Plastic Collapse of a Thin Annular Disk of Variable Thickness Subject to Thermo-Mechanical Loading

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Abstract. A semi-analytic solution for plastic collapse of a thin annular disk of variable thickness subject to thermo-mechanical loading is presented. It is assumed that the yield criterion depends on the hydrostatic stress. A distinguished feature of the boundary value problem considered is that there are two loading parameters. One of these parameters is temperature and the other is pressure over the inner radius of the disk. The general qualitative structure of the solution at plastic collapse is discussed in detail. It is shown that two different plastic collapse mechanisms are possible. One of these mechanisms is characterized by strain localization at the inner radius of the disk. The entire disk becomes plastic according to the other plastic collapse mechanism. In addition, two special regimes of plastic collapse are identified. According to one of these regimes, plastic collapse occurs when the entire disk is elastic, except its inner radius. According to the other regime, the entire disk becomes plastic at the same values of the loading parameters at which plastic yielding starts to develop.

Introduction

Thin annular disks subject to various loading conditions are a class of commonly used structures in mechanical engineering. The mechanical analysis and design of such disks may be based either on elastic or elastic-plastic solutions. In the latter approach, the state of plastic collapse is of great importance. The term plastic collapse is used here to denote the onset of unconstrained plastic flow. In most cases the state of plastic collapse is defined by the condition that the entire disk is plastic. In particular, analytical and semi-analytical solutions for fully plastic rotating disks as well as disks loaded by internal and external pressures have been proposed in [1 – 5]. A numerical technique to find the limit angular velocity of rotating disks has been developed [6]. However, another mechanism of plastic collapse may occur in thin disks and plates. According to this mechanism an unconstrained local thickening (or thinning) occurs at a certain radius of the disk. A comprehensive review of such solutions has been provided in [7]. The solution given in the present paper for an annular disk subject to thermo-mechanical loading demonstrates that both mechanisms of plastic collapse can occur. Moreover, the general qualitative structure of the plastic collapse solution is rather complicated. The formulation of the boundary value problem has been intentionally chosen as simple as possible to enable the semi-analytical solution. In particular, the temperature field is uniform and the initial thickness of the disk is constant. The pressure applied over the inner radius of the disk is also uniform. A review of solutions for a plate with a hole loaded by pressure has been given in [8]. An annular disk subject to thermal loading has been considered in [9]. The only possible mechanism of plastic collapse in this case occurs when the entire disk becomes plastic.

Other aspects related to thermal loading of thin disks have been presented in [10 – 13]. Collapse mechanisms have not been studied in these works.

A distinguished feature of the solution given in the present paper is that the disk is subject to the simultaneous action of both thermal and mechanical loads. This combination of loads applied to a long elastic cylinder has been considered in [14]. A solution for a rotating disk subject to thermal loading (therefore, there are also two loading parameters, temperature and angular velocity) has been proposed in [15]. The only collapse mechanism mentioned in this paper is associated with the fully plastic disk. General properties of limit load solutions in the case of multiple loading parameters have been considered in [16].

Another distinguished feature of the present solution is that the yield criterion is pressure-dependent. This property is inherent in many metallic materials (see, for example, [17 – 20]). Using the pressure-dependent yield criterion proposed in [21] solutions for thin disks subject to mechanical loading have been found in [22, 23].

Statement of the Problem

Consider a thin elastic-plastic annular disk of variable thickness subject to thermo-mechanical loading. The inner and outer radii of the disk are denoted by a_0 and b_0, respectively. It is convenient to introduce a cylindrical coordinate system (r, θ, z) whose z-axis coincides with the axis of symmetry of the disk. Then, the equations for the inner and outer radii of the disk are $r = a_0$ and $r = b_0$, respectively. The initial thickness of the disk varies according to the equation [24]

