# Numerical method for solving crack problems posed in terms of principal directions

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**Abstract.** This paper provides a tool for numerical modelling of crack problems with incomplete boundary conditions formulated in terms of principal directions given on the crack surfaces. Continuity of tractions across the crack contour is also assumed. The problem is reduced to a system of singular integral equations that can be fully homogenous; therefore the application of standard numerical methods does not allow one to obtain non-trivial solutions. It is proposed to apply the Carleman-Vekua regularization to transform the problem to a system of non-homogeneous Fredholm equations of the second kind. Numerical solution of the latter can be built by the method of mechanical quadratures. Solvability of the problem is also discussed and illustrated for the case of a single crack.

## Introduction

The development of effective numerical approaches for stress reconstruction in plane elastic media with cracks from incomplete boundary data is the main focus of this study.

Fracture in geological media occurs under compressive loads and it is quite often that the crack (discontinuity) surfaces stay in partial contact during the fracture development. This fact presents sufficient difficulties in formulations of well posed boundary value problems (BVP) due to absence of reliable information about stress/displacement distributions along the contact zones. Therefore, it is important to study alternative formulations of BVP that include supplementary information on stress indicators on the boundary as well as inside the domain considered. Examples of such formulations include: continuous boundary data on stress/displacement orientations, and/or discrete field measurements of these characteristics. The boundary conditions (BC) of such type are incomplete in the sense that solutions of such problems are not unique. It has been demonstrated in [1] that the BVP formulated in terms of principal directions of the plain stress tensor and their normal derivative on a closed contour may have finite number of solutions or be unsolvable depending on the behaviour of the boundary data. Similar conclusions have been found for the cases when orientations of the traction and displacement vectors are used as BC [2]. The solvability of BVP for an open contour has also been considered for the special case of incomplete BC in [3].

Despite the progress in investigation of the solvability of the BVP with incomplete BC no numerical approaches for has been suggested to address the case of open contours (specific for the crack problems). The present study intends to fill this gap.

## Integral equations for collinear shear cracks.

**Preliminaries.** The 2D crack problems in elastic plane are formulated in terms of Muskhelishvili approach [4]. Thus, the Kolosov's formulas provide the following relationships between the stress components $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ and the displacement vector $(u_x,u_y)$ and two holomorphic functions $\varphi(z)$ and $\psi(z)$ (and their derivatives) of the complex variable $z=x+iy$

$$\tfrac{1}{2}(\sigma_{yy}+\sigma_{xx})=2\mathrm{Re}[\varphi'(z)],\quad \tfrac{1}{2}(\sigma_{yy}-\sigma_{xx})+i\sigma_{xy}=\bar{z}\varphi''(z)+\psi'(z),\quad 2G(u_x+iu_y)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)} \qquad (1)$$

Here κ=3-4ν for plane strain, κ=(3-ν)/ (1+ν) for plane stress, *G* is the shear modulus. The harmonic function expressing the mean stress and the complex-valued stress deviator function are further denoted as

$$P(z,\bar{z}) = \tfrac{1}{2}(\sigma_{yy} + \sigma_{xx}), \quad D(z,\bar{z}) = \tfrac{1}{2}(\sigma_{yy} - \sigma_{xx}) + i\sigma_{xy} \tag{2}$$

The crack (a system of cracks) is modelled by the discontinuity of the displacements along the crack contour, Γ. For this reason one can introduce a complex-valued function that characterizes the generalized crack opening displacements

$$g(t) = 2G\left(u_x^+(t) - u_x^-(t) + iu_y^+(t) - iu_y^-(t)\right), \quad t \in \Gamma \tag{3}$$

where $u_x^\pm(t) + iu_y^\pm(t)$ is the complex displacement vector of Γ on the upper/lower surface of the cracks.

