# Modeling of heat transfer in a non-homogeneous material with a crack. The study of singularity at the vicinity of the crack tips

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**Key words:** crack, heat flow, singularity, asymptotics.

## Introduction

In this work three boundary value problems modeling steady heat distribution in a plane with a crack $l=[-1;1]\times\{0\}$ at variable coefficient of internal heat conductivity are studied. In all the problems it is assumed that the differences in temperatures and heat flows between the upper and lower tips of the crack $l$ are given.

Problem (1)-(3) is obtained under the assumption that the coefficient of internal heat conductivity is given by the function $G(x_2)=k\equiv const$; problem (8)-(10) is obtained under the assumption that $G(x_2)=G_0e^{kx_2}$, where $G_0\equiv const\neq 0,\ k\equiv const\neq 0$; problem (18)-(20) is obtained under the assumption that $G(x_2)=e^{k(x_2)}$, where the function $k(x_2)$ satisfies the conditions stated below. Note that problem (8)-(10) is a special case of problem (18)-(20).

The study of problem (1)-(3) and problem (8)-(10) was carried out as follows: the reduction of the initial problem to a generalized problem, the construction of a solution to the resulting generalized problem; the isolation of the components, which tend to infinity fastest when approaching the crack tips, in the representation of the first-order derivatives of the solution to the problem under consideration (obtaining an asymptotic representation for heat flows); the proof of the fact that the solution to the generalized problem is a solution to the problem under consideration.

Problem (18)-(20) has been studied by reducing it to problem (8)-(10).

## The study of stationary heat distribution in a plane with a crack with a constant coefficient of internal heat conductivity

Consider the problem:

$$\Delta v(x_1,x_2)=0,\ (x_1,x_2)\in\mathbb{R}^2/l, \tag{1}$$

$$v(x_1,+0)-v(x_1,-0)=q_0(x_1),\ x_1\in(-1;1), \tag{2}$$

$$\frac{\partial v(x_1,+0)}{\partial x_2}-\frac{\partial v(x_1,-0)}{\partial x_2}=q_1(x_1),\ x_1\in(-1;1). \tag{3}$$

*Definition.* The solution to problem (1)-(3) is a function $v(x_1,x_2)$ that belongs to $C^2\left(\mathbb{R}^2/l\right)$ and satisfies equation (1) in the region $\mathbb{R}^2/l$, for which in the sense of the principal value for $x_1$

belonging to $(-1;1)$, the boundary conditions (2), (3) are fulfilled, and such that the functions $v(x_1,x_2)$, $x_2\dfrac{\partial v(x_1,x_2)}{\partial x_2}$ and $\dfrac{\partial v(x_1,x_2)}{\partial x_2}-\dfrac{\partial v(x_1,-x_2)}{\partial x_2}$ are bounded in the vicinity of the crack $l$.

Similarly, the solution is determined for all other problems considered in this paper.

*Definition*. Let $q(x_1)$ belong to the space $C([-1;1])$. By $q(x_1)\delta_{[-1;1]}(x_1,x_2)$ we denote a generalized function from $D'(\mathbb{R}^2)$ acting according to the following rule: for any function $\varphi(x_1,x_2)$ belonging to the space $D(\mathbb{R}^2)$,

$$\left(q(x_1)\delta_{[-1;1]}(x_1,x_2),\varphi(x_1,x_2)\right)=\int_{-1}^{1}q(\sigma_1)\varphi(\sigma_1,0)d\sigma_1 .$$

*Remark 1*. In what follows we will assume that the functions $q_0(x_1)$ and $q_1(x_1)$ belong to the space $C^3([-1;1])$.

From the definition of the solution to problem (1)-(3) it follows that the function $v(x_1,x_2)$ belongs to the space $D'(\mathbb{R}^2)$. Standardly calculating the generalized derivatives of the function $v(x_1,x_2)$ (see [1]), we can prove the following theorem.

