# Mechanics of Small Cracks in Estimation of Endurance Limits

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The results of steel tests for high-cycle fatigue in normal environment serve as a basis for designing structures operating under corresponding conditions. The relationship between the endurance limits of cylindrical specimens (tubular and solid) in tension-compression, alternating bending and torsion is used in design.

A substantial contribution to the analysis of endurance in the above-mentioned loading modes was made by V.T. Troshchenko. He established the fact that the tension-compression endurance limit approximated the cyclic yield stress can be depends on a relatively small allowance for plastic strain. In his studies, as demonstrated by Refs. [1] and [2], much attention was paid to microplastic strains at the endurance limit.

The bending and shear stresses of a solid specimen are non-uniform and match the states of most structural components under their operating conditions. Solid specimens subjected to bending and torsion represent, in essence, the simplest types of structures. Besides the non-uniformity, the endurance limit is perceptibly affected by the asymmetry of loading cycle and stress state.

Allowance for such asymmetry in regulatory calculations is made based on the tension-compression test data. These results are generalized to the case of a complex stressed state by using the concept of stress intensity. Experiments show, however, that the effect of cycle asymmetry in the pure shear case is far less pronounced, than in tension-compression. Therefore, stress intensity involvement will not provide sufficiently accurate accounting for the influence of cycle asymmetry on the endurance limit. The recent progress of fracture mechanics in investigating fatigue crack growth makes it possible to use theoretical models to solve these problems.

At present, the analysis of fatigue-induced growth of small cracks whose depth is commensurable with the metal grain diameter found a wide range of applications. The approach taken in Ref. [3] is applied in this case for evaluating the endurance limits in different loading types.

It is assumed, that a small crack appears on the specimen surface in the maximum shear plane under cycling loading to develop later into a semicircular opening mode crack. Growing further in a non-uniform stress field, the crack will take a semi-elliptical shape.

Specimens will show unlimited endurance if the semicircular opening mode crack stops growing. Radius $a_0$ of this crack is roughly equal to twice the grain diameter and is governed by the material structure alone. Its value is independent of the specimen testing conditions. This crack model allows applying the methods of fracture mechanics in material strength analyses.

A power law diagram of cyclic deforming employed in the calculations appears as

$$\frac{\Delta\varepsilon}{2\varepsilon_{-1e}} = \frac{\Delta\varepsilon_e}{2\varepsilon_{-1e}} + \alpha_{-1}\left(\frac{\Delta\sigma}{2\sigma_{-1}}\right)^n, \tag{1}$$

where $\Delta\sigma$ and $\Delta\varepsilon$ – stress and total strain ranges, $\sigma_{-1}$– endurance limit in tension-compression, $\varepsilon_{-1e} = \frac{\sigma_{-1}}{E}$ – elastic strain at the endurance limit, *E*- modulus of elasticity, $n \approx 6$– strain-hardening exponent of the cyclic strain diagram.

According to Eq. (1), at $\frac{\Delta\sigma}{2} = \sigma_{-1}$, the plastic strain range is $\Delta\varepsilon_p = 2\alpha_{-1}\varepsilon_{-1e}$. As shown by experiments, the value of $\alpha_{-1}$ for steel varies within the limits of 0.2÷0.02, with the lower value being commensurable with the plastic strain measurement error.

The model under consideration suggests that in the case of unlimited endurance in tension-compression a specimen develops on its surface a semicircular crack of radius $a_0$, which will not grow further. Such a crack is characterized by the effective Stress Intensity Factor (SIF) range which is equivalent to the *J*-integral ($K_j = \sqrt{EJ}$) and is calculated by the following formula of Ref. [3]

$$\Delta K_{jef} = \sqrt{E\left(u_{th}^2 \Delta W_e + \Delta W_p\right)\pi a_0}\ , \tag{2}$$

where $u_{th} = \frac{u_{0th}}{1-R}$– coefficient of long crack (deep sharp notch) opening at the fatigue threshold with cycle asymmetry characteristic *R*; $u_{0th}$– coefficient of long crack opening at the fatigue threshold in zero-to-tension stress cycle; $\Delta W_e = \frac{\Delta\sigma^2}{2E}$, *E*– modulus of elasticity; $\Delta W_p \approx \frac{\Delta\sigma\Delta\varepsilon_p}{1+m}$, $m = \frac{1}{n}$.

