UNIVERSAL WEIGHT FUNCTIONS FOR LOADINGS AND SCREENINGS OF CRACKS

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Since for any particular specimen and crack geometry the stress intensity vector must be a functional of the loading or screening which are both of vectorial character, the stress intensity vector can be found by integration over the product of the dislocation and force density with two universal tensorial weight functions. The same technique is applicable to transformation and thermal stresses. Generalizations to three-dimensional universal weight functions are possible.

INTRODUCTION

For a two-dimensional situation the fact that the crack tip feels an elastic force implies that stress and distorsion near the crack tip must be proportional to $1/\sqrt{r}$, because the driving force must be obtainable as a surface integral over the energy-momentum tensor (Eshelby (1)) which has the dimension of an energy-density. In terms of the stresses σ_{i2} acting across the crack plane ($x_2=0$, $x_1>0$) the proportionality constant between stress and $1/\sqrt{r}$, known as stress intensity factor may be expressed as

$$K_i = \lim_{x_1 \to 0} (2\pi x_1)^{1/2} \sigma_{i2}(x_1,0) \tag{1}$$

K_1, K_2, K_3 are the stress intensity factors for crack loading modes II, I and III, respectively. The stress intensity vector K_i in eqn. (1) depends on the geometry of the specimen, the loading condition and the presence of dislocations.

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Every elastic state can be described by an arrangement of loadings $f_s(x)$ and/or the presence of dislocations with Burgers vector $b_s(x)$ where $x=(x_1,x_2)$ is a two-dimensional vector. More precisely, then, the stress intensity vector must be a functional of the loading by f_s and the screening by b_s in the form

$$K_i[\text{geometry};\ f_s(x);\ b_s(x)] \tag{2}$$

Here $f_s(x)$ is the source of stress according to

$$\partial_i \sigma_{is} + f_s = 0 \tag{3}$$

and $b_s(x)$ is the source of distorsion β according to

$$b_i = \varepsilon_{jk}\partial\beta_{ij}/\partial x_K \tag{4a}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. Eqn. (4a) is the two-dimensional form of the more familiar three-dimensional version

$$\alpha_{it} = \varepsilon_{jkt}\,\partial\beta_{ij}/\partial x_K \tag{4b}$$

where α_{it} is the dislocations density and ε_{jkt} the totally antisymmetric tensor. In eqn. (4a) b_i is a Burgers vector density with dimension [length/area.]

For brevity we have suppressed surface loadings and surface dislocations, and the surface boundary conditions corresponding to eqns. (3,4), but they can always be thought of as generalizations or rather, specializations of f_s and b_s.

WEIGHT FUNCTION TENSOR FOR LOADINGS

Bueckner (2) was the first to notice that for the case of unmixed loadings and elastic isotropy the functional behaviour of (2) implies that it must be possible to write the stress intensity vector as

$$K_i = \int F_{is}(\text{geometry};a;x) f_s(x)\ d^2x \tag{5}$$

where F_{is} (geometry;a;x) is a weight function that depends on the geometry, the crack length a, and the coordinate x, but is independent of the loading $f_s(x)$ itself.

Rice (3) gave a derivation of F_{is} much simpler than Bueckner's original one (2), but in the following we present an even simpler one: The elastic driving force dW/da, with W being the elastic energy, is known (4,5) to be a quadratic form in the stress intensity vector

$$dW/da = (8\pi)^{-1}K_i(B^{-1})_{ij}K_j \tag{6}$$

where $(B^{-1})_{ij}$ is a 3x3 matrix with dimension of a compliance. It is the inverse of the prelogarithmic factor of dislocation lines.

$$W = b_i B_{ij} b_j \ln R/r_o \tag{7}$$

where R and r_o are the outer and inner cut off radii (5). It can be obtained numerically (6) for any elastic anisotropy and any orientation of the crack front – what is required is the solution of a sextic polynomial. In eqn. (7) b_i is actually the Burgers vector of a singular dislocation line with dimension [length] .

On the other hand the elastic energy of the cracked body can also be written as

$$W = \frac{1}{2}\int u_s(x) f_s(x)\ d^2x \tag{8}$$

where $u_s(x)$ is the displacement.

