2 it is shown that - like in cases of uniaxial notch stresses - cyclic loading can be reduced to a sequence of monotonic loadings by taking into account the cyclic material behaviour described by the terms Masing- and memory behaviour. Thus, stress-strain analyses of notched components can be concentrated on monotonic loading.

For this problem an approximate procedure is proposed in chapter 3 delivering the complete state of elastic-plastic notch stresses and strains. The solution occurs in two steps. First, a relationship between applied load and equivalent notch stresses and strains is established, and, second, the principal notch stresses and strains are correlated to the equivalent quantities by applying theory of plasticity.

In chapter 4 special attention is given to the calculation of the principal quantities where Hencky's finite law is compared to the incremental flow rule of Prandtl-Reuss.

The method presented has been checked by experimental and numerical investigations (5,6) (structures investigated see Fig. 1). Application is illustrated in chapter 5 by the example of a shallow sharp notch under combined tensile and torsional loading.

A general discussion of multiaxial stressed notches including fatigue life predictions closes the paper.

2. CYCLIC BEHAVIOUR OF STRUCTURES UNDER PROPORTIONAL LOADING

Experimental investigations reveal (e.g. Wetzel (7), Dowling (8)) that the cyclic deformation behaviour of engineering materials can be described by a model consisting of the following elements:

1. Masing-behaviour[1)]
   The shape of hysteresis loop curves corresponds to the doubled cyclic stress-strain curve, Fig. 2.
2. Memory behaviour
   For an irregular load history the stress-strain response exhibits memory effects which can be characterized by (see e.g. Clormann and Seeger (10)):

   M1: Memory 1
   After forming a closed hysteresis loop the starting point of which has been on the cyclic stress-strain curve (Fig. 3, 1-2-1) the stress-strain path follows the cyclic stress-strain curve.

   M2: Memory 2
   After forming a closed hysteresis loop the starting point of which has been on a superior hysteresis loop (4-5-4) the stress-strain path follows the original hysteresis loop (3-4-6).

   M3: Memory 3
   A hysteresis loop started on the cyclic stress-strain curve ends in the opposite quadrant (between 4-6) when the stress or strain amount of its starting point is reached. Subsequent loading follows the cyclic stress-strain curve.

Assuming that a notched structure is only uniaxially stressed, or at least that multiaxial effects can be neglected, the similar cyclic behaviour of each material element of the structure leads to a structural behaviour that again can be described by Masing- and memory-behaviour. Hence, the complete notch strain history can be predicted by the load - notch strain relationship for monotonic loading. Let this relationship between applied load S and notch strain ε be expressed as

[1)] According to Masing (9) who described this phenomenon first.

$$\varepsilon = f(S). \tag{1}$$

Then, the hysteresis loops are described by

$$\frac{\varepsilon-\varepsilon_r}{2} = f\left(\frac{S-S_r}{2}\right) \quad \text{for increasing load} \tag{2a}$$

$$\frac{\varepsilon_r-\varepsilon}{2} = f\left(\frac{S-S_r}{2}\right) \quad \text{for decreasing load} \tag{2b}$$

To investigate whether this is valid for multiaxial situations, too, the shallow sharp notch, Fig. 1, under combined tensile and torsional loading was analyzed by Finite Element method. A bilinear stress-strain curve

$$\varepsilon = \begin{cases} \sigma/E & \sigma<\sigma_Y \\ \sigma_Y/E + (\sigma-\sigma_Y)/E_T & \sigma>\sigma_Y \end{cases} \tag{3}$$

with $E_T=0.05 \cdot E$ and a ratio $\tau_n/\sigma_n=2.5$ of the net section stresses was assumed.

The maximum principal strain history, Fig. 4, was determined in two different ways. First, the complete load sequence was calculated by FE-method using the kinematic hardening model (see symbols). Second, only a FE-analysis for monotonic loading was carried out resulting in a relationship like Eq. (1). Then, the load - strain path was constructed under consideration of Masing- and memory-behaviour from load reversal to load reversal point according to Table 1 leading to nearly the same results as with the cyclic FE-solution.

Fig. 5 reveals that the element at the notch root experiences relatively small changes of the stress ratio σ_2/σ_1 explaining that Masing- and memory-behaviour are fulfilled so well. It seems that the cyclic behaviour of multiaxially stressed notches under proportional loading is the same as in uniaxial situations, i.e. cyclic loading can be reduced to a sequence of monotonic loadings.

In the following a method for estimating multiaxial elastic-plastic notch stresses and strains is presented where the considerations can be concentrated on monotonic loading according to the above findings.

3. AN APPROXIMATE PROCEDURE FOR ESTIMATING MULTIAXIAL NOTCH STRESSES AND STRAINS

The structure of the approximate procedure for estimating notch stresses and strains at a traction free surface (for an example of triaxially stressed notches see reference (11)) is depicted in Fig. 6.