*Theorem 1*. The solution to problem (1)-(3) is a solution to the following generalized problem:

$$\Delta v(x_1,x_2)=q_1(x_1)\delta_{[-1;1]}(x_1,x_2)+\frac{\partial}{\partial x_2}\left(q_0(x_1)\delta_{[-1;1]}(x_1,x_2)\right). \tag{4}$$

*Remark 2*. The fundamental solution of the operator $\Delta$ in $\mathbb{R}^2$ is the function $\dfrac{1}{2\pi}\ln|x|$ (see [2]).

*Remark 3*. The generalized function $q(x_1)\delta_{[-1;1]}(x_1,x_2)$ is finite (see [1]), and for it $\operatorname{supp} q(x_1)\delta_{[-1;1]}(x_1,x_2)\subset l$.

Using Remark 2, Remark 3 and the convolution theorem with a finite functional (see [1]), we can prove the following theorem.

*Theorem 2*. Let $q_0(x_1),\ q_1(x_1)\in C([-1;1])$, then the solution to problem (4) can be represented in the form:

$$v(x_1,x_2)=\frac{x_2}{2\pi}\int_{-1}^{1}\frac{q_0(\sigma_1)}{(x_1-\sigma_1)^2+x_2^2}d\sigma_1+\frac{1}{4\pi}\int_{-1}^{1}q_1(\sigma_1)\ln\left[(x_1-\sigma_1)^2+x_2^2\right]d\sigma_1. \tag{5}$$

Using (5) and integration by parts, we can prove the following theorem.

*Theorem 3*. For first-order partial derivatives of the function $v(x_1,x_2)$ obtained in Theorem 2, for $(x_1,x_2)$ belonging to $\mathbb{R}^2/l$, the following representations are valid:

$$\frac{\partial v(x_1,x_2)}{\partial x_1} = -\frac{q_0(1)}{2\pi}\frac{x_2}{(1-x_1)^2+x_2^2} + \frac{q_0(-1)}{2\pi}\frac{x_2}{(1+x_1)^2+x_2^2} - \\ -\frac{q_1(1)}{4\pi}\ln[(1-x_1)^2+x_2^2] + \frac{q_1(-1)}{4\pi}\ln[(1+x_1)^2+x_2^2] + R_1(x_1,x_2), \tag{6}$$

$$\frac{\partial v(x_1,x_2)}{\partial x_2} = -\frac{q_0(1)}{2\pi}\frac{1-x_1}{(1-x_1)^2+x_2^2} - \frac{q_0(-1)}{2\pi}\frac{1+x_1}{(1+x_1)^2+x_2^2} + \\ +\frac{q_0'(1)}{4\pi}\ln[(1-x_1)^2+x_2^2] - \frac{q_0'(-1)}{4\pi}\ln[(1+x_1)^2+x_2^2] + R_2(x_1,x_2), \tag{7}$$

where $R_1(x_1,x_2)$, $R_2(x_1,x_2)$ are the functions bounded on any compact.

Using Theorem 2 and Theorem 3, we can prove the following theorem.

*Theorem 4.* The function $v(x_1,x_2)$ obtained in Theorem 2, belongs to the space $C^\infty\left(\mathbb{R}^2/l\right)$ and is a solution to problem (1)-(3).

**The study of stationary heat distribution in a plane with a crack with an exponential coefficient of internal heat conductivity**

Consider the problem:

$$\Delta\tilde{u}(x_1,x_2) + k\frac{\partial\tilde{u}(x_1,x_2)}{\partial x_2} = 0,\ \ x=(x_1;x_2)\in\mathbb{R}^2/l, \tag{8}$$

$$\tilde{u}(x_1,+0) - \tilde{u}(x_1,-0) = q_0(x_1),\ x_1\in(-1;1), \tag{9}$$

$$\frac{\partial\tilde{u}(x_1,+0)}{\partial x_2} + \frac{k}{2}\tilde{u}(x_1,+0) - \frac{\partial\tilde{u}(x_1,-0)}{\partial x_2} - \frac{k}{2}\tilde{u}(x_1,-0) = q_1(x_1),\ x_1\in(-1;1). \tag{10}$$