From Eq. (2), it follows that the effect of crack closing is possible only in the case of elastic strain. In subsequent calculations, it will be assumed that $u_{0th}$=0.5 in Eq. (2) [3].

The effective SIF range, equivalent to the *J*-integral, can be expressed with regards to Eq. (1) as

$$\Delta K_{jef} = \sqrt{\left(u_{th}^2 \frac{\Delta\sigma^2}{2} + \frac{\alpha_{-1}\Delta\sigma^2 \cdot}{1+m}\left(\frac{\Delta\sigma}{2\sigma_{-1}}\right)^{n-1}\right)\pi a_0}\ . \tag{3}$$

The threshold of crack fatigue growth under tension-compression is conditioned by

$$\Delta K_{jef} = \Delta K_{thc}, \tag{4}$$

where $\Delta K_{thc}$– critical fatigue threshold (which varies within $2 \div 4\ \mathrm{MPa}\sqrt{\mathrm{m}}$ for steel in air).

We shall now determine the relationship between the endurance limits in the symmetrical and pulsating axial stress cycles. By equating the right-hand sides of Eq. (3) for the symmetrical and pulsating cycles to each other and considering that $a_0$ has the same values in both cases and $n$=6, we arrive at:

$$\frac{\sigma_{a0}}{\sigma_{-1}} = \frac{\sqrt{\frac{u_{0th}^2}{8} + \frac{\alpha_{-1}}{1,17}}}{\sqrt{\frac{u_{0th}^2}{2} + \frac{\alpha_{-1}}{1,17}\left(\frac{\sigma_{a0}}{\sigma_{-1}}\right)^5}} . \tag{5}$$

If $\alpha_{-1}$=0,02 at the endurance limit, then $\frac{\sigma_{a0}}{\sigma_{-1}} = 0,62.$ This value may be treated as the lower boundary of the ratio between the endurance limits typical for brittle materials. With $\alpha_{-1}$=0.2, $\frac{\sigma_{a0}}{\sigma_{-1}} = 0,92.$ The average value may be taken to be $\alpha_{-1}$=0,1, hence $\frac{\sigma_{a0}}{\sigma_{-1}} = 0,85.$

The endurance limits will be now compared for the cases of uniform (axial tension-compression) and non-uniform (alternating bending) stressed states. With a bar bent to the point of plastic flow, nominal stresses ($\sigma_n = \frac{M}{W}$, $M$ – bending moment, $W$– section modulus) coincide with real stresses on its surface. When the outer fibers of the bar develop plastic strain, the nominal stresses will exceed the real ones.

Estimating the $J$-integral of a bending-induced small surface crack has found application in a method, which introduces a conditional diagram of cyclic deforming at nominal stresses-strains, similar to Eq. (1). In this diagram, the yield stress (in this case, $\sigma_{-1}$) is higher by a factor of $q$, with $q$ standing for the coefficient of bearing capacity. This coefficient is equal to the ratio between the stresses induced by incipient plastic hinge formation and by the onset of plastic flow when a bar of ideal plastic material is bent. In the case of a round solid bar $q$=1.7, while with bending of a bar of rectangular cross-section $q$=1.5. As with tension-compression the $J$-integral is calculated by Eq. (3), where the yield stress is increased $q$ times. Thus, the range of the effective SIF value, equivalent to the $J$-integral with the allowance for crack opening under nominal elastic strains, will take the following form for alternating bending:

$$\Delta K_{jef} = \sqrt{\left(u_{th}^2 \frac{\Delta\sigma_n^2}{2} + \frac{\alpha_{-1}\Delta\sigma_n^2}{1+m}\left(\frac{\Delta\sigma_n}{2q\sigma_{-1}}\right)^{n-1}\right)\pi a_0} . \tag{6}$$