The stress vector, *N*+*iT*, on Γ is expressed via the boundary values of the *P*(ζ) and *D*(ζ) are the boundary values of the stress functions in Eq. 2 as follows

$$N(\zeta) + iT(\zeta) = P(\zeta) + \exp(-2i\vartheta(\zeta))\overline{D(\zeta)}, \quad \zeta \in \Gamma \tag{4}$$

Here $\vartheta(\zeta)$ and is the angle between the tangent to the contour and the real axis.

The complex potentials can be expressed via the function *g*(*t*), which results in the following expressions for the stress functions that reflect the stress state of the infinite plate generated by the crack (provided that the tractions across the crack are continuous)

$$P_{cr}(z,\bar{z}) = \mathrm{Re}\left[\frac{1}{\pi i}\int_\Gamma \frac{g'(t)}{t-z}\,dt\right] \tag{5}$$

$$D_{cr}(z,\bar{z}) = \frac{-1}{2\pi i}\int_\Gamma \left[\frac{e^{-2i\vartheta(t)}}{t-z}\left(g'(t) + \overline{g'(t)}\right) + \frac{\bar{t}-\bar{z}}{t-z}\,g''(t)\right]dt \tag{6}$$

The boundary values of these functions are found as follows

$$P_{cr}^\pm(\zeta) = \mathrm{Re}\left[\pm g'(\zeta) + \frac{1}{\pi i}\int_\Gamma \frac{g'(t)}{t-\zeta}\,dt\right], \quad \zeta \in \Gamma \tag{7}$$

$$D_{cr}^\pm(\zeta) = \mp e^{-2i\vartheta(\zeta)}\,\mathrm{Re}[g'(\zeta)] - \frac{1}{\pi i}\int_\Gamma \frac{e^{-2i\vartheta(t)}\,\mathrm{Re}[g'(t)]}{t-\zeta}\,dt + \frac{1}{2\pi i}\int_\Gamma \left[\frac{e^{-2i\vartheta(t)}}{t-\zeta} - \frac{\bar{t}-\bar{\zeta}}{t-\zeta}\right]g'(t)\,dt, \quad \zeta \in \Gamma \tag{8}$$

In should be noted that the second integral in the right-hand side of Eq. 8 in not singular.

By superposition of the stress fields created by the cracks and the applied stress field one can present the equilibrium equation of the cracks contour from Eq. 4 in the form

$$P_{cr}^\pm(\zeta) + \exp(-2i\vartheta(\zeta))\overline{D_{cr}^\pm(\zeta)} = p(\zeta), \quad \zeta \in \Gamma \tag{9}$$

where the right hand side of this equation presents the loads applied to the crack surfaces (including the loads at infinity taken with the opposite sign).These loads are continuous across the contour.

Substitution of Eq. 7 and Eq. 8 into Eq. 9 leads to the standard singular integral equation (SIE) for crack systems, see e.g. [5]. The unknown density of crack opening displacements should satisfy the condition of single-valuedness of the displacements that for non-intersecting cracks has the form

$$\int_{\Gamma_k} g'(t)\,dt = 0 \tag{10}$$

where $\Gamma_k$ represent particular crack contours.
Solution of such SIE can be found in an analytical form for some simple cases of contours, in particular, for collinear cracks of a circular crack. For general geometries, the SIE can be solved numerically by a number of methods.
In should be noted that the SIE followed from Eq. 9 has a unique solution provided that the conditions of the single-uniqueness of displacements are satisfied. Therefore the choice of a numerical method is not a vital issue; usually it is convenient to discretize the SIE by applying a proper quadrature formula and to apply the collocation method to reduce it to a linear system of algebraic equations (e.g., BEM method [6] or the method of mechanical quadratures [5]). Other approaches are based on the method of moments or orthogonal polynomials. All these methods are applicable for non-homogeneous SIEs possessing unique solutions. However the uniqueness is violated if one considers the problem formulated in terms of principal directions [1]. Moreover, the system of SIE in this case is homogeneous, which restricts the application of direct numerical methods as those mentioned above. It has been recently proposed in [7] to apply the Carleman-Vekua regularization [8] to the SIE formulated on a closed contour, which results in a Fredholm equation of the second kind. The latter contain a non-homogeneous right hand side generated by the polynomials of the order determined from the solution of the corresponding Riemann problem (see [8] for detail). Here it is proposed to generalize this approach for the case of open contours.
**Problem formulation**. Let the entire contour $\Gamma$ consist of the union of $n$ non-interesting intervals that can be separated into two groups of $n_1$ intervals united in $\Gamma_1$ and $n_2$ intervals as $\Gamma_2$, i.e