The following input is required:

- o Elastic material constants and uniaxial stress-strain curve $\sigma=g(\varepsilon)$.
- o Elastic stress state at the notch, e.g. described by the two principal stresses σ_{e1} and σ_{e2} and the principal stress direction α_e (subscript e denotes elastic quantities).
- o Plastic limit load level S_p for elastic perfectly-plastic material.

The approximate solution occurs in two steps.

First, a relationship between applied load and equivalent notch stresses and strains is established[2)]. According to a proposal of Neuber (12,13) the known approximation formulas derived for uniaxially stressed notches (Dietmann (14), Neuber (15), Hardrath and Ohman (16), Saal (17), Seeger and Beste (18), Glinka (19), Kühnapfel (20)) are extended to multiaxial stress states by replacing the uniaxial notch stresses σ and strains ε involved as well as the stress concentration factor K_t by the corresponding equivalent quantities σ_q, ε_q and K_{tq}. For the definition of the equivalent quantities von Mises flow criterion is chosen.

In the second step the equivalent quantities obtained from the first step are correlated to the principal stresses and strains at the notch root. This correlation is established by a flow rule - either the incremental Prandtl-Reuss equations or Hencky's finite law - describing the plastic deformations of a multiaxially stressed volume element.

[2)] For a more detailed description of the load - equivalent strain relationship see reference (21) and (22).

Comparing the number of equations with the number of unknowns two boundary conditions of the notch element have to be known for solving the set of equations. Numerical and experimental investigations of notched round bars reveal (5,23) that the assumptions

o fixed principal stress direction α,

o and constant ratio $\varepsilon_2/\varepsilon_1$ of the surface strains[3])

describe the actual behaviour at the notch root with sufficient accuracy.

4. FINITE AND INCREMENTAL FORMULATION OF THE FLOW RULE

Starting point for the formulation of the flow rule represents the rule of normality postulating that the plastic strain increment $d\varepsilon_i{}^p$ (because of the boundary condition, fixed principal stress directions, principal quantities, i=1,2,3, can be considered) is perpendicular to the yield surface (e.g. Hill (24)). Assuming von Mises yield criterion the rule of normality yields the Prandtl-Reuss equations relating the plastic strain increments $d\varepsilon_i{}^p$ to the deviatoric stresses $\sigma_i{}'$.

$$d\varepsilon_i{}^p = \frac{3}{2}\frac{d\varepsilon_q{}^p}{\sigma_q} \cdot \sigma_i' \;, \quad i = 1,2,3 \tag{4}$$

Written in total strains and taking into account that $\sigma_3=0$ (traction free surface) leads to an incremental stress-strain relationship:

$$d\varepsilon_1 = \frac{1}{E}(d\sigma_1 - \nu \cdot d\sigma_2) + \frac{3}{2}\sigma_1' \cdot \frac{d\varepsilon_q{}^p}{\sigma_q} \tag{5}$$

$$d\varepsilon_2 = \frac{1}{E}(d\sigma_2 - \nu \cdot d\sigma_1) + \frac{3}{2}\sigma_2' \cdot \frac{d\varepsilon_q{}^p}{\sigma_q} \tag{6}$$

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[3]) Note that strain ε_2 and ε_3 are not ordered according to size. ε_1 and ε_2 denote surface strains and ε_3 strain normal to notch surface.

$$d\varepsilon_3 = -\frac{\nu}{E}(d\sigma_3 + d\sigma_2) + \frac{3}{2}\sigma_3' \cdot \frac{d\varepsilon_q{}^p}{\sigma_q} \quad (7)$$

with the deviatoric stresses

$$\sigma_1' = \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 \quad (8)$$

$$\sigma_2' = \frac{2}{3}\sigma_2 - \frac{1}{3}\sigma_1 \quad (9)$$

$$\sigma_3' = -\frac{1}{3}(\sigma_1 + \sigma_2) \quad (10)$$

Including von Mises yield criterion (in an incremental formulation)

$$\sigma_q \cdot d\sigma_q = \frac{3}{2} \cdot (\sigma_1' \cdot d\sigma_1 + \sigma_2' \cdot d\sigma_2) \quad (11)$$

there are four equations (Eq (5) to (7) and (11)) for the five unknown quantities $d\varepsilon_1$, $d\varepsilon_2$, $d\varepsilon_3$, $d\sigma_1$ and $d\sigma_2$. Incorporating the boundary condition

$$\varepsilon_2/\varepsilon_1 = \phi = \text{const.} \quad (12)$$

the set of equations can be solved. Note that the boundary condition, fixed principal stress direction, has already been employed in the formulation of the flow rule.

