Simulation of crack propagation using hybrid Trefftz method based on a strip-yield crack-tip plasticity model for automotive crash applications

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Abstract

This paper presents an advanced model for the simulation of crack propagation in thin walled high-strength steel sheets using special purpose elements, so called Trefftz-elements (T-elements) for crack tip regions. These T-elements are based on shape functions, which are exact solutions to the governing differential equations and to inner boundary conditions. This particular choice of shape functions results in high resolutions of the stress/strain fields without any refinement of the finite element mesh which would increase the computational time drastically, specifically for explicit finite element simulations. The T-elements can be combined with standard finite elements, that are used for rather uncritical regions, via the Hybrid Trefftz Method (HTM), a coupling formalism based on an extended variational principle. Beside this formalism the paper focuses on the derivation of particular solutions of the boundary value problem for a straight 2d-crack-problem including a Dugdale strip-yield zone. To complete the simulation procedure, a model, describing the materials behaviour of fracture toughness must be implemented additionally. This paper uses the crack-tip opening displacement (CTOD) as a capable parameter for fracture toughness, specifically for the application in automotive high strength steels. The parameterization of the CTOD-based model function is realized by fracture mechanical experiments based on single edge notched tension specimen. Beyond the parameterization of the materials fracture toughness model experimental results, like tensile forces, are used to validate the simulation results. Finally some future enhancements, like formulations for a cohesive zone model and crack deflection, are discussed.

Introduction

Under the aspects of rising demands in vehicle safety and increasing economic pressure, efficient methods for the simulation of car crashes play an important role in modern automotive industry. The explicit Finite Element Method (FEM) is established as the standard simulation tool in this field of engineering. Although this method provides a high prediction quality concerning deformations and crash intrusions, strong localized phenomena like failure of joints, crack initiation and propagation etc. cannot be described sufficiently without locally fine re-meshing. As a result of the Courant-Friedrich-Levy criterion for the critical simulation-time-step the elements have to be kept of specific minimum sizes to avoid escalating computational times. The mesh dependency of the achievable resolution in stress/strain fields as well as the inability to describe continuous crack propagation are considerable limitations of the standard FEM for the simulation of crack propagation.

An alternative to mesh-refinement is the use of special purpose elements, which are better adapted to the local conditions at the crack tip than conventional finite elements, based on polynomial shape functions.

This paper suggests the use of so called Trefftz-elements (T-elements) for the crack tip region. In contrast to conventional finite elements, a T-element uses particular solutions of the governing differential equations, which also satisfy inner boundary conditions [1, 2]. This approach results in high resolutions of the stress/strain fields in the vicinity of the crack tip without a mesh-refinement in the critical region. Additionally quasi continuous crack growth within the Trefftz-crack-tip-element can be realized, which is a second advantage compared to traditional crack simulation methods like nodal release- or element-elimination-technique.

After a short introduction to the Hybrid Trefftz Method, the paper presents an analytical derivation of particular solutions for a straight 2d-crack-problem including a Dugdale strip-yield zone in a linear elastic material. The results of this analytical treatment of the boundary value problem can be used to implement a moving local mesh procedure which will be shown later on.

Hybrid Trefftz Method (HTM)

Let Ω be a general 2d-solution domain containing a sharp crack (see Fig. 1) and two different boundary conditions, i.e. a displacement condition on $\partial\Omega_0$ and a force condition on $\partial\Omega_c$. To find an approximate solution, Ω is divided into two sub-domains, a crack tip region, denoted Ω_1, and a rather uncritical domain Ω_1. While Ω_0 is modeled by standard finite elements, the crack tip domain Ω_1 is modeled by a T-element.