By equating the right-hand sides of Equations (3) and (6), we obtain an equation for calculating the ratio between the endurance limits in alternating bending and tension-compression:

$$\frac{\sigma_{-1b}}{\sigma_{-1}}=\frac{\sqrt{\frac{u_{0th}^2}{8}+\frac{\alpha_{-1}}{1{,}17}}}{\sqrt{\frac{u_{0th}^2}{8}+\frac{\alpha_{-1}}{1{,}17}\left(\frac{\sigma_{-1b}}{q\sigma_{-1}}\right)^5}}\,. \tag{7}$$

Calculations show that with $\alpha_{-1}$ varying from 0.2 to 0.02, this ratio is reduced from 1.42 to 1.19. Thus, as plastic strains at the endurance limit in tension-compression decrease, so does the relative value of the endurance limit in alternating bending.

Note that in contrast to the traditional approach, here the effective SIF range values were equated rather than comparing the ranges of specimen surface strains in tension-compression and alternating bending. The discussed method provides a good fit with experimental data and eliminates the problems identified in Ref. [2].

Turning to alternating torsion of tubular specimens, we shall assume that in this case, as before, the fatigue threshold is governed by the opening fracture mode. With this assumption, there will be a semicircular crack of radius $a_0$ in the plane of the main normal stresses at the onset of fatigue crack growth.

Given pure shearing, the maximum normal stress is equal to the shear stress ($\Delta\sigma=\Delta\tau$). The elastic strain energy done by stresses normal to the crack edges in regards to the corresponding strains appears as $\Delta W_e=\frac{(1+\nu)\Delta\sigma^2}{2E}$, where $\nu$=0.3 is Poisson's ratio. The range of normal plastic strain is equal to half the range of shear plastic strain ($\Delta\varepsilon_p=0{,}5\Delta\gamma_p$). The plastic strain work done by stresses normal to the crack edges in regards to the corresponding plastic strain is $\Delta W_p=\frac{\Delta\sigma\Delta\varepsilon_p}{1+m}$. The diagram of cyclic deforming suggests that $\frac{\Delta\gamma_p}{\sqrt{3}}=\alpha_{-1}\frac{2\sigma_{-1}}{E}\left(\frac{\sqrt{3}\Delta\tau}{2\sigma_{-1}}\right)^n$ and, hence, $\Delta\varepsilon_p=\alpha_{-1}\frac{3\Delta\tau}{2E}\left(\frac{\sqrt{3}\Delta\tau}{2\sigma_{-1}}\right)^{n-1}$ Therefore, $\Delta W_p=\frac{1{,}5}{1+m}\alpha_{-1}\Delta\tau^2\left(\frac{\sqrt{3}\Delta\tau}{2\sigma_{-1}}\right)^{n-1}$.

With alternating pure shearing, the effective range of SIF is:

$$\Delta K_{jef}=\sqrt{\left(u_{th}^2\frac{(1+\nu)\Delta\tau^2}{2}+\frac{1{,}5}{1+m}\alpha_{-1}\Delta\tau^2\left(\frac{\sqrt{3}\Delta\tau}{2\sigma_{-1}}\right)^{n-1}\right)\pi a_0}\,. \tag{8}$$

By equating the right-hand sides of Equations (3) and (8) and given $u_{th}=\frac{u_{0th}}{2}$, we obtain the relationship between the endurance limits in symmetrical cycles of pure shearing and tension-compression ($n$=6):

$$\frac{\tau_{-1}}{\sigma_{-1}} = \frac{\sqrt{\frac{u_{0th}^2}{8} + \frac{\alpha_{-1}}{1{,}17}}}{\sqrt{\frac{1{,}3u_{0th}^2}{8} + 1{,}5\frac{\alpha_{-1}}{1{,}17}\left(\frac{\sqrt{3}\tau_{-1}}{\sigma_{-1}}\right)^5}}. \tag{9}$$