The steps required for calculating the principal stresses and strains for a given equivalent strain are listed in the appendix. Due to the incremental formulation of the Prandtl-Reuss equations a relatively complex solution strategy is necessary. For an approximate procedure easy to handle a formulation in total strains would be desirable.

A close integration of Eq (4) is only possible if the ratio between the deviatoric stresses remains constant during loading[4]. For this special case, the flow rule of Prandtl-Reuss reduces to Hencky's equations (also denoted as deformation theory) relating the plastic strains to the deviatoric stresses

[4] The integration condition, proportional principal stresses, given by Nadaj (25) represents an unnecessary restriction for triaxial stress states.

$$\varepsilon_i{}^p = \frac{3}{2}\frac{\varepsilon_q{}^p}{\sigma_q} \cdot \sigma_i' \ , \quad i = 1,2,3 \tag{13}$$

Notating this equation in total strains a generalized formulation of Hooke's law is obtained ($\sigma_3=0$ is incorporated in the following equations).

$$\varepsilon_1 = \frac{\varepsilon_q}{\sigma_q}(\sigma_1-\nu'\cdot\sigma_2) \ , \tag{14}$$

$$\varepsilon_2 = \frac{\varepsilon_q}{\sigma_q}(\sigma_2-\nu'\cdot\sigma_1) \ , \tag{15}$$

$$\varepsilon_3 = -\nu'\frac{\varepsilon_q}{\sigma_q}(\sigma_1+\sigma_2) \ , \tag{16}$$

$$\nu' = \frac{1}{2}-(\frac{1}{2}-\nu)\frac{\sigma_q}{E\,\varepsilon_q} \ , \tag{17}$$

Though the deviatoric stresses do not behave proportional for the problems considered, Hencky's equations are employed as approximate solution for correlating principal stresses and strains to equivalent quantities. Incorporating the von Mises yield criterion and the boundary condition, $\varepsilon_2/\varepsilon_1$=const., yields

$$a = \frac{\sigma_2}{\sigma_1} = \frac{\varepsilon_2/\varepsilon_1+\nu'}{1+\nu'\varepsilon_2/\varepsilon_1} \ , \tag{18}$$

$$\frac{\varepsilon_3}{\varepsilon_1} = -\nu'\frac{1+a}{1-\nu' a} \ , \tag{19}$$

$$\sigma_1 = \frac{1}{\sqrt{1-a+a^2}}\sigma_q \ , \tag{20}$$

$$\varepsilon_1 = \frac{1-\nu' a}{\sqrt{1-a+a^2}}\varepsilon_q \ . \tag{21}$$

For comparison of Hencky's equations with the incremental flow rule of Prandtl-Reuss a material element under constant $\varepsilon_2/\varepsilon_1$-straining is considered being typical for the constraint prevailing at notches. The strain ratio $\phi=\varepsilon_2/\varepsilon_1$ is chosen in such a way that a maximum variation of the deviatoric stress ratio σ_2'/σ_1' occurs.

For estimating this value ϕ^* Hencky's equations are used.

With Eqs. (7), (8) and (18) the σ_2'/σ_1'-variation from elastic loading ($\nu'=\nu=0.3$) to large plastic strains ($\nu'=0.5$) results to

$$\frac{\sigma_2'}{\sigma_1'}(\nu'=0.5) - \frac{\sigma_2'}{\sigma_1'}(\nu'=\nu) = \frac{(1-2\cdot\nu)\cdot(1-\phi^2)}{2-\nu - \phi\cdot(1-2\cdot\nu)} \tag{22}$$

and becomes a maximum for

$$\phi^* = [2 - \nu-\sqrt{3\cdot(1-\nu^2)}]/(1-2\nu) = 0.12 \tag{23}$$

For this strain ratio ϕ^* the σ_2'/σ_1'-variation amounts to 0.24.

Principal stresses and strains calculated for a given equivalent strain by Hencky's and Prandtl-Reuss' equations, respectively, are depicted in Fig. 7 (elastic perfectly-plastic material). The variable Poisson's ratio ν', defined by Eq (17), is taken as loading measurement. $\nu'=\nu=0.3$ represents elastic behaviour and $\nu'=0.5$ large plastic deformations. Though there is a relatively large variation of the deviatoric stress ratio σ_2'/σ_1' deformation theory delivers nearly the same results as the exact solution.

Further investigations reveal (23) that even for small rotations of the principal stress directions Hencky's equations can successfully be applied. It seems that the capability of the deformation theory is linked with

- o a steady increase of the plastic deformations,
- o and monotonic behaviour of the stresses and strains.

Summarizing can be stated that for the boundary conditions investigated it is sufficient to use Hencky's equations instead of the more complex equations of Prandtl-Reuss.

