Model of Symmetric Crack Formation in a Plate and a Wedge under Bending by a Point Indenter

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Keywords: crack, crack formation, fracture, bending, plate, wedge, model, indenter.

Abstract. The energy approach is used to propose a model of brittle fracture of a thin plate (and a wedge) under bending by a point indenter, which permits studying some possible mechanisms determining the number of sectors into which the plate breaks. Since the energy necessary to form new cracks and the total elastic bending energy of the n triangular sectors-beams arising under bending vary in opposite directions with variation in both the crack length L and n, it follows that the total energy required to form n sectors has a minimum depending on L and n, and it is this minimum that determines the number n of the arising sectors. In the simplest scheme, the number of developing cracks turns out to be independent of the plate physical-mechanical characteristics, and its thickness and varies from 2 to 4 as the wedge opening angle varies from 0 to 2π. An analysis is performed and a qualitative interpretation of the obtained results is given. Possible refinements of the proposed model in various directions are discussed.

1. Introduction. Statement of the Model. Energy Relations

When studying the interaction of ice fields with icebreakers, ice-resistant structure footings, and other objects and in several other cases (fracture of glass and other brittle materials), there arise problems leading to the scheme of fracture of a plate made of a brittle material by a point indenter or by a lumped force in which several cracks begin to develop under the indenter and cut out the corresponding number of sectors in the plate [1,2], being different in different cases.

For the theoretical estimate of the number of sectors arising in crack formation in a plate under the action of an indenter, we assume that

(1) The plate is loaded by a point indenter.

(2) As the plate strength is exhausted, fracture occurs instantaneously with the formation of a symmetric system of radial cracks.

(3) One can neglect the irreversible (nonelastic, thermal, etc.) losses (i.e., the plate behavior is quasibrittle) and the possible dynamics (vibrations and waves).

(4) The main contribution to the energy balance equation is made by the energy of formation of new surfaces (cracks) and by the elastic bending energy of the arising sectors. In this case, for simplicity, we assume that the strain of the undisturbed (and hence preserving the former rigidity) peripheral part of the plate is small and its contribution to the energy balance equation can be neglected. Thus, in fact, it becomes an unstrained foundation for the arising sectors, which are rigidly fixed to it by their bases.

(5) The minimum-energy-consuming fracture scheme is realized; i.e., the total energy is minimal in this case.

First, consider the case in which the load is applied at the plate center. Under the assumption that the arising sectors are equal to each other, we can write

$$W = nLh\gamma + nU, \tag{1}$$

where $W = W(n, L)$ is a function of the total energy expenditure in the crack formation, n is the number of arising cracks (and sectors), L is the length of arising cracks, h is the plate thickness, γ is the effective surface energy of fracture, and U is the bending energy of each of the arising triangular sectors-beams.
Write out the expression for the elastic bending energy U of one sector

$$U = u^2 /(2\Pi) \tag{2}$$

where Π is the bending compliance of the sector, u – the indenter vertical displacement (descent).

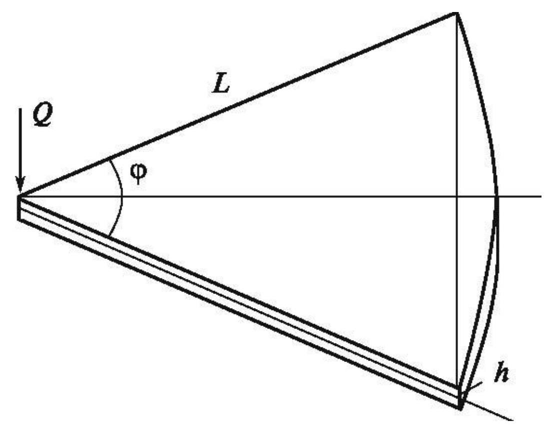

Fig. 1

Treating the sector as a cantilever beam triangular in plan (i.e., a cantilever of variable width) working in bending (Fig. 1), we write out the expression for its compliance in the form ([3], Table 18)

$$\Pi = \frac{u}{Q} = \frac{3L^2 \cos^3(\varphi/2)}{Eh^3 \sin(\varphi/2)} \tag{3}$$

where Q is the force acting at the end of each beam, L is the length of the lateral surface of the sector (equal to the length of the arising cracks), E is the Young modulus, h is the plate thickness, and φ is the central angle of the sector.
In the case of formation of n equal cracks in a solid plate, $\varphi = 2\pi/n$. Taking into account this relationship and substituting successively of Eq. 3 in Eq. 2 and then in Eq. 1 we obtain for the function of total energy expenditures

$$W = n\left[Lh\gamma + \frac{Eh^3 \sin(\pi/n)}{6L^2 \cos^3(\pi/n)} u^2 \right] \tag{4}$$