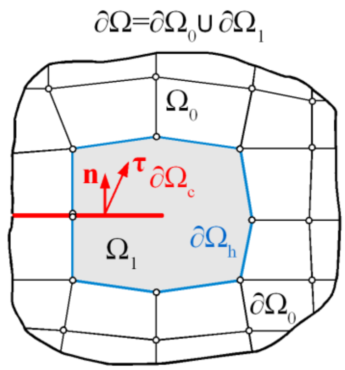


Fig.1. Solution domain containing a Trefftz-crack tip element and surrounding standard elements

With the shape functions U_{ki} in Ω_1, which per definition fulfill the governing differential equations and a force boundary condition on $\partial\Omega_c$, an ansatz for the displacements u_i in Cartesian coordinates *(i=1,2)*,

$$u_i = \sum_{k=0}^{K} c_k U_{ki} + u_p \,, \tag{1}$$

can be found, where the first term represents a superposition of the homogenous solutions U_{ki} of the differential equations and the second term, u_p, denotes a particular solution, satisfying a possible inhomogeneous boundary condition on $\partial\Omega_c$. The connection between the two sub-domains is accomplished via a displacement frame $\partial\Omega_h$ along the edges of all adjacent standard elements. While C_0-continuity holds for the standard elements in Ω_0 it's not possible to achieve this continuity on $\partial\Omega_h$. Instead of a strong geometrical boundary condition on $\partial\Omega_h$ a weaker form can be found via an extension of the first variation of the potential energy [3] by an additional term,

$$\int_{\partial\Omega_i} \delta\tau_i \left(\tilde{u}_i - u_i\right) t ds \,, \tag{2}$$

where τ_i are the analytic expressions for the traction densities in Ω_1, $\tilde{u}_i$ are the polynomial shape functions of the adjacent standard elements and t is the plate thickness. The element stiffness matrix and the element nodal force vector, which does not vanish if there are external forces acting on $\partial\Omega_i$, can be obtained by minimization of the extended potential energy.

Analytic Trefftz-solutions
For linear elastic materials, the governing differential equations are the Navier-Cauchy equations. For plane problems in Cartesian coordinates x_i $(i=1,2)$ and under the assumption of absent volume forces they take the form

$$\left(\lambda+\mu\right)\frac{\partial^2 u_r}{\partial x_r \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_r^2} = 0 \,, \tag{3}$$

where $r=1,2$, λ and μ are the Lame' constants. The equations can be formulated in the Complex plane with the relations

$$z = x_1 + i x_2 \,, \tag{4}$$
$$q = u_1 + i u_2 \tag{5}$$

for the coordinates and displacements, respectively, and a complex representation of (3),

$$2\left(\lambda+\mu\right)\frac{\partial}{\partial \bar{z}}\left(\frac{\partial q}{\partial z} + \overline{\frac{\partial q}{\partial z}}\right) + 4\mu \frac{\partial^2 q}{\partial z \partial \bar{z}} = 0 \,, \tag{6}$$

can be specified. It can be shown [4, 5, 6] that the general solution for the displacement- and stress-fields are related to two complex potential functions, $\phi(z)$ and $\psi(z)$, and their complex derivatives according to

$$q = \frac{1}{2\mu}\left(\kappa\phi(z) - z\overline{\phi'(z)} - \psi\left(\overline{z}\right)\right), \tag{7}$$

and

$$\sigma_{xx} + i\tau_{xy} = \phi'(z) + \overline{\phi'(z)} - z\overline{\phi''(z)} - \overline{\psi'(z)} \,, \tag{8}$$
$$\sigma_{yy} - i\tau_{xy} = \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)} \,, \tag{9}$$

where $\kappa=(3-\nu)/(1+\nu)$ for plane stress. $\phi(z)$ and $\psi(z)$ are holomorphic in Ω_1, and the resulting complex displacements and stresses must be single valued. To obtain a set of particular solutions according to Eq. 1, the force boundary condition on the edges of the crack must be fulfilled.