5. EXAMPLE OF A NOTCHED ROUND BAR UNDER COMBINED LOADING

Application of the proposed procedure is illustrated by the example of the shallow sharp notch under combined tensile and torsional loading already discussed in chapter 2. The prediction of the notch stresses and strains follows a solution scheme given in (23) for routine application.

1. Definition of the material stress-strain curve $\sigma=g(\varepsilon)$ and selection of von Mises yield criterion:

The material stress-strain law be a bilinear curve according to Eq (3).

2. Description of the elastic stress state at the notch:

Poisson's ratio amounts to $\nu=0.30$. Elastic analysis for tensile loading results to

$K_{t\sigma} = \sigma_{ez}/\sigma_n = 3.89, \quad \bar{a}_e = \sigma_{e\theta}/\sigma_{ez} = 0.27$

and for torsional loading

$K_{t\tau} = \tau_{ez\theta}/\tau_n = 2.19$

where σ_n and τ_n represent net section stresses for tensile and torsional loading, respectively.

The elastic principal stresses and stress direction are calculated by

$$\sigma_{e1} = \left\{1+\bar{a}_e+\sqrt{(1-\bar{a}_e)^2+4\left(\frac{K_{t\tau}\cdot\tau_n}{K_{t\sigma}\cdot\sigma_n}\right)^2}\right\}\cdot\frac{K_{t\sigma}\cdot\sigma_n}{2} = 4.44\cdot\sigma_n \quad (24)$$

$$a_e = \frac{\sigma_{e2}}{\sigma_{e1}} = \frac{1+\bar{a}_e-\sqrt{(1-\bar{a}_e)^2+4\left(\frac{K_{t\tau}\cdot\tau_n}{K_{t\sigma}\cdot\sigma_n}\right)^2}}{1+\bar{a}_e+\sqrt{(1-\bar{a}_e)^2+4\left(\frac{K_{t\tau}\cdot\tau_n}{K_{t\sigma}\,\sigma_n}\right)^2}} = -0.39 \quad (25)$$

$$\tan 2\alpha_e = \frac{2\cdot K_{t\tau}\cdot\tau_n}{(1-\bar{a}_e)\cdot K_{t\sigma}\cdot\sigma_n} = 37.8^{\circ} \quad \text{(radian: 0.66)} \quad (26)$$

3. Estimation of the plastic limit load level S_p for elastic perfectly-plastic material based on elementary equilibrium considerations:

For combined tensile and torsional loading of a round bar elementary equilibrium considerations, assuming constant distribution of normal and shear stresses, lead to (23)

$$\frac{S_p}{\sigma_Y} = \frac{4}{3}\sqrt{\frac{3 + (\sigma_n/\tau_n)^2}{3 + 16/9\cdot(\sigma_n/\tau_n)^2}} \qquad (27)$$

Substituting τ_n/σ_n in Eq (28) results in $S_p/\sigma_Y = 1.31$.

4. Definition of nominal stress and calculation of equivalent stress concentration factor K_{tq}:

The equivalent stress at the notch obtained from common structural analysis is chosen as nominal stress S

$$S = \sqrt{\sigma_n^2 + 3\tau_n^2} \qquad (28)$$

The equivalent stress concentration factor, defined by the ratio of the elastic equivalent notch stress σ_{eq} to the nominal stress S, is calculated with von Mises yield criterion.

$$K_{tq} = \sigma_{e1}/S\cdot\sqrt{1 - a_e + a_e^2} = 2.27 \qquad (29)$$

5. Calculation of equivalent notch stress and strain by Neuber's rule:

With the extension to multiaxial stress states Neuber's rule reads

$$E\cdot\sigma_q\cdot\varepsilon_q = (K_{tq}\cdot S)^2\cdot\frac{E\cdot e^*}{S^*} \qquad (30)$$

$$\text{with } S^* = \frac{S}{S_p/\sigma_Y}, \qquad e^* = g^{-1}(S^*) \qquad (31)$$

where the term $E\cdot e^*/S^*$ takes nonlinear net section behaviour into account (26). For a given load level S the right-hand side of Eq (30) is known (with Eq (31)) and the elastic plastic notch stresses σ_q and strains ε_q can be calculated under consideration of the material's stress-strain curve, Eq (3). Note, that the term $E\cdot e^*/S^*$ equals 1 for $S<S^*=0.76\sigma_Y$.

6. Definition of notch element's boundary conditions:

I : fixed principal stress direction

$$\alpha = \alpha_e = 37.8^o = \text{const.} \qquad (32)$$

II: constant surface strain ratio

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{e2}}{\varepsilon_{e1}} = \frac{a_e - \nu}{1-\nu \cdot a_e} = \text{const.} \qquad (33)$$

7. Calculation of the principal stresses and strains according to Hencky's equations, Eq. (17) to (22).

The results of the approximate solution are compared in Fig. 8 to FE-calculations. The left-hand diagram shows a plot of the nominal stress S versus the maximum strain ε_1. Additionally in this diagram the corresponding stress σ_1 is plotted versus ε_1. Logarithmic scales are used because linear scales overemphasize high load levels. The range of application in general is limited by the plastic limit load S_p.