By replacing $\tilde{u}(x_1,x_2) = e^{-\frac{kx_2}{2}}\tilde{V}(x_1,x_2)$ problem (8)-(10) is reduced to the problem:

$$\Delta\tilde{V}(x_1,x_2) - \frac{k^2}{4}\tilde{V}(x_1,x_2) = 0,\ \ x\in\mathbb{R}^2\setminus l, \tag{11}$$

$$\tilde{V}(x_1,+0) - \tilde{V}(x_1,-0) = q_0(x_1),\ x_1\in(-1;1), \tag{12}$$

$$\frac{\partial\tilde{V}(x_1,+0)}{\partial x_2} - \frac{\partial\tilde{V}(x_1,-0)}{\partial x_2} = q_1(x_1),\ x_1\in(-1;1). \tag{13}$$

By analogy with Theorem 1 we prove the following theorem.

*Theorem 5.* The solution to problem (11)-(13) is a solution to the following generalized problem:

$$\Delta\tilde{V}(x_1,x_2) - \frac{k^2}{4}\tilde{V}(x_1,x_2) = q_1(x_1)\delta_{[-1;1]}(x_1,x_2) + \frac{\partial}{\partial x_2}\left(q_0(x_1)\delta_{[-1;1]}(x_1,x_2)\right). \tag{14}$$

*Remark 4.* The fundamental solution of the operator $\Delta-\frac{k^2}{4}$ in $\mathbb{R}^2$ is the function $E(x_1,x_2)=-\frac{1}{2\pi}K_0\left(\frac{|k|}{2}|x|\right)$, where $K_0(z)$ is the Macdonald function (see [2]).

In what follows, by $K_n(z)$ we will denote the Macdonald function.

Proceeding in the same way as in Theorem 2, we can prove the following theorem.

*Theorem 6.* Let $q_0(x_1),\ q_1(x_1)\in C([-1;1])$, then the solution to problem (14) can be represented in the form:

$$\tilde{V}(x_1,x_2)=\frac{|k|\,x_2}{4\pi}\int_{-1}^{1}K_1\left(\frac{|k|}{2}\sqrt{(x_1-\sigma_1)^2+x_2^2}\right)\frac{q_0(\sigma_1)}{\sqrt{(x_1-\sigma_1)^2+x_2^2}}d\sigma_1-$$
$$-\frac{1}{2\pi}\int_{-1}^{1}K_0\left(\frac{|k|}{2}\sqrt{(x_1-\sigma_1)^2+x_2^2}\right)q_1(\sigma_1)d\sigma_1. \tag{15}$$

Proceeding in the same way as in Theorem 3, from (15) and asymptotic estimates for the Macdonald functions (see [3]):

$$K_0(z)=\ln\frac{1}{z}+O(1),\ K_n(z)=\frac{1}{2}\frac{(n-1)!}{(z/2)^n}+O(z^{2-n}),$$

where $0<z<1,\ n\in\mathrm{N}$, we obtain the following theorem.

*Theorem 7.* For the first-order partial derivatives of the function $\tilde{V}(x_1,x_2)$ obtained in Theorem 6, for $(x_1,x_2)$ belonging to $\mathbb{R}^2/l$, the following representations are valid:

$$\frac{\partial\tilde{V}(x_1,x_2)}{\partial x_1}=-\frac{q_0(1)}{2\pi}\frac{x_2}{(1-x_1)^2+x_2^2}+\frac{q_0(-1)}{2\pi}\frac{x_2}{(1+x_1)^2+x_2^2}-$$
$$-\frac{q_1(1)}{4\pi}\ln[(1-x_1)^2+x_2^2]+\frac{q_1(-1)}{4\pi}\ln[(1+x_1)^2+x_2^2]+R_1(x_1,x_2), \tag{16}$$

$$\frac{\partial\tilde{V}(x_1,x_2)}{\partial x_2}=-\frac{q_0(1)}{2\pi}\frac{1-x_1}{(1-x_1)^2+x_2^2}-\frac{q_0(-1)}{2\pi}\frac{1+x_1}{(1+x_1)^2+x_2^2}+$$
$$+\frac{q_0'(1)}{4\pi}\ln[(1-x_1)^2+x_2^2]-\frac{q_0'(-1)}{4\pi}\ln[(1+x_1)^2+x_2^2]+R_2(x_1,x_2), \tag{17}$$

where $R_1(x_1,x_2)$, $R_2(x_1,x_2)$ are the functions bounded on any compact.