2. Minimization of the Expression for the Energy Expenditure. The Case of a Solid Plate
Let minimize the obtained expression for W with respect to the crack length L and their number n. We rewrite Eq. 4 as

$$W = AL\left(1+\frac{B}{L^3}\right) \tag{5}$$

$$A = nh\gamma \tag{6}$$

$$B = \frac{Eh^2 \sin(\pi/n)}{6\gamma \cos^3(\pi/n)} u^2 \tag{7}$$

By computing the derivative $\partial W/\partial L$ and by equating it with zero, we obtain

$$\frac{B}{L^3} = \frac{1}{2} \tag{8}$$

$$L = \sqrt[3]{(2B)} = \left[\frac{Eh^2 \sin(\pi/n)}{3\gamma \cos^3(\pi/n)} u^2\right]^{\frac{1}{3}} \tag{9}$$

Now we substitute A, L, and B/L^3 computed by Eqs. 6, 9 and 8 into Eq. 5 and obtain

$$W = Cn\left[\frac{\sin(\pi/n)}{\cos^3(\pi/n)}\right]^{1/3} \tag{10}$$

$$C = \frac{3}{2} h\gamma \left(\frac{Eh^2 u^2}{3\gamma}\right)^{1/3} \tag{11}$$

Since C is independent of n, it is convenient to divide W by πC and consider the inverse function of $\pi C/W$. In Eq. 10, we pass from the discrete variable n to the continuous variable x by the formulas

$$\pi/n \to x,\ n > 2,\ 0 < x < \pi/2 \tag{12}$$

and from Eq. 10 we obtain the following expression for the cube of this new function

$$\left(\frac{\pi C}{W(n)}\right)^3 = \left(\frac{\pi}{n}\right)^3 \frac{\cos^3(\pi/n)}{\sin(\pi/n)} \to \left|\frac{\pi}{n} \to x\right| \to \frac{(x \cos x)^3}{\sin x} \equiv \Omega(x) \tag{13}$$

We replace the minimization of the function $W(n)$ by the maximization of the function $\Omega(x)$ with respect to x. One can readily show that this function has a single maximum at $x_0 \approx 0.84$. Since, according to Eq. 12, the discrete variable n and the continuous variable x are related as $x \leftrightarrow \pi/n$, it follows that the extreme value of n is one of the two integers nearest to $\pi/x_0 \approx \pi/0.84 \approx 3.74$. By checking the minimum of the function

$$w(n) \equiv \left[\frac{W(n)}{C}\right]^3 = n^3 \frac{\sin(\pi/n)}{\cos^3(\pi/n)}$$

for $n = 3, 4$, we obtain

$$w(4) < w(3), \quad n_{\min} = 4 \tag{14}$$

3. The Case of a Wedge (*n*–1 Cracks and *n* Sectors)

We assume that the plate has the shape of a wedge with opening angle Φ, $0 \le \Phi \le 2\pi$, and the point indenter acts at the vertex of this wedge. Then the appearance of n-1 cracks in this plate corresponds to the formation of n sectors with opening angle Φ/n. In problems on an icebreaker in ice fields, the case $\Phi = 2\pi$ corresponds to the case of an icebreaker in the mouth of the channel crushed by it ([1], p. 72), and $\Phi = \pi$ corresponds to the case of an icebreaker coming over the ice field edge or a slant smooth support. Operating similar to paragraph 2, we obtain in this case instead of Eqs. 10, 13

$$w_\Phi(n,\Phi) \equiv \left[\frac{W_\Phi(n,\Phi)}{C}\right]^3 = n^3\left(1-\frac{1}{n}\right)^2 \frac{\sin[\Phi/(2n)]}{\cos^3[\Phi/(2n)]} \tag{15}$$

$$\Omega_\Phi(x,\Phi) = \frac{(x\cos x)^3}{(\Phi/2-x)^2 \sin x} \tag{16}$$

and replace the minimization of $W_\Phi(n, \Phi)$ by the maximization of $\Omega_\Phi(x, \Phi)$ with respect to x.
Then, for the complete plane (plate) with half-infinite cut $\Phi = 2\pi$, Eq. 16 acquires the form

$$\Omega_\Phi(x,2\pi) = \frac{(x\cos x)^3}{(\pi-x)^2 \sin x} = \frac{\Omega(x)}{(\pi-x)^2} \tag{17}$$

for which the relation of the type Eq. 14 remains valid, $w_\Phi(4, 2\pi) < w_\Phi(3, 2\pi)$. Thus, for $\Phi = 2\pi$ the function $W_\Phi(n, 2\pi)$ of energy expenditures in crack formation attains its minimum for $n_{min} = 4$ as well.
For a plate-half-plane, $\Phi = \pi$, and Eq. 16 becomes

$$\Omega_\Phi(x,\pi) = \frac{(x\cos x)^3}{(\pi/2-x)^2 \sin x} = \frac{\Omega(x)}{(\pi/2-x)^2} \tag{18}$$