$$h = h_0 \left(\frac{r}{a} \right)^n . \tag{1}$$

The strains are supposed to be small. Let σ_r, σ_θ and σ_z be the normal stresses in the cylindrical coordinate system. The state of stress is plane, $\sigma_z = 0$. It is assumed that the thickness of the disk is everywhere sufficiently small for the stresses to be averaged through the thickness. The disk is loaded by a uniform pressure P over its inner radius. This boundary condition can be written as

$$\sigma_r = -P \tag{2}$$

for $r = a_0$. It is assumed that P is a monotonically increasing function of a time-like parameter and $P = 0$ at the initial instant. Let T be the increase in temperature from its initial value. Thus, by assumption, both $P \ge 0$ and $T \ge 0$ and the disk is stress free at the initial instant. The outer radius of the disk is fixed. Therefore,

$$u_r = 0 \tag{3}$$

for $r = b_0$. Here u_r is the radial displacement. It is evident that the problem is axisymmetric and the solution is independent of θ. Moreover, the normal stresses in the cylindrical coordinates are the principal stresses.

It is assumed that plastic yielding is influenced by the hydrostatic stress, σ, according to the Drucker-Prager yield criterion [21]

$$\alpha\sigma + \sigma_{eq} = \sigma_0 \tag{4}$$

where σ_{eq} is the equivalent stress, α and σ_0 are material constants. In the case under consideration the hydrostatic stress and the equivalent stress are defined by

$$\sigma = \frac{\sigma_r + \sigma_\theta}{3}, \quad \sigma_{eq} = \sqrt{\frac{3}{2}}\sqrt{(\sigma_r - \sigma)^2 + (\sigma_\theta - \sigma)^2 + \sigma^2}\ . \tag{5}$$

No specific relation between stress and strain (or strain rate) in the plastic zone is needed for limit analysis.

It is convenient to normalize the stresses by σ_0 without changing the notation and to introduce the dimensionless radius by $\rho = r/b_0$. Then, Eqs. 2 and 3 become

$$\sigma_r = -p \ \text{ for } \ \rho = a \tag{6}$$

and

$$u = 0 \ \text{ for } \ \rho = 1. \tag{7}$$

Here $u = u_r/b_0$, $a = a_0/b_0$, and $p = P/\sigma_0$.

Elastic Solution

Taking into account Eq. 1 the only non-trivial equilibrium equation can be written in the form

$$\rho\frac{\partial \sigma_r}{\partial \rho} + (1+n)\sigma_r = \sigma_\theta\ . \tag{8}$$

Solving this equation along with the Duhamel-Neumann law yields

$$\begin{aligned} &\sigma_r = A\rho^{\gamma_1} + B\rho^{\gamma_2}, \quad \sigma_\theta = A(1+n+\gamma_1)\rho^{\gamma_1} + B(1+n+\gamma_2)\rho^{\gamma_2}, \\ &\frac{u}{\rho q} = A(1+n-\nu+\gamma_1)\rho^{\gamma_1} + B(1+n-\nu+\gamma_2)\rho^{\gamma_2} + \tau, \end{aligned} \tag{9}$$

where $q = \sigma_0/E$, E is Young modulus, ν is Poisson's ratio, $\tau = \gamma TE/\sigma_0$, γ is the thermal coefficient of linear expansion, and A and B are constants of integration. Also,

$$\gamma_1 = -\left(1+\frac{n}{2}\right) - \frac{1}{2}\sqrt{(2-n)^2 + 4n(1-\nu)}, \quad \gamma_2 = -\left(1+\frac{n}{2}\right) + \frac{1}{2}\sqrt{(2-n)^2 + 4n(1-\nu)}\ .$$

When p and τ are small enough, the entire disk is elastic. In this case A and B are determined from the solution Eq. 9 using the boundary conditions Eq. 6 and Eq. 7 as

$$\begin{aligned} &A = A_e = \frac{p(1+n-\nu+\gamma_2) - \tau a^{\gamma_2}}{(1+n-\nu+\gamma_1)a^{\gamma_2} - (1+n-\nu+\gamma_2)a^{\gamma_1}}, \\ &B = B_e = \frac{-p(1+n-\nu+\gamma_1) + \tau a^{\gamma_1}}{(1+n-\nu+\gamma_1)a^{\gamma_2} - (1+n-\nu+\gamma_2)a^{\gamma_1}}. \end{aligned} \tag{10}$$

As a result of an increase in τ, or p or both, a plastic zone can appear at the inner radius of the disk.