From (15), (16) and (17) follows Theorem 8.

*Theorem 8.* The function $\tilde{V}(x_1,x_2)$ obtained in Theorem 6, belongs to the space $C^\infty(\mathbb{R}^2/l)$ and is a solution to problem (11)-(13).

**The study of stationary heat distribution in a plane with a crack with a variable coefficient of internal heat conductivity**

Consider the problem:

$$\Delta U(x_1,x_2)+k'(x_2)\frac{\partial U(x_1,x_2)}{\partial x_2}=0,\ \ x=(x_1;x_2)\in\mathbb{R}^2/l, \tag{18}$$

$$U(x_1,+0)-U(x_1,-0)=e^{-\frac{k(0)}{2}}q_0(x_1),\ x_1\in(-1;1), \tag{19}$$

$$\frac{\partial U(x_1,+0)}{\partial x_2}+\frac{k'(0)}{2}U(x_1,+0)-\frac{\partial U(x_1,-0)}{\partial x_2}-\frac{k'(0)}{2}U(x_1,-0)= \\ =e^{-\frac{k(0)}{2}}q_1(x_1),\ x_1\in(-1;1). \tag{20}$$

*Remark 5.* In what follows we will assume that the function $k(x_2)$ belongs to the space $C^4(\mathbb{R})$; there exist the constants $\varepsilon_1$ and $\varepsilon_2$ such that for $x_2$, belonging to $\mathbb{R}$, the following estimates $\varepsilon_2>\tilde{k}^2(x_2)>\varepsilon_1>0$, where $\tilde{k}^2(x_2)=\left(k'(x_2)\right)^2+2k''(x_2)$ are fulfilled.

By replacing $U(x_1,x_2)=e^{-\frac{k(x_2)}{2}}V(x_1,x_2)$ problem (18)-(20) is reduced to the problem:

$$\Delta V(x_1,x_2)-\frac{\tilde{k}^2(x_2)}{4}V(x_1,x_2)=0,\ \ x\in\mathbb{R}^2\setminus l, \tag{21}$$

$$V(x_1,+0)-V(x_1,-0)=q_0(x_1),\ x_1\in(-1;1), \tag{22}$$

$$\frac{\partial V(x_1,+0)}{\partial x_2}-\frac{\partial V(x_1,-0)}{\partial x_2}=q_1(x_1),\ x_1\in(-1;1). \tag{23}$$

The solution to problem (21)-(23) will be sought in the form:

$$V(x_1,x_2)=u(x_1,x_2)+W(x_1,x_2), \tag{24}$$

where the function $u(x_1,x_2)$ is a solution to the problem:

$$\Delta u(x_1,x_2)-\frac{\tilde{k}^2(0)}{4}u(x_1,x_2)=0,\ \ x\in\mathbb{R}^2\setminus l, \tag{25}$$

$$u(x_1,+0)-u(x_1,-0)=q_0(x_1),\ \ x_1\in(-1;1), \tag{26}$$

$$\frac{\partial u(x_1,+0)}{\partial x_2}-\frac{\partial u(x_1,-0)}{\partial x_2}=q_1(x_1),\ \ x_1\in(-1;1), \tag{27}$$

and the function $W(x_1,x_2)$ is a solution to the problem:

$$\Delta W(x_1,x_2)-\frac{\tilde{k}^2(x_2)}{4}W(x_1,x_2)=0,25\left(\tilde{k}^2(x_2)-\tilde{k}^2(0)\right)u(x_1,x_2),\ \ x\in\mathbb{R}^2\setminus l, \tag{28}$$

$$W(x_1,+0)-W(x_1,-0)=0,\ \ x_1\in(-1;1), \tag{29}$$

$$\frac{\partial W(x_1,+0)}{\partial x_2}-\frac{\partial W(x_1,-0)}{\partial x_2}=0,\ \ x_1\in(-1;1). \tag{30}$$

Note that problem (25)-(27) coincides with problem (11)-(13) for $k=\tilde{k}(0)$.