The Dugdale model. In [7, 8] a strip yield zone was proposed for a through crack in an infinite plate for a non-hardening material in plane stress. The elastic-plastic behavior is approximated by the superposition of the linear elastic solution for traction free crack edges and a solution for a closure stress with a magnitude of the yield stress σ_y at the crack tip such, that the stress singularity of the linear elastic near field solution is removed. Fig. 2 shows a Dugdale-like configuration with a crack from -∞ to $+r_D$ on the x-axes, where r_D is the size of the Dugdale-zone, i.e. for $0<x< r_D$ a constant crack closing traction density $\tilde{\tau} = \pm i\sigma_y$ acts while for $x\leq 0$ the crack is unloaded.

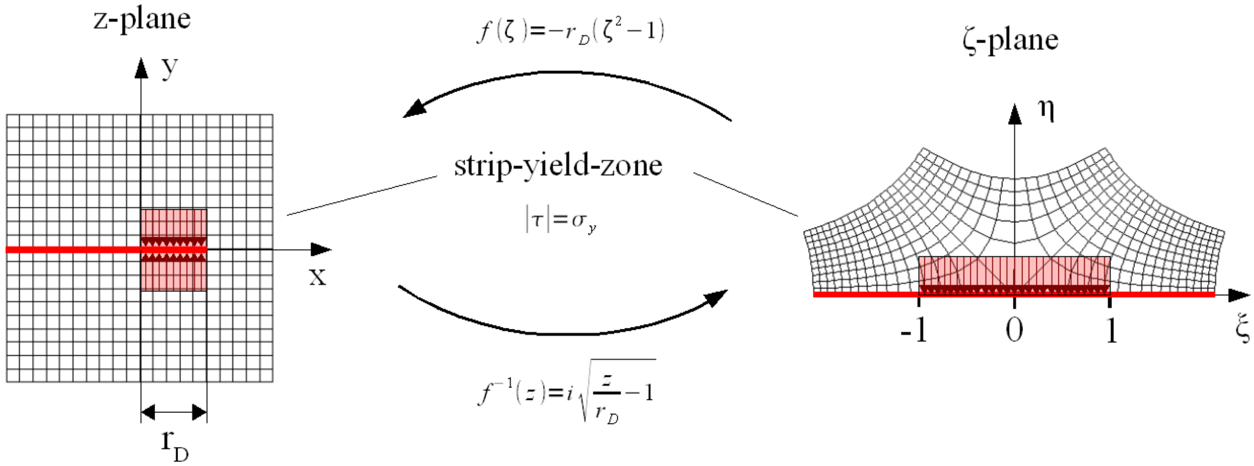


Fig.2. Conformal Map of the solution domain

To apply this condition, the physical solution domain in the z-plane is considered as a map from the complex upper half of the ζ-plane, see Fig. 2, via

$$f(\zeta) = -r_D\left(\zeta^2 - 1\right) \tag{10}$$

such, that the Dugdale-zone is now located in the interval (-1,1) on the ξ-axes. By integration of the traction densities along the crack with arbitrary lower and upper integration limits, a simpler form of the boundary condition,

$$\phi(\xi) + \frac{f(\xi)}{\overline{\dot{f}}(\xi)}\overline{\dot{\phi}}(\xi) + \bar{\psi}(\xi) = \begin{cases} \sigma_y f(\xi) & \text{for } |\xi|<1 \\ 0 & \text{for } |\xi|\geq 1 \end{cases}, \tag{11}$$

is obtained. Using the Schwarz reflection principle for ϕ and f, an ansatz for ψ,

$$\psi(\zeta) = -\overline{\phi(\bar{\zeta})} - \frac{\overline{f(\bar{\zeta})}}{\dot{f}(\zeta)}\dot{\phi}(\zeta), \tag{12}$$