General Stress Solution in the Plastic Zone

In order to find the general stress solution in the plastic zone, it is necessary to combine the yield criterion given in Eq. 4 and Eq. 8.

In the case under consideration, the yield criterion is satisfied by the following substitution [23]

$$\sigma_r = 3\beta_0 + \beta_1 \sin\psi + 3\beta_2 \cos\psi, \quad \sigma_\theta = 3\beta_0 - \beta_1 \sin\psi + 3\beta_2 \cos\psi \tag{11}$$

where

$$\beta_0 = \frac{2\alpha}{4\alpha^2 - 9}, \quad \beta_2 = \frac{3}{9 - 4\alpha^2}, \quad \beta_1 = \sqrt{\beta_2}\,. \tag{12}$$

Substituting equation Eq. 11 into equation Eq. 8 gives

$$\rho\left(\beta_1 \cos\psi - 3\beta_2 \sin\psi\right)\frac{\partial\psi}{\partial\rho} = -\left(2+n\right)\beta_1 \sin\psi - 3n\left(\beta_0 + \beta_2 \cos\psi\right). \tag{13}$$

Using Eq. 11 the boundary condition given in Eq. 6 transforms to

$$3\beta_0 + \beta_1 \sin\psi_a + 3\beta_2 \cos\psi_a = -p \tag{14}$$

where ψ_a is the value of ψ at $\rho = a$. The solution to Eq. 13 satisfying the condition $\psi = \psi_a$ at $\rho = a$ is

$$\rho = a \exp\left\{\int_{\psi_a}^{\psi}\left[\frac{3\beta_2 \sin\chi - \beta_1 \cos\chi}{\left(2+n\right)\beta_1 \sin\chi + 3n\left(\beta_0 + \beta_2 \cos\chi\right)}\right]d\chi\right\} \tag{15}$$

where χ is a dummy variable of integration.

Consider the mechanism of plastic collapse according to which the plastic zone occupies the entire disk. At the instant when plastic collapse occurs, the elastic zone reduces to the curve $\rho = 1$ in the $\rho\theta$-plane. The general solution given in Eq. 9 is valid in this vanishing elastic zone, though A and B are not given by the relations in Eq. 10. Since the stresses σ_r and σ_θ as well as the displacement u must be continuous across the elastic plastic boundary $\rho = 1$, it follows from Eqs. 7, 9 and 11 that

$$\begin{aligned} &A + B = 3\beta_0 + \beta_1 \sin\psi_m + 3\beta_2 \cos\psi_m, \\ &A\left(1+n+\gamma_1\right) + B\left(1+n+\gamma_2\right) = 3\beta_0 - \beta_1 \sin\psi_m + 3\beta_2 \cos\psi_m, \\ &A\left(1+n-\nu+\gamma_1\right) + B\left(1+n-\nu+\gamma_2\right) + \tau = 0. \end{aligned} \tag{16}$$

where ψ_m is the value of ψ at $\rho = 1$. Using Eq. 15 this value is determined in implicit form as

$$1 = a \exp\left\{\int_{\psi_a}^{\psi_m}\left[\frac{3\beta_2 \sin\chi - \beta_1 \cos\chi}{\left(2+n\right)\beta_1 \sin\chi + 3n\left(\beta_0 + \beta_2 \cos\chi\right)}\right]d\chi\right\}. \tag{17}$$

Taking into account Eq. 14 the solution to Eqs. 16 and 17 gives a relation between p and τ when the entire disk becomes plastic. However, a difficulty is that this system may have no solution.