Using the results obtained in the study of problem (11)-(13), we can prove the following theorem (see [4]).

*Theorem 9.* Let $k(x_2)\in C^{k+2}(\mathbb{R})$, where $k=2,...$, then equation (28) has a solution, once continuously differentiable in the vicinity of $l$ and $k$ times continuously differentiable outside of $l$.

From Theorem 9 and the results obtained in the study of problem (11)-(13), we obtain the following theorem.

*Theorem 10.* Let $k(x_2)\in C^{k+2}(\mathbb{R})$, where $k=2,...$, then problem (18)-(20) has the solution $U(x_1,x_2)$ and $U(x_1,x_2)\in C^k\left(\mathbb{R}^2\backslash l\right)$. The functions $U(x_1,x_2)$, $\dfrac{\partial U(x_1,x_2)}{\partial x_1}$, $\dfrac{\partial U(x_1,x_2)}{\partial x_2}$ in the vicinity of $l$ have the same asymptotic representation as the functions $e^{-\frac{k(x_2)}{2}}u(x_1,x_2)$, $e^{-\frac{k(x_2)}{2}}\dfrac{\partial u(x_1,x_2)}{\partial x_1}$, $e^{-\frac{k(x_2)}{2}}\dfrac{\partial u(x_1,x_2)}{\partial x_2}$, respectively, where $u(x_1,x_2)$ is a solution to problem (25)-(27).

## Analysis of the results

From Theorem 3, Theorem 7 and Theorem 10 follows the coincidence, up to a constant cofactor, of the principal terms of the asymptotic expansion of heat flows in each of the problems considered.

Also from these theorems it follows that the rate, at which heat flows tend to infinity, depends on the way of approaching the crack tips.

We show this by the example of the behavior of the function $\dfrac{\partial v(x_1,x_2)}{\partial x_1}$ in the vicinity of the left tip of the crack $l$, the point with the coordinates $(-1;0)$. From Theorem 3 it follows that in this case the rate of convergence to infinity is determined by the values $A=\dfrac{x_2}{(1+x_1)^2+x_2^2}$ and $B=\ln[(1+x_1)^2+x_2^2]$.

Consider the behavior of the values $A$ and $B$, when the point $(-1;0)$ is approached along the curve:

$$\begin{cases} x_1=-1+t^{\alpha}, \\ x_2=t,\ t\in\left(0,\delta\right],\ \alpha>0,\ \delta<1. \end{cases} \tag{31}$$

From (31) we obtain that if $\alpha \le \frac{1}{2}$ , then the value $A = \frac{t}{t^{2\alpha} + t^2}$ is bounded for $t \to 0$. Consequently, the function $\frac{\partial v(x_1, x_2)}{\partial x_1}$ tends to infinity as $\ln[t^{2\alpha} + t^2]$ for $t \to 0$.

From (31) we obtain that if $\frac{1}{2} < \alpha \le 1$, then $A = \frac{t}{t^{2\alpha} + t^2} = \frac{1}{t^{2\alpha - 1}} \cdot \frac{1}{1 + t^{2-2\alpha}} \sim ct^{1-2\alpha}$ for $t \to 0$, where $c = 1$ for $\frac{1}{2} < \alpha < 1$ and $c = \frac{1}{2}$ for $\alpha = 1$. Consequently, the function $\frac{\partial v(x_1, x_2)}{\partial x_1}$ tends to infinity as $ct^{1-2\alpha}$ for $t \to 0$.

From (31) we obtain that if $1 < \alpha$, then $A = \frac{t}{t^{2\alpha} + t^2} = \frac{1}{t} \cdot \frac{1}{1 + t^{2\alpha - 2}} \sim t^{-1}$ for $t \to 0$ . Consequently, the function $\frac{\partial v(x_1, x_2)}{\partial x_1}$ tends to infinity as $t^{-1}$ for $t \to 0$.