can be found. Substituting of Eq. 12 in Eq. 11 results in

$$\lim_{\zeta\to\xi\in\mathbb{R}}\left(\phi(\zeta) - \phi(\bar{\zeta}) + \frac{f(\zeta) - f(\bar{\zeta})}{\overline{\dot{f}}(\zeta)}\overline{\dot{\phi}}(\zeta)\right) = \begin{cases} \sigma_y f(\xi) & \text{for } |\xi|<1 \\ 0 & \text{for } |\xi|\geq 1 \end{cases}, \tag{13}$$

which is used for the determination of ϕ. While for the homogeneous part of Eq. 13 a power series can be used, for the inhomogeneous part a more specific ansatz is chosen by means of complex base functions

$$\hat{W}_n(\zeta) = \left(\zeta - \sqrt{\zeta^2 - 1}\right)^n . \tag{14}$$

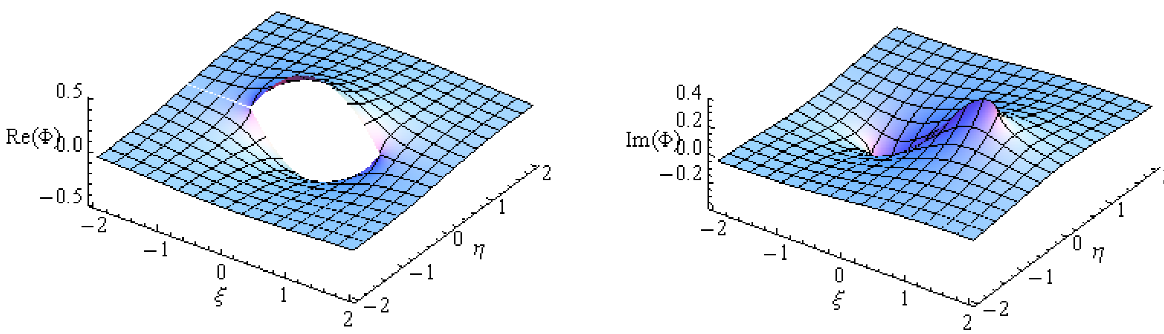
In summary, the ansatz for the complex potential is

$$\phi(\zeta) = \underbrace{\sum_{k=0}^{K} A_k \zeta^k}_{\phi_h} + \underbrace{\sum_{n=0}^{\infty} a_n \hat{W}_n(\zeta)}_{\phi_p} . \tag{15}$$

The functions $\hat{W}_n(\zeta)$ form a complete set of base functions in the considered interval (-1, 1) on the ξ-axis, which is used to fulfill the force boundary condition within this interval. For the constant traction density a closed form,

$$\phi_p(\zeta) = \frac{i}{\pi} \sigma_y r_D \left(\zeta - 2\left(\zeta^2 - 1\right) \tanh^{-1}\left(\zeta - \sqrt{\zeta^2 - 1}\right)\right), \tag{16}$$

of the inhomogeneous part ϕ_p can be found.



(a) (b)

Fig.3. (a) Real part of the complex potential ϕ_p. (b) Imaginary part of the complex potential ϕ_p

In Fig. 3a and Fig. 3b the real- and the imaginary- part of ϕ_p, Eq. 16, are shown. It can be seen, that ϕ_p has a branch cut in the interval (-1, 1) on the ξ-axes. This branch cut exactly satisfies the boundary condition, given in Eq. 13, and ϕ_p tends to zero for $|\zeta| \to \infty$. Due to this characteristic, the choice of this particular potential ϕ_p is ideally suited for the T-element formulation of the Dugdale crack problem. Mapping of the ansatz ϕ, Eq. 15, to the z-plane by Eq. 10 can be used to derive an expression for the displacement-field, according to Eq. 1, with Eq.7 and Eq. 5.