Here the function $W_\Phi(n,\pi)$ attains its minimum at $n_{min} = 2$.
Thus, as the opening angle Φ of the loaded wedge decreases, the number n of sectors minimizing the total energy expenditures necessary for their formation decreases from $n = 4$ for $\Phi = 2\pi$ to $n = 2$ for $\Phi = \pi$. The natural question arises: How does n_{min} vary as Φ varies from 0 to 2π; in particular, for what values of the wedge opening angle Φ does n_{min} vary from $n = 2$ to $n = 3$ ($\Phi_{2\to3}$) and from $n = 3$ to $n = 4$ ($\Phi_{3\to4}$)? To answer this question, it suffices to compute $w_\Phi(n,\Phi)$ for Φ varying from π to 2π by Eq. 15 for $n = 2,3,4$. In Fig. 2, we present the graph of the dependence of $\ln[w_\Phi(n,\Phi)]$ on the wedge opening angle Φ. The points of intersection of $w_\Phi(2,\Phi)$ with $w_\Phi(3,\Phi)$ and of $w_\Phi(3,\Phi)$ with $w_\Phi(4,\Phi)$ give precisely the values of the wedge opening angles Φ at which the number n_{min} of the formed sectors (or cracks) is changed, $\Phi_{2\to3} \approx 4.43$ and $\Phi_{3\to4} \approx 5.94$. The sectors with maximum opening angle (near $\Phi_{2\to3}/2 \approx 2.21$ rad) are formed for Φ close to $\Phi_{2\to3} \approx 4.43$.

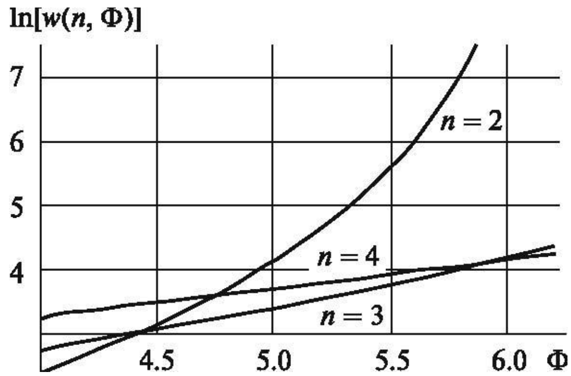

Fig. 2

4. Example

After the number n of cracks formed in fracture is found, one can use the Eq. 9 to estimate the order of the lengths of these cracks L, for example, in the case of a solid plate made of window glass. But here we encounter another difficulty. By Eq. 9, the length of the formed cracks is determined by the value of the critical deflection u^*. In the framework of this model, nothing can be said about u^*, since we do not specify any local fracture criterion and do not study the stress field distribution. But if for some u^* the plate is destroyed according to the above model, then in this plate there arise four symmetric cracks of length determined by Eq. 9 with $n = 4$.

This implies an interesting observation. Suppose that an artificial stress concentrator, a conic cave (countersinking) is placed on the lower part of the plate under the indenter. Then different u^* are realized depending on the dimensions (depth and opening angle) of this cave, and, respectively, systems of cracks of different L will be formed.

In a similar way, in the case of symmetric extension of the strip edges (Fig. 3) by u under the action of loads applied on a small part of dimension $d > \delta$ (δ is the unknown dimension of the defect in the material), we assume that the plate is mechanically isotropic, in strength and in imperfection, and we do not precisely know what defects are contained in the plate. But since the crack-like defects perpendicular to the load direction are most dangerous, the fracture occurs for different displacements u^* of the force application points depending on the maximum initial dimension δ of such defects. The lengths of the arising cracks L are different and correspond to the energy U accumulated at this time.