Below, Fig. 1 shows the behavior of the principal terms of the asymptotic expansion of the function $\frac{\partial v(x_1, x_2)}{\partial x_1}$ in the vicinity of the point $(-1;0)$, provided that $q_0(-1) = q_1(-1) = 1$.

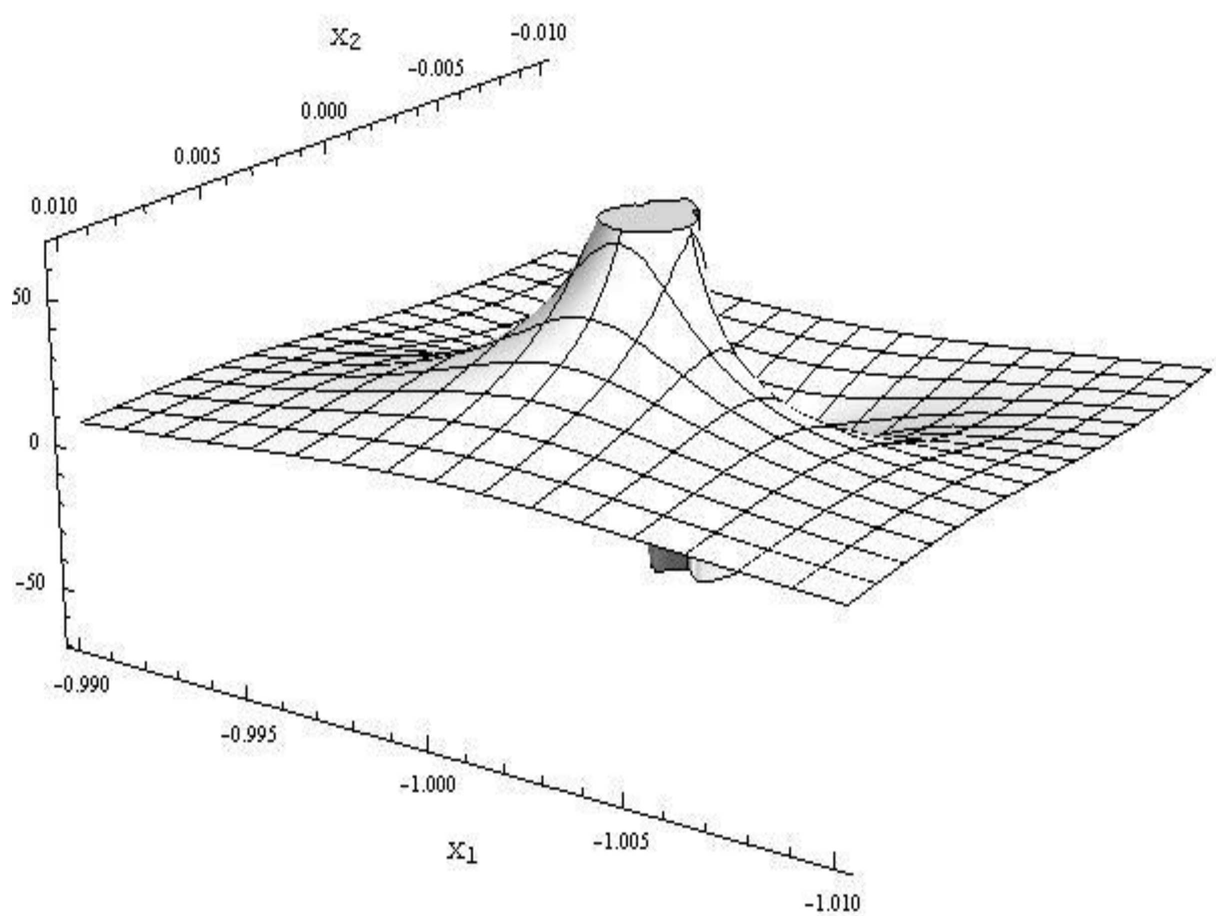


Fig. 1

Below, Fig. 2 shows the behavior of the principal terms of the asymptotic expansion of the function $\frac{\partial v(x_1, x_2)}{\partial x_2}$ in the vicinity of the point $(-1;0)$, provided that $q_0(-1) = q_0'(-1) = 1$.