The effect of cycle asymmetry can be described by the ratio of shear stress in pulsating cycle to that in a symmetrical cycle. To this end, it is possible to use the following expression resulting from relation (8):

$$\frac{\tau_{a0}}{\tau_{-1}} = \frac{\sqrt{\frac{1{,}3u_{0th}^2}{8} + 1{,}5\frac{\alpha_{-1}}{1{,}17}\left(\frac{\sqrt{3}\tau_{-1}}{\sigma_{-1}}\right)^5}}{\sqrt{\frac{1{,}3u_{0th}^2}{2} + 1{,}5\frac{\alpha_{-1}}{1{,}17}\left(\frac{\sqrt{3}\tau_{0}}{\sigma_{-1}}\right)^5}}. \tag{10}$$

This equation has a numerical solution, with the $\tau_{a0}$ value calculated using quantity $\tau_{-1}$ found from Eq. (9). Calculations show (Table 1) that the effect of cycle asymmetry is smaller in this case compared to tension-compression. This can be explained by the fact that the function of elastic strains in pure shearing is smaller due to lower normal stresses.

Torsion experiments normally involve solid cylinders rather than tubular specimens, and the results of such tests are usually compared with the data obtained in alternating bending tests. The endurance limits in torsion and bending can be compared with the ones of Eq. (11) below, where $q_b$=1.7 and $q_t$=4/3, while $\sigma_{-1b}$ is evaluated by Eq. (7).

$$\frac{\tau_{-1t}}{\sigma_{-1b}} = \frac{\sqrt{\frac{u_{0th}^2}{8} + \frac{\alpha_{-1}}{1{,}17}\left(\frac{\sigma_{-1b}}{q_b\sigma_{-1}}\right)^5}}{\sqrt{\frac{1{,}3u_{0th}^2}{8} + 1{,}5\frac{\alpha_{-1}}{1{,}17}\left(\frac{\sqrt{3}\tau_{-1}}{q_t\sigma_{-1}}\right)^5}}. \tag{11}$$

The results of the above calculations are presented in Table 1, which also contains averaged data for carbon steel of various grades as given in Refs. [4] and [5].

Table 1. Relative values of endurance limits as found for cylindrical specimens under different loading conditions.

| $\alpha_{-1}$ | $\frac{\sigma_{a0}}{\sigma_{-1}}$ | $\frac{\sigma_{-1b}}{\sigma_{-1}}$ | $\frac{\tau_{-1}}{\sigma_{-1}}$ | $\frac{\tau_{a0}}{\tau_{-1}}$ | $\frac{\tau_{-1}}{\tau_{-1t}}$ | $\frac{\tau_{-1t}}{\sigma_{-1}}$ | $\frac{\tau_{-1t}}{\sigma_{-1b}}$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.92 | 1.42 | 0.64 | 0.96 | 0.82 | 0.78 | 0.55 |
| 0.1 | 0.85 | 1.38 | 0.65 | 0.92 | 0.82 | 0.79 | 0.57 |
| 0.06 | 0.77 | 1.33 | 0.66 | 0.89 | 0.83 | 0.80 | 0.60 |
| 0.02 | 0.62 | 1.19 | 0.69 | 0.76 | 0.85 | 0.81 | 0.68 |
| Average value (experiment) | 0.82 | 1.25 | 0.6 | 0.9 | 0.84 | 0.75 | 0.6 |

This table shows a good agreement between calculation and experiment. Therefore, the model discussed here allows to properly describe endurance limits under various stresses, using only data on endurance limits in tension-compression without making any special assumptions. This demonstrates (Such an outcome may be viewed as a case for) the efficiency of using small crack mechanics in structural analyses.

**References.**


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