The complete information about the stress and strain state is given by the right-hand diagram showing the strain and stress ratios $\varepsilon_2/\varepsilon_1$, $\varepsilon_3/\varepsilon_1$, σ_2/σ_1 and the angle of the principal stress direction α (radians) in relation to the nominal stress.

Accuracy studies reveal that the differences between FE-analysis and the approximate solution are mainly caused by the inaccuracy of Neuber's rule. Note that the assumptions, fixed principal stress direction and constant surface strain ratio $\varepsilon_2/\varepsilon_1$ describe the actual behaviour at the notch with sufficient accuracy.

6. GENERAL DISCUSSION

6.1 Simplified handling of kinematic hardening

Investigations on notched round bars under proportional loading have shown that the complex relations of kinematic hardening can be reduced to a model which consists of Masing- and memory-behaviour, and Hencky's finite formulation of the flow rule.

This is founded on the fact that the structures investigated experience only relatively small stress redistributions and small changes of the deviatoric stress ratios.

To clarify the cyclic behaviour of more complex structures it is sufficient to run a single FE-analysis for monotonic loading. Determine the deviatoric stresses for elastic loading and for the maximum load. If the deviatoric stress ratios do not change considerably and the plastic deformations increase steadily it is expected that Masing- and memory-behaviour is valid as well as Hencky's flow rule can be applied.

The above considerations refer to the assumption of stabilized material stress-strain behaviour which is sufficient for most fatigue life predictions. Cyclic softening or hardening can be included, in principle, by adopting a variable stress-strain curve in the approximate procedure where the variation can depend e.g. on maximum strain reached and numbers of load cycles applied. However, the problem then arises how to assess the stress-strain curve with unclosed hysteresis loops.

It has to be pointed out that for nonproportional loading the simplified handling of kinematic hardening does not hold anymore, see chapter 6.5.

6.2 Load - equivalent strain relationship

Extensive studies concerning the accuracy of the method presented show (23) that the major deviations are based on the chosen approximation formula, here Neuber's rule.

As Neuber's rule describes a nontangential transition from the elastic into the elastic-plastic regime it tends to overestimate notch strains (19,27). This is especially true for low hardening materials. Therefore, a formula proposed by Seeger (18) (with tangential transition) should be employed for low hardening materials, and if highly accurate notch strains are wanted near yield initiation.

$$\varepsilon_q = \frac{\sigma_q}{E} \cdot \left[\left(\frac{\sigma_{e,q}}{\sigma_q} \right)^2 \cdot \frac{2}{u^2} \cdot \ln \frac{1}{\cos u} + 1 - \frac{\sigma_{e,q}}{\sigma_q} \right] \cdot \frac{E \cdot e^*}{S^*} \qquad (34)$$

with $u = \frac{\pi}{2} \cdot \frac{\sigma_{e,q}/\sigma_q - 1}{K_p - 1}$, $K_p = K_{tq} \cdot S_p/\sigma_Y$, $\sigma_{e,q} = K_{tq} \cdot S$

6.3 Boundary conditions of the notch element

If a traction free surface is considered two boundary conditions of the notch element must be known for the evaluation of the notch principal stresses and strains.

Concerning the first condition, it is assumed that possible changes of the principal stress directions can be neglected. For a lot of structures and loading situations principal axes are fixed because of symmetry conditions. Experimental and numerical investigations of combined loading show that even in these cases there are only small changes in the principal stress directions.

The second condition is obtained by taking into account the geometrical constraint at notches. For example, circumferential strains of the notch are controlled by the circumferential strains at the gross area. Based on the investigations of the notched round bars, Fig. 1, it is assumed that the ratio $\varepsilon_2/\varepsilon_1$ of the surface strains remains constant during loading. The limiting case, pure torsional loading ($\varepsilon_2/\varepsilon_1 = -1$), is satisfied exactly.

The constraint assumption should not be applied to structures with extremly shallow and mild notches which resemble smooth specimens. For these cases an assumption concerning a stress ratio should be applied (23).

6.4 Fatigue life predictions

The previous considerations have been focused mainly on the estimation of the local stress-strain histories. Fatigue life predictions furthermore require the definition of a multiaxial fatigue parameter relating multiaxial damage behaviour to the uniaxial behaviour observed on smooth specimen, and employing a damage accumulation theory for variable amplitude loading.