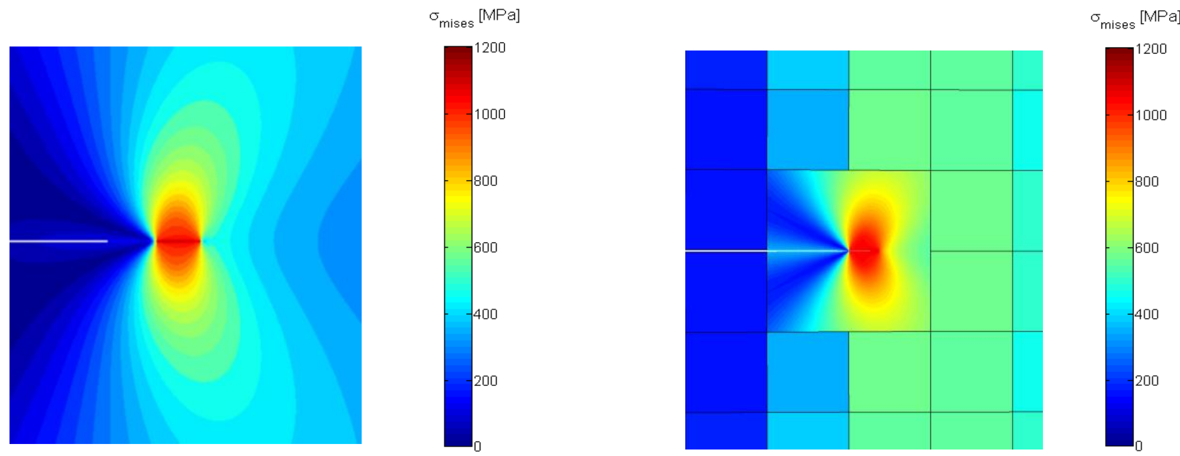
Length of the Dugdale-zone r_D. In order to remove the singularity in the stress-field at the crack tip, which results from the homogeneous part of ϕ, a relation between the coefficient A_1 and the length of the Dugdale-zone r_D,

$$r_D = \frac{i\pi}{2\sigma_y} A_1, \tag{17}$$

is found by calculating the limit of the von Mises-stress for $z \rightarrow r_D$. It should be noted, that for a pure Mode I condition, the complex coefficient A_1 becomes negative imaginary, resulting in a positive real value for r_D. For Mixed Mode conditions, Eq. 17 is also valid, but r_D becomes complex. Because of the relation between A_1 and r_D, which determines the conformal map, Eq. 10, the element-stiffness matrix and the element nodal force vector can only be calculated iteratively.

Results for a stationary crack

Fig 3a,b show von Mises stresses for a stationary crack in a single edge notched tension specimen under Mode I loading. While Fig. 3a is calculated with a very fine standard element mesh (edge length of the elements 0.1mm), Fig 3b shows the result of a calculation via HTM using a T-element with analytic shape functions and a coarse residual mesh with 5mm edge lengths.



(a) (b)

Fig.3. Examples for the von Mises Stress under Mode I loading calculated with: (a) fine standard finite element-mesh and (b) a T-element

The results show a good accordance in the relevant region in front of the physical crack tip (Dugdale-zone). Due to the high resolution of the analytic displacement- and stress-fields, they can be used to evaluate as fracture mechanical propagation criteria.

Simulation of Crack Propagation

A simulation of the crack propagation based on HTM can be realized by a change of the crack tip position in arbitrary small steps inside the T-elements domain. In each simulation step a crack driving force is evaluated and compared against an empirical model function for the crack growth resistance obtained from experimental data. If the simulated value exceeds that of the empirical model function, the crack is extended by a predefined small step and the criterion is evaluated once again. This procedure is repeated until the simulated value is lower than the empirical one. Subsequently the next external load step can be computed. The T-element usually replaces more than one standard element. Therefore the crack can grow for some distances inside the element. If the crack tip approaches the element boundary, the T-element is re-positioned in propagation direction, replacing the corresponding elements in front of the T-element. The occurring gaps in the back of the re-positioned T-element can be refilled with standard elements.