Consider now one and the same specimen geometry but with two different loadings (+) and (–). The total stress intensity vector, displacement and loadings are

$$K_i^+ + K_i^- \tag{9}$$

$$u_s^+ + u_s^- \tag{10}$$

$$f_s^+ + f_s^- \tag{11}$$

Equations (6) and (8) for the sum of the two loadings read

$$dW/da = (8\pi)^{-1}(K_i^+ + K_i^-)(B^{-1})_{ij}(K_j^+ + K_j^-) = \frac{1}{2}\frac{d}{da}\int [u_s^+(x) + u_s^-(x)]\,[f_s^+ + f_s^-(x)]\ d^2x \tag{12}$$

The purely quadratic terms of (12) cancel because, by definition,

$$(8\pi)^{-1}K_i^+(B^{-1})_{ij}K_j^+ = \frac{1}{2}\frac{d}{da}\int u_s^+(x) f_s^+(x)\ d^2x \tag{13}$$

$$(8\pi)^{-1}K_i^-(B^{-1})_{ij}K_j^- = \frac{1}{2}\frac{d}{da}\int u_s^-(x) f_s^-(x)\ d^2x \tag{14}$$

Since

$$(B^{-1})_{ij} = (B^{-1})_{ji} \tag{15}$$

also

$$K_i^+(B^{-1})_{ij}K_j^- = K_i^-(B^{-1})_{ij}K_j^+ \tag{16}$$

and of eqn. (12) there remains only

$$2(8\pi)^{-1}K_i^+(B^{-1})_{ij}K_j^- = \frac{1}{2}\frac{d}{da}\int[u_s^+(x)f_s^-(x) + u_s^-(x)f_s^+(x)]\ d^2x \qquad (17)$$

According to the theorem of reciprocity the two terms under the integral are equal. The differentiation d/da acts only on the displacement field, so that

$$2(8\pi)^{-1}K_i^+(B^{-1})_{ij}K_j^- = \int\frac{du_s^-(x)}{da}\ f_s^+(x)d^2x \qquad (18)$$

If eqn. (18) is written not for one reference state f^- but for three reference states loaded by

$$f_s^{(1)},\ f_s^{(2)},\ f_s^{(3)} \qquad (19)$$

with displacement and stress intensity vectors

$$u_s^{(1)},\ u_s^{(2)},\ u_s^{(3)} \qquad (20)$$

$$K_i^{(1)},\ K_i^{(2)},\ K_i^{(3)} \qquad (21)$$

one has the three equations

$$2(8\pi)^{-1}K_i^+(B^{-1})_{ij}K_j^{(z)} = \int\frac{du_s^{(z)}(x)}{da}f_s^+(x)\ d^2x \qquad z=1,2,3 \qquad (22)$$

If the three stress intensity vectors $K_i^{(1)}$, $K_i^{(2)}$, $K_i^{(3)}$ are written as a matrix,

$$K_j^{(z)} = K \qquad (23)$$

Eqn. (21) can be solved

$$K_i = \int 4\pi B_{ij}(K^{-1})_j^{(z)}\frac{du_s^{(z)}(x)}{da}f_s(x)\ d^2x \qquad (24)$$

where we have now suppressed the superscript (+) for the state of interest. Comparison with eqn. (5) shows that the weight function is

$$F_{is}(a;x) = 4\pi B_{ij}(K^{-1})_j^{(z)}\ \frac{du_s^{(z)}}{da} \qquad (25)$$

Although $F_{is}(a;x)$ was obtained from three reference states $f_s^{(1)}$, $f_s^{(2)}$, $f_s^{(3)}$, eqn. (5) implies that it is independent of the three reference states chosen. This is so because both the matrix K and the displacement u_s are linear in the loadings. Bueckner's (2) function for unmixed loading was a universal (vector) weight function; as it stands F_{is} is a universal (tensor) weight function for loadings.

WEIGHT FUNCTION TENSOR FOR SCREENINGS

In the derivation of the weight function tensor $F_{is}(a;x)$ of eqn. (25) the existence of a displacement field $u_s(x)$ was assumed. According to eqn. (8) this displacement is conjugate to the forces $f_s(x)$ with respect to the energy W. In the presence of dislocations (that colloquially are said to "screen" or "shield" the crack) the elastic energy is not only the one of eqn (8), which, as it refers to an external loading $f_s(x)$ is called the external energy, but there is also a contribution from the presence of the (two-dimensional) dislocation density $b_s(x)$. Since $b_s(x)$ is, according to eqn. (4), the source of internal stresses, that contribution is often called "internal". According to the theorem of Collonetti (7) there is, however, no interaction between the applied stresses and the internal ones, there is no cross term in $f_s(x)$ and $b_s(x)$ that contributes to W. The elastic energy, with both $f_s(x)$ and $b_s(x)$ present is given by

$$W = \frac{1}{2}\int [u_s(x)f_s(x) + \phi_s(x)b_s(x)]\ d^2x \qquad (26)$$

The quantity $\phi_s(x)$ is the dislocation potential (8), since it is conjugate to $b_s(x)$ with respect to W. It is known as the Airy vector stress function and is related to the stresses by

$$\sigma_{i1} = -\partial\phi_1/\partial x_2 \qquad \sigma_{i2} = \partial\phi_2/\partial x_1 \qquad (27)$$

It should be noted that the stresses can be obtained either from eqn. (27) or from Hooke's law and the displacement gradient. Essentially one can say that it is as easy or difficult to find the displacement $u_s(x)$ for an elastic problem as it is to find the Airy function $\phi_s(x)$ for the same problem.