General Structure of the Solution at Plastic Collapse

It will be seen later that the set of parameters at which the plastic zone starts to develop is also of importance for the solution at plastic collapse. When plastic yielding initiates the dependence of the radial and circumferential stresses on ψ is given by Eq. 11 at $\rho = a$. The solution in Eq. 9 with A and B determined from Eq. 10 is valid in the range $a \le \rho \le 1$. The stresses σ_r and σ_θ as well as the displacement u must be continuous across the elastic plastic boundary $\rho = a$. Therefore,

$$\begin{aligned} &A_e a^{\gamma_1} + B_e a^{\gamma_2} = 3\beta_0 + \beta_1 \sin\psi_0 + 3\beta_2 \cos\psi_0, \\ &A_e\left(1+n+\gamma_1\right)a^{\gamma_1} + B_e\left(1+n+\gamma_2\right)a^{\gamma_2} = 3\beta_0 - \beta_1 \sin\psi_0 + 3\beta_2 \cos\psi_0 \end{aligned} \qquad (18)$$

where ψ_0 is the value of ψ_a at the instant of the initiation of plastic yielding. Since A_e and B_e are expressed through p and τ, the dependence of p on τ corresponding to the initiation of plastic yielding is determined from Eq. 18. Using the imposed restrictions $p \ge 0$ and $\tau \ge 0$ it is possible to find the range of possible values of ψ_0, say $\psi_0^{(1)} \le \psi_0 \le \psi_0^{(2)}$. A typical dependence of p on τ corresponding to the initiation of plastic yielding is illustrated in Fig. 1. The specific values of parameters used to find this curve are $a = 1/2$, $\alpha = 0.2$, $n = 0$ and $\nu = 0.3$. It is seen from Fig. 1 that there is a local maximum of the function $p(\tau)$ at some value of $\tau = \tau_k$ (point k in Fig. 1). It is evident that $dp(\tau)/d\tau = 0$ at $\tau = \tau_k$. Replacing A_e and B_e in Eq. 18 with p and τ by means of Eq. 10, differentiating and excluding $d\psi_0$ yield

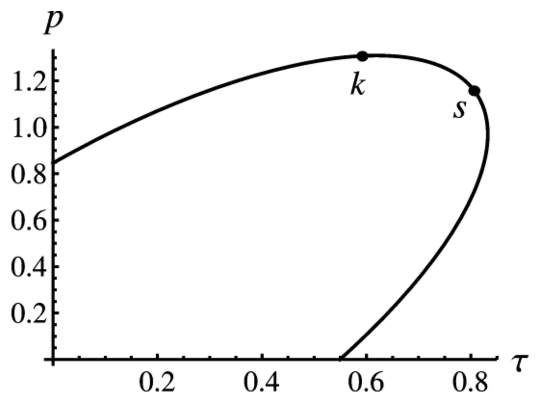

Fig.1. Curve corresponding to the initiation of plastic yielding.

$$\frac{dp}{d\tau} = \frac{\left(3\beta_1 \sin\psi_0 - \cos\psi_0\right)\left(\gamma_2 - \gamma_1\right)a^{\gamma_1+\gamma_2}}{\left\{\begin{aligned} &\left(3\beta_1 \sin\psi_0 + \cos\psi_0\right)\left[\left(1+n-\nu+\gamma_2\right)a^{\gamma_1} - \left(1+n-\nu+\gamma_1\right)a^{\gamma_2}\right] + \\ &\left(3\beta_1 \sin\psi_0 - \cos\psi_0\right)\left[\left(1+n-\nu+\gamma_1\right)\left(1+n+\gamma_2\right)a^{\gamma_2} - \left(1+n+\gamma_1\right)\left(1+n-\nu+\gamma_2\right)a^{\gamma_1}\right] \end{aligned}\right\}}. \qquad (19)$$