Crack tip opening displacement (CTOD). This paper uses the CTOD as a capable parameter to simulate the crack growth. According to different experimental determination methods, various definitions for CTOD exist [9]. This paper uses the definition of CTOD at the end of the Dugdale-zone [10], which can be evaluated directly by the Trefftz-displacement solution (see Fig. 2 and Fig. 4).

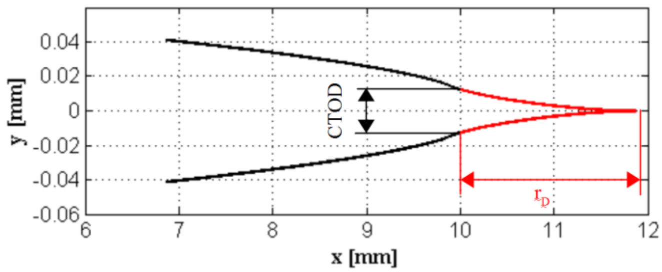


Fig.4. Vertically zoomed contour of the crack edges in the crack tip region, calculated via HTM

Fracture mechanical testing. The materials fracture toughness is modeled by an empirical CTOD-Δa model function,

$$CTOD = c_1 \Delta a + c_2 \Delta a^{c_3} , \tag{18}$$

where Δa is the crack extension and c_1, c_2, c_3 are constants, which obtained from quasi-static displacement controlled tensile tests performed on fatigue pre-cracked single edge notched specimen (SENT-tests). Since direct methods to determine the materials CTOD-Δa-curve are rather cumbersome, an alternative, based on CTOD-δ5 [10], is realized. The measured CTOD-δ5-Δa-curve can be used to control the crack propagation for an HTM-simulation to obtain the CTOD-Δa-curve as described above.

Validation. Fig. 5b shows the simulated load-crack-extension-curve (F-Δa-curves) compared against experimental data.

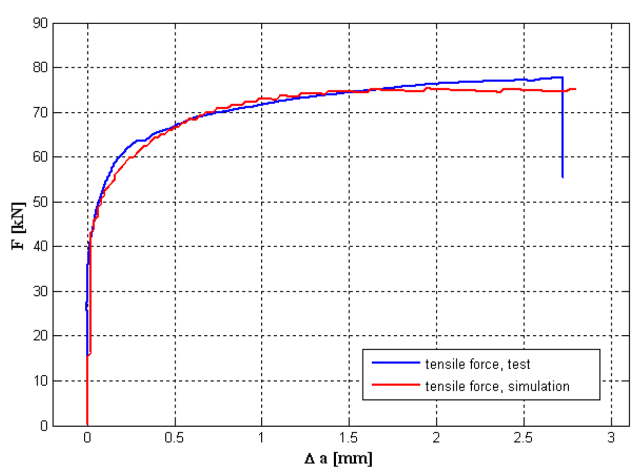


Fig.5 Validation of the simulation results based experimental load-crack extension curves for quasi static tensile tests

It is found, that the simulated results for the tensile forces as well as the crack length for which instability occurs are in good accordance with experimental data.

Conclusion

This paper presents a simulation method for 2d crack propagation in high strength steel sheets using a Hybrid-Trefftz-element. The method provides a high resolution of the stress- and displacement fields in the vicinity of the crack tip by means of particular solutions of the governing differential equations and inner boundary conditions. These analytic solutions for a crack-problem with a Dugdale-strip-yield-zone can be derived, introducing a specific function base for ϕ_p, the inhomogeneous part of the complex potential to satisfy the strip yield force boundary condition.

Furthermore the paper presents a complete crack propagation algorithm, based on the Crack tip opening displacement (CTOD) criterion. The materials fracture toughness is included by means of an empirical model function for the CTOD-Δa-curve. The simulation results are validated against experimental data via load-crack-extension curves and show a good accordance.

The introduced function base for ϕ_p enables further enhancements like the implementation of cohesive zone models for hardening materials or the deflection of cracks using Schwarz-Christoffel mapping.

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