As, at the moment, no consensus exists neither on the most appropriate multiaxial fatigue parameter nor on a damage accumulation theory a simplified analysis of multiaxially stressed notches is proposed in (28) with

- o maximum shear strain as multiaxial fatigue parameter (tending to conservative estimates as maximum shear strain is the most severe one of the parameters known from literature),
- o and employment of linear damage accumulation with damage sums usually less than one for variable amplitude loading (for definition of the allowable damage sum the experience with the Relative Miner rule gathered in crack initiation life predictions of smooth specimens under variable amplitude loading should be incorporated).

6.5 Nonproportional loading

The method presented is restricted to proportional loading.

For external loads acting at the same frequency but with a constant phase difference the following procedure may lead to reasonable estimates of the stress and strain amplitudes:

1. Calculate the notch stresses and strains under the assumption of proportional loading.
2. Take the influence of the phase difference into account by defining correction functions depending on phase difference, load ratio and load amplitudes.

For arbitrarily nonproportional loading, e.g. external loads acting with different frequencies, further developments are necessary. In this case Masing- and memory-behaviour cannot be assumed anymore. Extension of the approximate solution may be possible by replacing

- o approximate formula by a differential formulation of Neuber's rule or Glinka's strain energy density approach (19),
- o Hencky's flow rule by the kinematic hardening model of Mroz (29),
- o and constant values of the boundary conditions by, e.g.:
  I: $\alpha = \alpha_e(t)$ II: $\varepsilon_2/\varepsilon_1 = \varepsilon_{e2}(t)/\varepsilon_{e1}(t)$
  where (t) denotes time dependency.

7. CONCLUSIONS

1. Cyclic, proportional loading of multiaxially stressed components can be reduced to a sequence of monotonic loadings by taking Masing- and memory-behaviour into account.

2. Load - equivalent notch strain relationship can be established by known approximation formulas.

3. Hencky's rule in combination with boundary conditions of the notch element proves to be an excellent tool for relating principal stresses and strains to equivalent stresses and strains.

4. The procedure proposed delivers complete information about the multiaxial elastic-plastic stress and strain state at the notch. Incorporating a multiaxial fatigue parameter it can be employed for fatigue life predictions with no need of extensive nonlinear FE-analyses.

5. The method is restricted to proportional loading. Arbitrarily nonproportional loading requires further developments.

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SYMBOLS USED

E = Young's modulus
E_T = Tangent modulus
K_{tq} = equivalent stress concentration factor
S = nominal stress
S^* = modified nominal stress
S_p = plastic limit load
a = stress ratio σ_2/σ_1
$\bar{a}$ = stress ratio σ_θ/σ_z
e^* = modified nominal strain
α = principal stress direction
ϕ = surface strain ratio $\varepsilon_2/\varepsilon_1$
ε = strain
$d\varepsilon$ = strain increment
ε^p = plastic strain
ε_q = equivalent strain
ν = Poisson's ratio
ν' = variable Poisson's ratio
σ = stress
σ' = deviatoric stress
$d\sigma$ = stress increment
σ_n, τ_n = net section stresses according to common structural analysis
σ_q = equivalent stress
σ_Y = yield stress
$\sigma = g(\varepsilon)$ = stress-strain relationship

Subscripts

1,2,3 principal stresses,strains
e elastic quantities
q equivalent quantities
r,θ,z cylindrical coordinates

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APPENDIX

Calculation of the principal quantities with Prandtl-Reuss equations

The steps required for calculating principal stresses and strains for given equivalent stresses and strains are listed below.

1. Incrementation of ε_q. $\Delta\varepsilon_q = (\varepsilon_q - \varepsilon_Y)$/number of increments.

2. Stresses and strains at yield initiation represent the initial values.

3. Calculation of the plastic equivalent strain increment $\Delta\varepsilon_q^{\,p}$

$$\varepsilon_{q,j+1} = \varepsilon_{q,j} + \Delta\varepsilon_q, \quad \sigma_{q,j+1} = g(\varepsilon_{q,j+1}) \tag{A1}$$

$$\Delta\varepsilon_q^{\,p} = \varepsilon_{q,j+1} - \varepsilon_{q,j} - (\sigma_{q,j+1} - \sigma_{q,j}) \tag{A2}$$

4. Calculation of the stress and strain increments

$$\Delta\sigma_1 = \frac{\frac{2}{3}\cdot(1+\nu\phi)\cdot\sigma_q\cdot\Delta\sigma_q - \frac{3}{2}\cdot(\phi\sigma_1' - \sigma_2')\cdot\sigma_2'\cdot\frac{E\cdot\Delta\varepsilon_q^{\,p}}{\sigma_q}}{(1+\nu\phi)\cdot\sigma_1' + (\nu+\phi)\cdot\sigma_2'} \tag{A3}$$

$$\Delta\sigma_2 = \frac{\nu + \phi}{1 + \nu\phi}\cdot\Delta\sigma_1 + \frac{3}{2}\cdot\frac{(\phi\sigma_1' - \sigma_2')\cdot E\cdot\Delta\varepsilon_q^{\,p}}{(1+\nu\phi)\cdot\sigma_q} \tag{A4}$$

$\Delta\varepsilon_1$, $\Delta\varepsilon_2$, $\Delta\varepsilon_3$ according to Eq. (5) to (7).