Because the form of eqn. (26) is symmetric with respect to $u_s(x)$ and $\phi_s(x)$, and symmetric with respect to $f_s(x)$ and $b_s(x)$, one can repeat the argument of the last section almost word by word to define a new universal (tensor) weight function for screenings. In analogy to eqn. (25) it is

$$D_{is}(a;x) = 4\pi B_{ij}(K^{-1})_j^{(z)} \frac{d\phi_s^{(z)}}{da} \qquad (28)$$

and the generalization of eqn. (5) to the presence of both loadings $f_s(x)$ and a dislocation density $b_s(x)$ being present is

$$K_i = \int [F_{is}(a;x) f_s(x) + D_{is}(a;x) b_s(x)] d^2 x \tag{29}$$

with the two universal tensor weight functions given by eqns. (25) and (28).

PRECIPITATES AND THERMAL STRESSES

Since every elastic state can be thought of as being caused by a distribution of forces $f_s(x)$ and/or dislocations $b_s(x)$, the two weight tensors F_{is} and D_{is} cover every possible situation. If, for example, one wants to consider solute atoms that are characterized by a dipole tensor M_{st}, which has the dimension of an energy (8), the analogy of eqns. (5) and (25), or of (5) and (28) becomes

$$K_i = \int P^t_{is}(a;x)\; M_{st}(x)\; d^2 x \tag{30}$$

where

$$P^t_{is}(a;x) = 4\pi B_{ij} (K^{-1})^{(z)}_j \frac{d}{da} \frac{\partial u^{(z)}_s}{\partial x_t} \tag{31}$$

A dipole tensor M_{st} which is proportional to the unit tensor describes a situation where temperature gradients are present in a thermoelastic material. Since P^t_{is} is obtained by differentiation of F_{is} with respect to x_t, the influence of thermal stresses on the stress intensity vector of a cracked specimen is trivial so solve. It is not necessary to solve the thermoelastic equations, but the thermal and the elastic problem split up conveniently: once the (easier) thermal problem is solved and the temperature gradients $dT(x)/dx_s$ have been found, they can be put proportional to fictional forces $f_s(x)$. For these the elastic part of the problem is solved with the universal tensor weight function $F_{is}(a;x)$.

DISCUSSION

Although it is true that it is as easy to find $\emptyset_s(x)$ as it is to find $u_s(x)$ or its gradient $du_s(x)/dx_t$, this is actually of little help as long as the displacement $u_s(x)$ and its derivative with respect to the crack length a is not known. So far the universal weight functions seem to be explicitly available only for infinite isotropic specimens with half plane cracks. The construction of $F_{is}(a;x)$ and $D_{is}(a;x)$ for more complicated specimen geometries, especially for finite specimens, remains a challenging problem. One can, however, say that once $F_{is}(a;x)$ has been found (or, what amounts to the same, $D_{is}(a;x)$), any loading can be put on and any dislocations or precipitates can be put into the specimen, or the specimen can be heated, and the influence on K_i of all that can be found by mere integrations - without solving the complete elastic problem. One can also employ the freedom one has in choosing the reference state to get a little more than just the stress intensity vector. The method of Cardew, Goldthorpe, Howard and Kfouri (10) to find the first nonsingular term of the stresses around a crack amounts to selecting the reference state in a clever way and using the idea of reciprocity which is the foundation of the present paper.

THREE-DIMENSIONAL CRACKS

Rice (3) was the first to notice that the basic idea of Bueckner (2) can be extended to three-dimensional situations. Recently he elaborated this idea in a series of papers (11,12,13) and also Bueckner (14) discussed three-dimensional weight functions. For a two-dimensional situation the force on the crack is obtained by taking the derivative d/da with respect to the crack length. For the three-dimensional situation this has to be replaced by the functional derivative d/da(s) where s is the parameter along the crack perimeter. The situation is similar to finding the self-force of a curved dislocation (15). Although the three-dimensional theory is analogous in its development to the two-dimensional one, in view of the fact that finding the two-dimensional weight functions for non-trivial geometries is already difficult, it seems unlikely that the required three-dimensional weight functions could be constructed easily for, say, finite cracks in finite bodies. A possible way for such constructions might be to link the weight function approach to the integral equation approach (16).

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