Here the relation between β_1 and β_2 given in Eq. 12 has been taken into account. It follows from Eq. 19 that the condition $dp(\tau)/d\tau = 0$ is equivalent to

$$\tan\psi_0 = \left(3\beta_1\right)^{-1}. \tag{20}$$

Substituting this value of ψ_0 in Eq. 18 it is possible to find the corresponding values of A_e and B_e and, then, using Eq. 10 the values of p and $\tau = \tau_k$. Moreover, it is evident that the coefficient of the derivative in Eq. 13 vanishes at $\psi = \psi_0$ if ψ_0 is determined from Eq. 20. For further convenience, it is advantageous to consider a general case assuming that $\psi = \psi_c$ and $\cos\psi_c - 3\beta_1\sin\psi_c = 0$. In a particular case $\psi_c = \psi_0$ where ψ_0 is determined from Eq. 20. Assume that $\rho = \rho_c$ at $\psi = \psi_c$. In the vicinity of this point Eq. 13 transforms to

$$\left(\psi - \psi_c\right)\frac{\partial\left(\psi - \psi_c\right)}{\partial\left(\rho - \rho_c\right)} = \frac{\left(2+n\right)\beta_1\sin\psi_c + 3n\left(\beta_0 + \beta_2\cos\psi_c\right)}{\rho_c\left(\beta_1\sin\psi_c + 3\beta_2\cos\psi_c\right)}. \tag{21}$$

Integrating yields

$$\left(\psi - \psi_c\right)^2 = \left[\frac{\left(2+n\right)\beta_1\sin\psi_c + 3n\left(\beta_0 + \beta_2\cos\psi_c\right)}{\rho_c\left(\beta_1\sin\psi_c + 3\beta_2\cos\psi_c\right)}\right]\left(\rho - \rho_c\right) \tag{22}$$

to leading order. The plastic zone occupies the domain $\rho \le \rho_c$. Assume that $\psi_c \to \psi_0$ where ψ_0 is determined from Eq. 20. Then, it is possible to verify by inspection that the coefficient of $\rho - \rho_c$ in Eq. 22 is positive at $\psi_c = \psi_0$ and, therefore, the right hand side of this equation is negative in the range $a \le \rho \le \rho_c = a + \delta$ where $\delta << 1$. This contradicts the left hand side of Eq. 22. Therefore, the plastic zone cannot start to develop. The physical interpretation of this mathematical feature of the solution is that plastic deformation is localized within a layer of infinitesimal thickness at $\rho = a$. This corresponds to another mechanism of plastic collapse as compared to the state in which the entire disk is plastic. A remarkable property of the set of loading parameters at point *k* (Fig. 1) is that the disk losses its load carrying capacity without any plastic deformation in the domain $a < \rho \le 1$.

Returning to the curve shown in Fig. 1, another point of great interest (point *s* in Fig. 1) corresponds to the value of $\psi_0 = \psi_s$ determined from the following equation $\left(2+n\right)\beta_1\sin\psi_s + 3n\left(\beta_0 + \beta_2\cos\psi_s\right) = 0$. The corresponding values of τ and p can be found from Eqs. 10 and 18. It is evident that Eq. 13 has a special solution $\psi = \psi_s$ which is not obtainable from the solution given in Eq. 15. Then, it follows from Eq. 11 that the stresses σ_r and σ_θ are independent of ρ. The physical meaning of this mathematical feature of Eq. 13 is that the plastic zone occupies the entire disk once plastic yielding has initiated at $\rho = a$.

If the initiation of plastic yielding corresponds to any point of the curve shown in Fig. 1 other than points *k* or *s*, than the elastic/plastic boundary propagates from the surface $\rho = a$ until the plastic collapse occurs. As in the special cases considered, the same two plastic collapse mechanisms are possible. In particular, once the value of ψ_a has attained the value of $1/\left(3\beta_1\right)$, the coefficient of the derivative in Eq. 13 vanishes. Eqs. 21 and 22 are valid. Assuming that $\psi_c \to \psi_a$ it is possible to arrive at the same contradiction as before. Therefore, the solution cannot be extended beyond this value of ψ_a. The corresponding collapse mechanism is localization of plastic deformation at $\rho = a$.