$$\Gamma = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 = \bigcup_{k=1}^{n_1} (a_k, b_k), \qquad \Gamma_2 = \bigcup_{j=1}^{n_2} (c_j, d_j), \quad n = n_1 + n_2 \,. \tag{11}$$

The boundary conditions of the problem are formulated as follows: there is no jump of normal displacements, $u_y(x,y)$, across $\Gamma$; shear stresses, $T(x)$, are known on $\Gamma_1$ and the principal directions, specified by the angle $\theta = \theta(x,0+)$, are known on $\Gamma_2$, i.e.

$$u_y(x,0+) - u_y(x,0-) = 0, \quad x \in \Gamma \tag{12}$$

$$\mathrm{Im} D(x,0) = -T(x), \quad x \in \Gamma_1 \tag{13}$$

$$\mathrm{Im}\left[e^{-i\alpha(x,0)} D(x,0)\right] = 0, \quad x \in \Gamma_2 \tag{14}$$

Here $\alpha=\pi-2\theta$ is the argument of the complex-valued stress deviator function.
Due to Eq. 12 the complex-valued function $g(t)$ in Eq. 3 becomes real-valued and its contour derivative can be presented in the form

$$\mu(t) = \begin{cases} \mu_1(t), & t \in \Gamma_1 \\ \mu_2(t), & t \in \Gamma_2 \end{cases}, \quad \mu(t) = g'(t) = 2G \frac{d}{dt}\left(u_x^+(t) - u_x^-(t)\right), \quad t \in \Gamma, \quad \mathrm{Im}\left(\mu(t)\right) = 0 \tag{15}$$

The boundary values of the stress functions in Eq. 7-8 assume the form

$$P_{cr}^{\pm}(x) = \pm\mu(x), \quad x \in \Gamma \tag{16}$$

$$D_{cr}^{\pm}(x) = \mp\mu(x) - \frac{1}{\pi i}\int_{\Gamma_1}\frac{\mu_1(t)}{t-x}dt - \frac{1}{\pi i}\int_{\Gamma_2}\frac{\mu_2(t)}{t-x}dt, \quad x \in \Gamma \tag{17}$$

Then the system of equations derived from Eq 13 and Eq 14 assume the form

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_{\Gamma_1}\frac{\mu_1(t)}{t-x}dt + \frac{1}{\pi}\int_{\Gamma_2}\frac{\mu_2(t)}{t-x}dt = -T(x), & x \in \Gamma_1 \\ \sin\alpha(x)\mu_2(x) - \cos\alpha(x)\left[\dfrac{1}{\pi}\displaystyle\int_{\Gamma_2}\frac{\mu_2(t)}{t-x}dt + \frac{1}{\pi}\int_{\Gamma_1}\frac{\mu_1(t)}{t-x}dt\right] = 0, & x \in \Gamma_2 \end{cases} \tag{18}$$

For completeness, this system is complemented by the conditions given by Eq. 10.