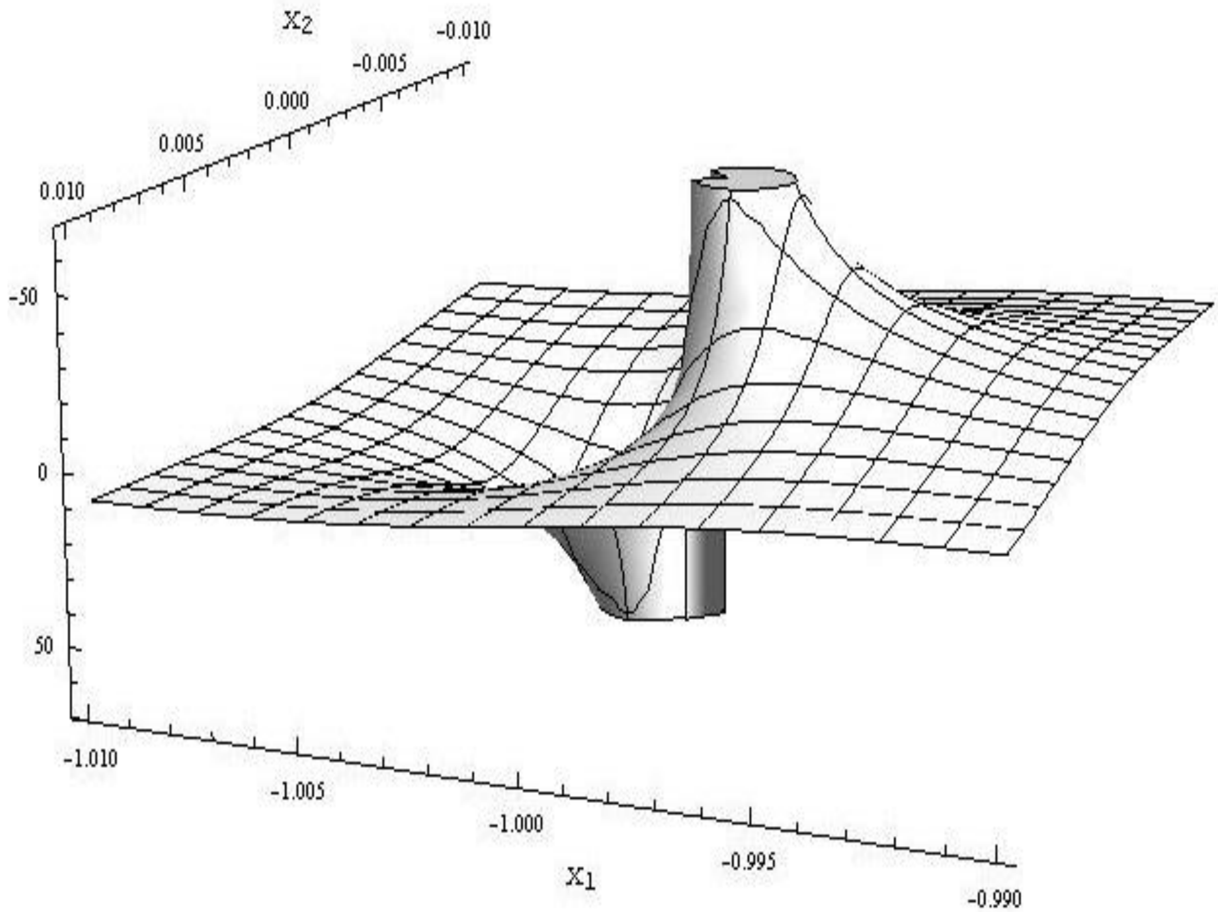


Fig. 2

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