5. Summation yields to the stresses and strains referring to $\varepsilon_{q,j+1}$.

$$\sigma_{j+1} = \sigma_j + \Delta\sigma_j, \quad \varepsilon_{j+1} = \varepsilon_j + \Delta\varepsilon_j \tag{A5}$$

6. If the given equivalent strain is not reached yet, calculate the deviatoric stresses σ_i', Eq (8) to (10), increase j by one and go to step 3.

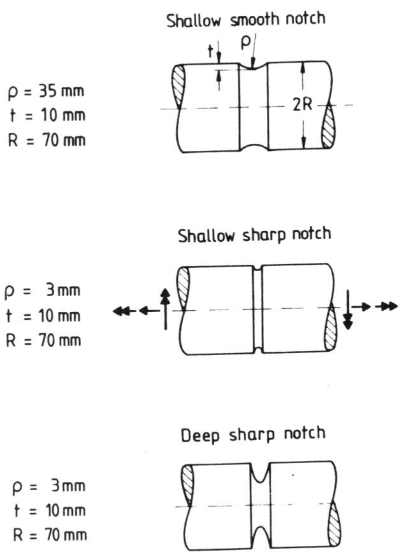

Figure 1 Structures and loading cases investigated in (5).

$S - \varepsilon_1$ path	Equation used	$S - \varepsilon_1$ - point used for (S_r, ε_r)
0 - 1	1	-
1 - 2	2b	1
2 - 2'	2a	2
2' - 3	1	-
3 - 4	2b	3
4 - 5	2a	4
5 - 6	2b	5
6 - 6'	2a	6
6' - 7	2a	4

Table 1: Construction of the load strain path in Fig. 4

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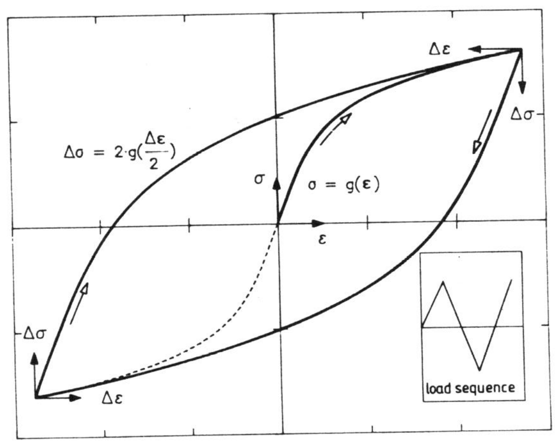

Figure 2 Masing-behaviour.

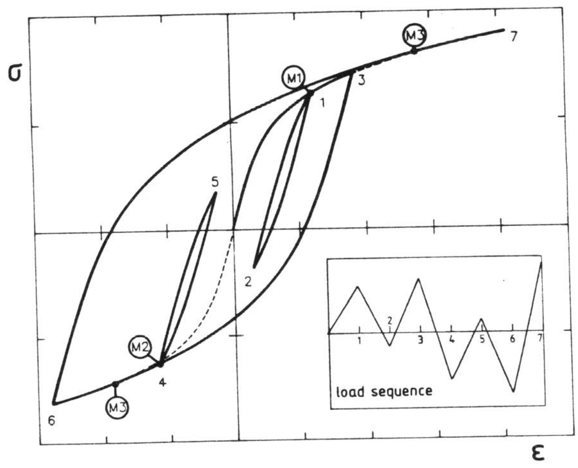

Figure 3 Memory-behaviour.

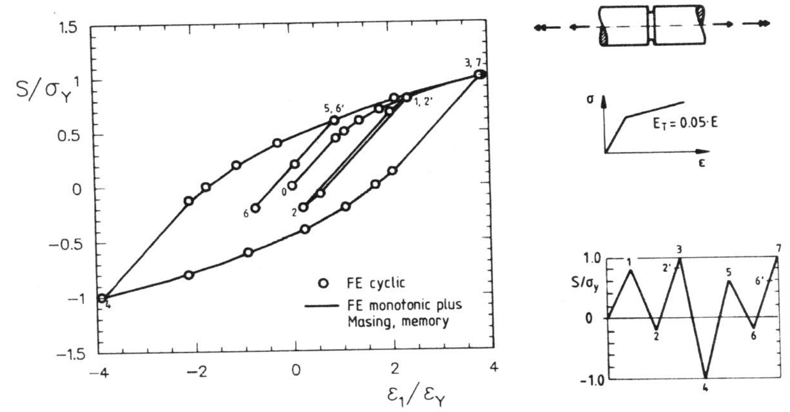

Figure 4 Cyclic behaviour of a notched component. Nominal stress versus notch maximum strain.