Since β_1 is constant for a given material, the critical value of ψ_a is also constant. Then, it follows from Eq. 14 that the value of p at plastic collapse is independent of τ. Therefore, this collapse mechanism is interpreted geometrically as a straight line parallel to the τ – axis. This line is illustrated in Fig. 2 for $\alpha = 0$ and $\nu = 0.3$ (line 2). The curve corresponding to the initiation of plastic yielding (curve 1) is tangent to this line at point *k*. Curve 1 has been found for $a = 1/2$, $\alpha = 0$, $n = 0$ and $\nu = 0.3$.

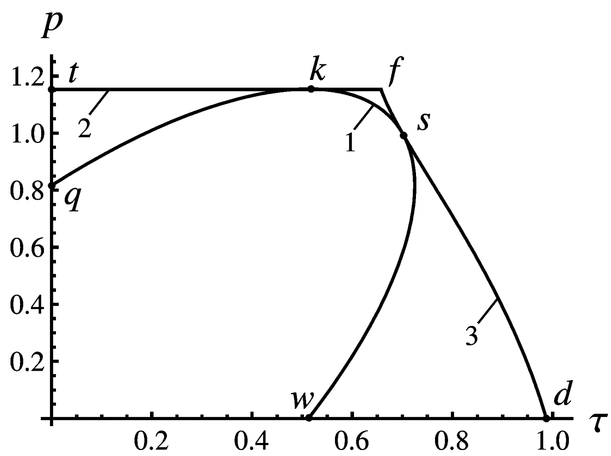

Fig.2. Illustration of the general structure of the solution. Curve 1 corresponds to the initiation of plastic yielding. Straight line 2 and curve 3 correspond to different plastic collapse mechanisms.

In order to determine the curve corresponding to the other plastic collapse mechanism, it is necessary to solve Eq. 17 for ψ_m numerically assuming that the value of ψ_a is given. Then, the value of *p* immediately follows from Eq. 14. Eliminating *A* and *B* in Eq. 16 yields

$$\tau = (1+\nu)\beta_1 \sin\psi_m - 3(1-\nu)(\beta_0 + \beta_2 \cos\psi_m). \tag{23}$$

Since the value of ψ_m has been determined, the corresponding value of τ can be found from this equation. Thus the dependence of p on τ is obtained in parametric form. This dependence is illustrated by curve 3 in Fig. 2 for $a = 1/2$, $\alpha = 0$, $n = 0$ and $\nu = 0.3$. Curves 1 and 3 have the same tangent line at point *s*.

Conclusions

A new semi-analytical solution for the state of plastic collapse of a thin annular disk of variable thickness subject to thermo-mechanical loading has been found. The numerical part of the solution reduces to solving Eq. 17 for ψ_m. Plastic yielding is influenced by the hydrostatic stress according to the Drucker-Prager yield criterion. The study has emphasized qualitative features of the plastic collapse solution whose general structure is illustrated in Fig. 2. It has been shown that there are two plastic collapse mechanisms. According to one of these mechanisms the load bearing capacity of the disk is lost because of strain localization at its inner radius. This mechanism is illustrated by line 2 in Fig. 2. It is seen that it is solely controlled by the dimensionless pressure over the inner radius of the disk. According to the other plastic collapse mechanism, the entire disk becomes plastic. The dependence between *p* and τ corresponding to this mechanism is shown by curve 3 in Fig. 2. In addition to these two general cases, there are three special cases of great interest for both numerical solution of similar problems and interpretation of elastic/plastic plane stress solutions for thin-walled structures. These special cases are denoted by symbols *k*, *s*, and *f* in Fig. 2. If the state of stress corresponds to point *k* then the disk losses its load bearing capacity by plastic strain localization at its inner radius whereas the entire disk (except the inner radius) is elastic. If the state of stress corresponds to point *s* then the entire disk becomes plastic at the same values of the loading parameters at which plastic yielding appears (i.e. the plastic zone does not propagate from the inner radius of the disk as the loading parameters increase but occupies the entire disk instantly as the state of stress from the elastic solution attains point *s*). A distinguished feature of point *f* is that both of the aforementioned plastic collapse mechanisms occur simultaneously. It is expected that these qualitative features of the solution are rather common for a class of thin-walled structures and they can cause some difficulties with finding numerical solutions for such structures.