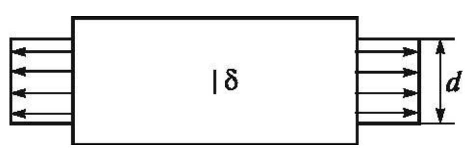

Fig. 3

To obtain estimates of the length L of the arising cracks by Eq. 9 in the framework of the proposed model, it is necessary to introduce some reasonable values of the critical deflection u^* and some actual values of the glass mechanical characteristics E, h, and γ. The effective surface fracture energy γ can be expressed in terms of the crack growth resistance K_{IC} by the Irwin formula

$$\gamma = \frac{K_{IC}^2(1-\mu^2)}{E} \qquad (19)$$

where μ is the Poisson ratio. For glass, we set $E = 6\times10^{10}$ N/m^2, μ = 0.3 ([4], p. 116), and $K_{IC} = 0.8$ kg/mm$^{3/2}$ = 0.8·10 N/(10^{-3} m)$^{3/2}$ = 8×10$^{4.5}$ N/m$^{3/2}$ ([5], p. 620); the typical values of the glass thickness h are $h \cong (1 \div 10)$ mm = (10^{-3} ÷ 10^{-2}) m; for u*, we take several values proportional to h by the formula $u^* = \alpha h$, where α = 1; 10^{-1}; 10^{-2}; 10^{-3} .
By substituting γ expressed by Eq. 19 into Eq. 9 and by taking $n = 4$, we see that for such parameter values the lengths of the cracks arising in glass can be of the order of several centimeters already for α = 10^{-3}. For the thickness $h = 4\cdot10^{-3}$ m typical of window glass, the relative deflections α = 10^{-3}, 10^{-2} imply the values $L \cong 0.1$ m and $L \cong 0.5$ m, respectively, for L.
The above model can be extended to the case of a circular plate of finite dimensions. In this case the dimensionless function of the energy of crack $w(n, l)$, similar to the function Eq. 4, takes the form: for a clamped plate

$$w(n,l) = nl + B\frac{1}{\frac{\kappa}{n}\frac{\cos^3(\pi/n)}{\sin(\pi/n)}l^2 + \left(1 - l^2 + 6l^2\ln l\right) - C_3\left(3 - 3l^2 + 2\ln l\right)}10^{-2m}$$

for a freely supported plate

$$w(n,l) = nl + B\frac{10^{-2m}}{\frac{\kappa}{n}\frac{\cos^3(\pi/n)}{\sin(\pi/n)}l^2 + \left(6l^2\ln l - 4l^2 + 2\ln l + 4\right) + C\left[3(1-\mu)l^2 - 2(1+\mu)(1+\ln l) + (1-\mu)\right]}$$

for a clamped annular plate

$$w = nl + B\frac{10^{-2m}}{\frac{\kappa}{n}\frac{\cos^3(\pi/n)}{\sin(\pi/n)}\left(2r_i l + l^2\right) + \left[1 - (r_i + l)^2 + 6(r_i + l)^2\ln(r_i + l)\right] - C_3\left[3 - 3(r_i + l)^2 + 2\ln(r_i + l)\right]}$$

where this time

$$w(n,l) = \frac{W}{R^* h\gamma}$$

$l = L/R^*$

$$B \equiv \frac{2\pi}{3}\frac{E^2 h^4}{(1-\mu^2)^2 R^{*3} K_{Ic}^2}$$

$$\kappa = \frac{4\pi}{1-\mu^2} \geq 13$$

$$C_3(\mu, l) = 2l^2\frac{1 - (\mu+1)\ln l}{(1-\mu) + (1+\mu)l^2}$$

$$C = \frac{(3+\mu)l^2 + (1-\mu) - 2(1+\mu)l^2\ln l}{\left(1-\mu^2\right)\left(1-l^2\right)},$$

R^* is the radius of the annular plate, $r_i = R_i/L$ – dimensionless inner radius of the annular plate, m – the parameter that determines the magnitude of the critical plate deflection or elastic energy stored in the plate at the crash moment.

5. Discussion of the Obtained Results and Accepted Assumptions

First, we note that at the first glance it seems rather strange that the obtained "optimal" values of the number of sectors (or cracks) are independent (except for the wedge opening angle Φ) of any geometrical and physical parameters of the model: the plate thickness, its rigidity, and fracture viscosity. To understand this fact, we recall that, for a given wedge (with angle Φ at the vertex), it is required to find a system of cracks of number and length such that the energy necessary to create such a system (this energy is the sum of the energies of formation of new surfaces and the energy of bending of the arising sectors) be minimal over all n and L. This minimization with respect to L implies the condition that the energy required to form the cracks is equal to the doubled energy used to bend the arising sectors (Eqs. 5, 8) and the "optimal" crack length L is expressed in terms of the thickness and the plate physical characteristics by a power law (Eq. 9). As a result, it follows from these relations that the total energy W is proportional to the crack formation energy whose expression contains all the above parameters only as factors raised to various powers and which then disappear in the process of optimization. Since we only take into account the strain of the plate central part cut by cracks, the solution does not contain the plate fixation conditions in any way.