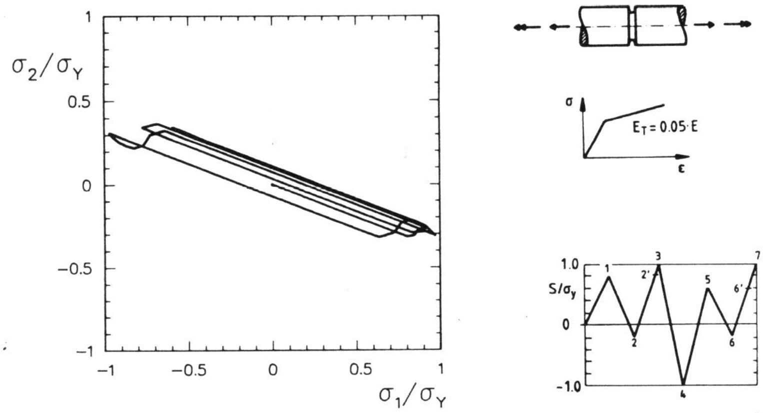

Figure 5 Cyclic behaviour of a notched component. Principal stress path.

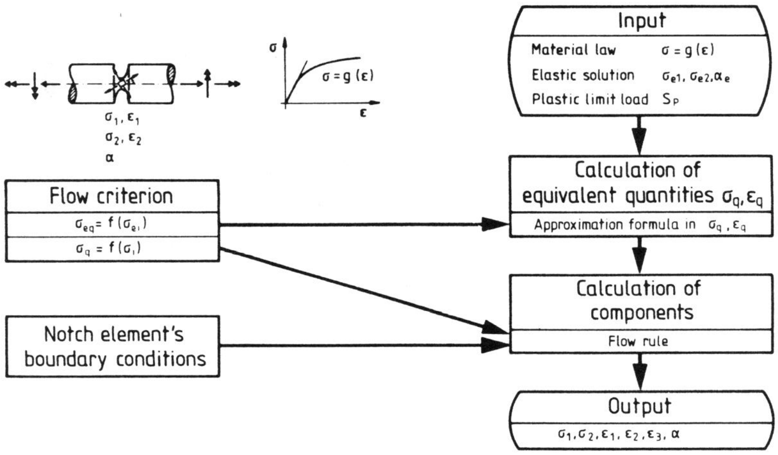

Figure 6 Structure of the approximation procedure.

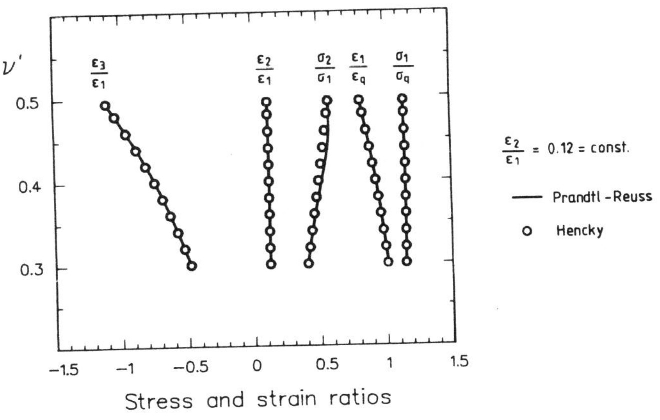

Figure 7 Comparison of Hencky's and Prandtl-Reuss' flow rule.

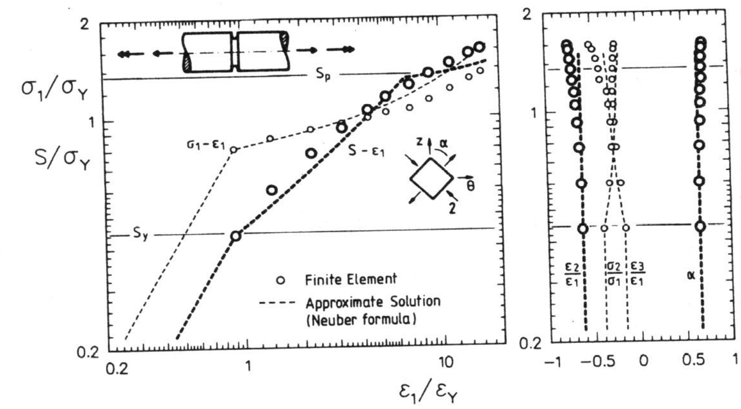

Figure 8 Notch stress and strain estimates using Neuber's rule.