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References

[1] J. Lenard and J. B. Haddow: Int. J. Mech. Sci. Vol. 14 (1972), p. 285–292
[2] K. Hein and M. Heinloo: Int. J. Solids Struct. Vol. 26 (1990), p. 617–630
[3] X.-L. Gao and S. N. Atluri: Math. Mech. Solids Vol. 2 (1997), p. 335–349
[4] G. Ma, H. Hong and Y. Miyamoto: Int. J. Mech. Sci. Vol. 43 (2001), p. 1137–1153
[5] A. N. Eraslan and H. Argeso: Int. J. Solids Struct. Vol. 39 (2002), p. 3109–3130
[6] S. Bhowmick, D. Misra and K. N. Saha: Int. J. Mech. Sci. Vol. 50 (2008), p. 163–174
[7] R. Debski and M. Zyczkowski: Z. Angew. Math. Mech. Vol. 82 (2002), p. 655–669
[8] R. Masri, T. Cohen and D. Durban: Q. Mech. Appl. Math. Vol. 63 (2010), p. 589–616
[9] S. Alexandrov and N. Alexandrova: J. Stain Anal. Eng. Des. Vol. 36 (2001), p. 169–176
[10] H. Lippmann: Int. J. Plast. Vol. 8 (1992), p. 567–582
[11] W. Mack: Arch. Appl. Mech. Vol. 63 (1993), p. 42–50
[12] M. Bengeri and W. Mack: Acta Mech. Vol. 103 (1994), p. 243–257
[13] W. Mack and M. Bengeri: Int. J. Mech. Sci. Vol. 36 (1994), p. 669–705
[14] J.-Q. Tarn: Int. J. Solids Struct. Vol. 38 (2001), p. 8189–8206
[15] A. N. Eraslan and T. Akis: Mech. Based Des. Struct. Mach. Vol. 31 (2003), p. 529–561
[16] Hodge P. G. Jr. and Sun C.-K.: Trans. ASME J. Appl. Mech. Vol. 35 (1968), p. 107–110
[17] W. A. Spitzig, R .J. Sober and O. Richmond: Metallurg. Trans. Vol. 7A (1976), p. 1703-1710
[18] A. S.Kao, H. A. Kuhn, W. A. Spitzig and O. Richmond: Trans. ASME J. Eng. Mater. Technol. Vol. 112 (1990), p. 26-30
[19] C. D.Wilson: Trans. ASME J. Appl. Mech. Vol. 69 (2002), p. 63-68
[20] P. S. Liu: Mater. Sci. Eng. Vol. A422 (2006), p. 176-183
[21] D. C. Drucker and W.Prager: Q. Appl. Math. Vol. 10 (1952), p. 157-165
[22] S. E. Alexandrov, E. V. Lomakin and Y.-R. Jeng: Dokl.-Phys. Vol. 55 (2010), p. 606-608
[23] S. Alexandrov, Y.-R. Jeng and E. Lomakin: Trans. ASME J. Appl. Mech. Vol. 78 (2011), Paper 031012
[24] J. Chakrabarty: Int. J. Mech. Sci. Vol. 13 (1971), p. 439-444