The character of variation in the number of arising sectors n with varying wedge opening angle Φ may also be explained qualitatively. The function W of total energy expenditures is the sum of the crack formation energy $nLh\gamma$ and the energy nU of elastic bending of sectors-beams. For small $\Phi < \Phi_{2\to3}$, the arising sectors are narrow, and their total elastic energy weakly decreases as n increases, but the crack formation energy always increases linearly in n. Therefore, the minimum of W is realized for the minimum feasible value $n = 2$ at which the energy is minimal. For large Φ and small n, the elastic energy U is very sensitive to variations in n (moreover, as $\Phi \to 2\pi$, in the framework of the accepted scheme, the value $n = 2$ is associated with $U \to \infty$). As a result, the minimum point moves upwards, first, towards $n = 3$ for $\Phi = \Phi_{2\to3}$ and then towards $n = 4$ for $\Phi = \Phi_{3\to4}$. In this case, $U_n = 2$, $\Phi \to 2\pi$, which conceptually reflects the fact of a sharp increase in the rigidity of the arising sectors and hence in the accumulated elastic energy and formally shows that the beam model cannot be used.

As follows from the results in Sec. 4, the computed length of the arising cracks can be comparable with the general dimensions of the plate L_p (for example, for typical window glass). This means that, on the one hand, there is a natural upper limit for possible values of lengths of the arising cracks, and on the other hand, it is necessary to take into account the plate dimensions and the corresponding boundary conditions.

Consider one purely kinematic consequence of the boundedness of possible crack lengths. As the crack length $L = L_p$ is attained in the energy balance Eq. 4, the further increase in W in the left-hand side can be counterbalanced in the right-hand side for fixed $L = L_p$ only by an increase in n. For a small excess over the calculated $L > L_p$, the energy excess is small and obviously can be radiated as elastic vibrations and waves (which is not detected by the proposed model). But, starting from a certain moment, the accumulated energy becomes sufficient for the formation of a picture with five rather than four symmetric cracks, and then with six, etc. Then, in general, the number n of arising cracks is always determined as the integral part of the solution of an equation of the form Eq. 4 with respect to n for L_p and given values of W (or u^*) and the other quantities contained in it. Thus, for a sufficiently small imperfection (high strength) of the plate, which permits accumulating a large amount of elastic energy, the finiteness of its dimensions may result in an increase in the number of cracks arising in it.

In the case of nonsymmetrical conditions of the plate support (when the lengths of the arising cracks are limited only in several directions), the symmetry of the crack formation picture is generally violated.
For example, consider a plate in the form of a long strip clamped along the long sides. Let us trace the evolution of the crack formation picture as the accumulated elastic energy and, respectively, the lengths of the arising cracks increase. As long as these lengths are much less than the characteristic dimensions of the plate, the picture remains symmetric (for simplicity, we assume that the cracks are oriented as in Fig. 4). But for sufficiently large cracks such that $L/2^{1/2} \cong b/2$ (where b is the plate width), the cracks in the symmetric picture cannot grow further outside the plate boundaries. The energy expenditure of the original symmetric fracture scheme with $n = 4$ becomes exhausted. Then, we obtain the problem of minimal-power-consuming fracture scheme under the conditions that two transverse sectors are bounded in height by the plate half-width, i.e., the problem of minimization of W with respect to L and n with constraints in the form of inequalities such as $L_{i,y} \leq b/2$. The picture begins to distort. If we formally remain in the class of rectilinear solutions-cracks, then we obtain solutions-intervals with ends sliding along the long sides of the plate away from the ordinate axis (the cracks begin to bend towards the plate axis). Just as above, starting from certain values of W (or u^*), a symmetric solution with $n > 4$ may appear, etc.

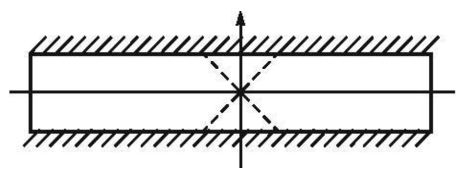

Fig. 4

In practice, the arising cracks are obviously curvilinear, and this fact must be taken into account by more realistic models of crack formation.

This research was supported by the RFBR under project No. 11-08-01243.

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