**Solvability of the system**. The system of SIE is further reduced to a single SIE by solving the first equation of the system with respect to the unknown function $\mu_1(t)$ by using the inversion formulas for open contours [8]. This solution can be presented in the form

$$\mu_1(t) = \frac{1}{\pi R(t)}\int_{\Gamma_1}\frac{R(x)}{x-t}\left[T(x) + \frac{1}{\pi}\int_{\Gamma_2}\frac{\mu_2(s)}{s-x}ds\right]dx + \frac{P_{n_1-1}(t)}{R(t)}, \quad t \in \Gamma_1 \tag{19}$$

where $P_{n1-1}(t)$ is an arbitrary polynomial of the degree ($n_1$-1) and the function $R(z)$ is

$$R(z) = \prod_{k=1}^{n_1}\sqrt{(z-a_k)(z-b_k)} \tag{20}$$

It should be noted that the polynomial term in Eq 19 should vanish due to the conditions of single valuedness specified by Eq. 10, which is evident from the evaluation of the integrals of the form

$$\int_{a_k}^{b_k}\frac{dt}{R(t)(t-x)} = 0, \quad x \in (a_k, b_k), \quad k = 1\ldots n_1. \tag{21}$$

Then the expression for the unknown function $\mu_1(t)$ can be transformed to the form

$$\begin{aligned} &\mu_1(t) = \frac{1}{\pi R(t)}\int_{\Gamma_2}K(s,t)\mu_2(s)ds + f(t), \quad t \in \Gamma_1 \\ &f(t) = \frac{1}{\pi R(t)}\int_{\Gamma_1}\frac{R(x)T(x)}{x-t}dx, \quad K(s,t) = \frac{1}{\pi}\int_{\Gamma_1}\frac{R(x)}{(s-x)(x-t)}dx \end{aligned} \tag{22}$$

Substitution of Eq. 22 into the second formula in Eq. 18 results in the following SIE

$$\sin\alpha(x)\mu_2(x) - \cos\alpha(x)\frac{1}{\pi}\int_{\Gamma_2}\left[\frac{1}{s-x} + M(s,x)\right]\mu_2(s)ds = F(x), \quad x \in \Gamma_2 \tag{23}$$

where the following notations are introduced

$$M(s,x)=\frac{1}{\pi}\int_{\Gamma_1}\frac{K(s,t)}{R(t)(t-x)}dt\,,\quad F(x)=-\frac{1}{\pi}\int_{\Gamma_1}\frac{f(t)}{t-x}dt \tag{24}$$

The solvability of Eq. 23 depends on the index of the corresponding Riemann BVP. It has been shown [7] that the total index of the problem for $M$ cracks is equal to 2K+$M$, where 2K is the difference of the number of revolutions of the principal directions counterclockwise and clockwise that in the considered can be determined as follows

$$2\mathrm{K}=Index\left(-\exp 2i\alpha(x)\right)\Big|_{-\infty}^{+\infty}=\tfrac{1}{\pi}[\alpha(+\infty)-\alpha(-\infty)] \tag{25}$$

For solvability the total index must not be negative. Then, in general, the number of free parameters for the Riemann problem should be equal to 2K+$n_2$+1, however due to the presence of $n_2$ conditions of single valuedness of the displacements ($n_1$ conditions have already been taken into account in transformation of Eq. 19 to Eq. 22) the total number of free parameters is reduced to 2K+1, i.e. the solution of the characteristic SIE contains an arbitrary polynomial of the 2K degree, provided that 2K ≥0 (otherwise the problem has no solutions). We further use this fact to build up the numerical approach for solving Eq. 23.

**Solution for a single crack**. For illustration, let us consider the case of a single crack with the given principal directions on the crack surfaces. In this case $\Gamma_1$=0 and one can select $\Gamma_2$=(-1,1). Then $\mu_1$(t)=0, $\mu_1$(t)=μ(t) and the system (Eq. 18) is reduced to a single homogeneous SIE

$$\sin\alpha(x)\mu(x)-\cos\alpha(x)\frac{1}{\pi}\int_{-1}^{1}\frac{\mu(t)}{t-x}dt=0,\quad |x|<1 \tag{26}$$

with the condition

$$\int_{-1}^{1}\mu(t)\,dt=0 \tag{27}$$

It is evident that the application of any direct numerical approach to the problem given by Eq. 26-27 will result in the trivial solution. However, the problem can have non-trivial solutions depending on the conditions imposed on the boundary values of $\alpha(x)$. Since Eq. 26 presents itself as the dominant SIE for an open contour, its solution is found in accordance with the following formula [8]

$$\mu(x)=\frac{1}{\sqrt{1-x^2}}\exp\left(\frac{1}{2i\pi}\int_{-1}^{1}\frac{\ln G(s)}{s-x}ds\right)\sum_{k=0}^{2\mathrm{K}+1}C_k x^k,\quad x\in(-1,1),\quad G(s)=\frac{i\sin\alpha(s)-\cos\alpha(s)}{i\sin\alpha(s)+\cos\alpha(s)}=-e^{2i\alpha(s)} \tag{28}$$

One of the arbitrary real coefficients $C_k$ can be excluded to satisfy Eq. 27. Therefore the total number of linearly independent solutions is 2K+1.

Let us, for example, assume that $\cos(\alpha(x))=x$, then Eq. 26 yields

$$\sqrt{1-x^2}\mu(x)-\frac{x}{\pi}\int_{-1}^{1}\frac{\mu(t)}{t-x}dt=0,\quad |x|<1 \tag{29}$$

Then evaluating the integral in Eq. 28

$$\frac{1}{2i\pi}\int_{-1}^{1}\frac{\ln G(s)}{s-x}ds=\frac{1}{i\pi}\int_{-1}^{1}\frac{\ln\left(\sqrt{1-s^2}+is\right)}{s-x}ds=\frac{1}{\pi}\int_{-1}^{1}\frac{s}{\sqrt{1-x^2}(s-x)}ds=1 \tag{30}$$

and taking into account that 2K=0 one obtains

$$\mu(x)=\frac{C_0+C_1x}{\sqrt{1-x^2}},\quad x\in(-1,1) \tag{31}$$

From Eq. 27 it is evident that $C_0$=0. Thus the solution is

$$\mu(x)=\frac{C_1x}{\sqrt{1-x^2}},\quad x\in(-1,1) \tag{32}$$

This solution present the case when the crack surfaces are loaded by a constant shear stresses $T_0$, i.e. when the stress deviator is as follows

$$D(z,\bar{z})=iT_0\left(1-\frac{z}{\sqrt{z^2-1}}\right) \tag{33}$$

The constant $C_1$, however, cannot be identified because no data on the stress magnitudes have been specified in the boundary conditions.

**Numerical approach.**

It is evident from the example considered for a single crack that the application of the Carleman-Vekua regularization [8] to the second SIE in the system specified in Eq. 18 is capable to take into account all homogeneous solutions. The regularization of the non-homogeneous SIE in Eq. 18, in principal, is not necessary because this equation processes a unique solution (provided that the conditions of single-uniqueness of the displacements are satisfied). However the regularization of the first equation reduces the system to a single SIE of the form presented by Eq. 23. Thus, the computational cost of calculating additional iterative integrals can partly be mitigated by the smaller dimension of a system of the linear algebraic equations (SLAE) resulting from the single SIE. Let us further consider the case of discretisation of Eq. 23 and it reduction to a SLAE.

Firstly one can rewrite Eq. 23 in the form of a new system of $n_2$ SIE for each crack contour

$$\sin\alpha_j(x)\mu_j(x)-\cos\alpha_j(x)\left\{\frac{1}{\pi}\int_{c_j}^{d_j}\frac{\mu_k(s)}{s-x}ds+\sum_{m=1}^{n_2}\frac{1}{\pi}\int_{c_m}^{d_m}L(s,x)\mu_m(s)ds\right\}=F_j(x),\quad x\in(c_j,d_j),\quad j=1\ldots n_2 \tag{34}$$

where $L(s,x)=\frac{1-\delta_{mj}}{s-x}+M(s,x)$ and $\delta_{mj}$ is Kronecker delta.

The conditions of single-valuedness take the form

$$\int_{c_j}^{d_j}\mu_j(t)\,dt=0,\quad j=1\ldots n_2 \tag{35}$$

The application of the Carleman-Vekua regularization converts the system of Eq. 34 into the following system of Fredholm equations (see [8] for the proof of regularity of the kernels in the obtained system)

$$\mu_j(x) = \sin\alpha(x) g(x) + \frac{\cos\alpha_j(x) Z_j(x)}{\pi} \int_{c_j}^{d_j} \frac{g_j(s)}{Z_j(s)} \frac{ds}{s-x} + \cos\alpha_j(x) Z_j(x) \sum_{k=0}^{2K_j+1} C_{j,k} x^k, \quad x \in (c_j, d_j),\ j = 1 \ldots n_2$$
$$g_j(x) = F_j(x) - \cos\alpha_j(x) \sum_{m=1}^{n_2} \frac{1}{\pi} \int_{c_m}^{d_m} L(s,x) \mu_m(s) ds,\ \ Z_j(x) = \frac{1}{\sqrt{(x-c_j)(x-d_j)}} \exp\left( \frac{1}{2i\pi} \int_{c_j}^{d_j} \frac{\ln e^{-2i\alpha(s)}}{s-x} ds \right) \quad (36)$$

Here $2K_j$ are particular indices for the $j$-contour determined in the same way as in the considered case of a single crack (previous section), $C_{j,k}$ are unknown constants.
Further the method of mechanical quadrature [5] is applied to the system in Eq. 36 and to Eq. 35, which leads to a SLAE of the form

$$\mathbf{AM} = \mathbf{G_{load}} + \mathbf{C}^T \mathbf{G_{homo}} \quad (37)$$

Here the matrix **A** is quadratic ($n_2$x$N$)x($n_2$x$N$), where $N$ is the number of roots of the Chebyshev polynomials used in quadrature, the vector **M** consists of the set of sought values of the unknown functions $\mu_j^0(t) = \mu_j(t)\sqrt{(d_j - t)(t - c_j)}$ at nodal points, the vector $\mathbf{G_{load}}$ addresses the known loads applied while the matrix $\mathbf{G_{homo}}$ is the set of the values of the eigen-functions $\cos\alpha_j(x) Z_j(x) x^k$ of the dominant SIE at collocation points, $\mathbf{C} = \{C_{j,k}\}$ is the matrix of arbitrary coefficients.
The solution of the SLAE is built as the sum of the solution responsible for the applied loads and the homogeneous solutions. The latter can be presented via the set of 2K basis vectors (1,0,….0), (0,1,…0),…(0,0,…1). Therefore the total solution can be presented in the following symbolic form

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{G_{load}} + \mathbf{C}^T \mathbf{A}^{-1} \mathbf{G_{homo}} \quad (38)$$

The mode II stress intensity factors at the crack tips can be found by interpolation to obtain the end values of $\mu_j^0(t)$ (see [5] for detail).

**Concluding remarks**.

The paper suggests a numerical approach for solving singular integral equations for collinear cracks with the boundary conditions posed in terms of the principal directions given on the crack surfaces. Since the SIE of the problem is homogeneous its solution cannot be obtained by direct application of well-established numerical methods, which necessitates regularization. The latter is performed by using the analytical solution of the dominant SIE (the Carleman-Vekua regularization, [8]), which allows one to transform the original system of SIE into a system of Fredholm equations of the second kind. As the result of this transformation, the new system becomes non-homogeneous with the right hand side dependent of the eigen-functions of the dominant SIE. This allows one to build numerically a non-trivial solution of the problem by superposition of the solutions of the Fredholm equations with the full set of the eigen-functions in the right hand side.

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