# Application of thermo-elastic dislocation on a cracked layer under thermal field

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**Abstract.** The effect of steady-state thermal loading on a cracked layer is investigated. A Volterra type thermo-elastic dislocation is introduced in a layer which is free of traction on the boundaries. The assumptions of quasi-static, steady-state condition are employing and the uncoupled theory of thermo-elasticity is considered. The Fourier transformation is utilized to obtain temperature distribution and stress fields in the layer containing dislocation. The temperature field is also derived in the layer with specified temperature at the boundaries and in the absence of any defects. By means of the distributed dislocation technique, the dislocation solution is introduced into the layer to derive integral equations for dislocation density functions on the surfaces of cracks. These equations are Cauchy singular and are solved numerically. The solutions are employed to determine stress intensity factors (SIFs) for cracks in both cases of the impermeable and partially permeable heat flux.

## Introduction

Structures containing defects are vulnerable to thermal loading. The mutual effects of heat flux and thermal stress on interacting cracks may induce excessive SIFs resulting in the instability of cracks and the failure of structures. The stress analysis of cracked elastic bodies subjected to thermal loads was carried out by several researchers. Some investigations regarding uncoupled thermal analysis of half-planes and layers containing cracks relevant to the present study, are enumerated here. A half-plane weakened by an insulated crack under uniform heat flow was solved by Sekine [1] and modes I and II SIFs were determined for a crack with arbitrary orientation. Nied [2] studied the effects of thermal shocks in a strip weakened by an edge crack. The strip was insulated at a boundary and cooled by surface convection at the other boundary. Lam et al. [3] analyzed mixed mode fracture of cracked strips with varying crack surface heat conductivity under uniform heat flow. Transient thermal stress in a strip with an edge crack was the subject of investigation by Rizk and Radwan [4].The elastic strip was insulated at one face and cooled suddenly on the other boundary. A strip containing a crack perpendicular to the boundary under sudden surface heating was solved by Rizk [5]. Jin and Noda [6] considered a graded half-plane with exponentially varying material properties having an edge crack. The medium was under steady heat flux and thermal SIF for various material constants was determined. Liu and Kardomateas [7] modeled an insulated crack in an anisotropic half-plane under a uniform heat flux by a continuous distribution of dislocations to determine thermal SIFs. Their solution was based on the formulations derived by Clements [8] and Sturla and Barber [9] in conjunction with the image method. The problem of two periodic edge cracks in an isotropic elastic strip located symmetrically along the free boundaries and quenched by a ramp function temperature change was investigated by Rizk [10].

In the present article, a layer with free boundaries is considered. A Volterra type thermo-elastic dislocation is introduced and the temperature distribution and stress fields are derived in the layer under the assumptions of quasi-static, steady-state employing the uncoupled theory of thermo-elasticity. The stress components and heat flux are Cauchy singular at dislocation location. The distributed dislocation technique is utilized to perform a set of integral equations for the layer weakened by multiple cracks subjected to general temperature distribution at the boundaries. These equations are Cauchy singular. Only the case of the complete opening of cracks is studied. A crack surface may be partially heat permeable. The integral equations are solved numerically and SIFs are determined. Several examples are solved for layers with cracks having different geometries and the effect of the ratio of heat permeability on SIFs is studied.

**Solution of thermo-elastic dislocations.** An elastic isotropic layer which is free at the boundaries is considered, Fig.1. The layer is stress free at the ambient temperature $T = 0$. A thermo-elastic dislocation is located at a point with coordinates $(0, d)$.The dislocation line is extended in the positive direction of the x-axis. In the uncoupled theory of thermo-elasticity, utilizing Fourier's law of heat conduction and under steady-state situation, the temperature field is governed by the Laplace equation

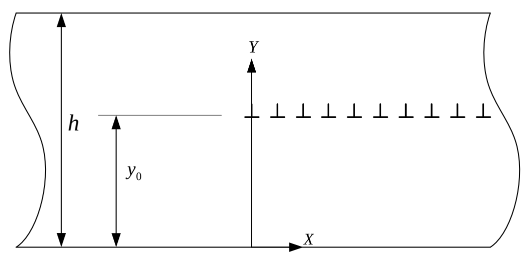


Fig.1. Schematic view of a layer with edge dislocations

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

The temperature field of the thermo-elastic dislocation in the layer is defined by the following equations

$$T(x, y_0^+) - T(x, y_0^-) = \delta_T H(x)\,, \qquad T(x, 0) = T(x, h) = 0, \qquad Q_y(x, y_0^+) = Q_y(x, y_0^-) \tag{2}$$

where $H(x)$ is the Heaviside step function, $\delta_T$ is the temperature discontinuity on the line of dislocation and $Q_y$ is the heat flux in the y-direction. The last equation of Eqs 2 implies the continuity of heat flux crossing the dislocation line and by means of Fourier law reads as

$$\partial T(x, y_0^+)/\partial y = \partial T(x, y_0^-)/\partial y \tag{3}$$

The solution to Eq. 1 subject to the above-mentioned boundary conditions is accomplished by means of the complex Fourier transformation with respect to the variable $x$, Sneddon [11]. Carrying out a straightforward manipulation, the temperature field is obtained. In a material with thermal conductivity $K$, the heat flux $Q_n$ in the direction of a unit vector $\bar{n}$ is

$$Q_n^T = -K\overline{\nabla T}.\bar{n} \tag{4}$$

Substituting the temperature field into Eq. 4 , results in

$$\frac{Q_n^{1T}(x,y)}{\delta_T} = q_n^1(x,y)$$
$$= -\frac{Kn_x}{2\pi}\left(\int_0^\infty \frac{cos(\zeta x)}{\sinh \zeta h}\left(e^{-\zeta h}\sinh\big(\zeta(y-y_0)\big) + \sinh\big(\zeta(y+y_0-h)\big)\right)d\zeta - \frac{y-y_0}{x^2+(y-y_0)^2}\right)$$
$$-\frac{Kn_y}{2\pi}\left(\int_0^\infty \frac{sin(\zeta x)}{\sinh \zeta h}\left(\cosh\big(\zeta(y+y_0-h)\big) + \cosh\big(\zeta(y-y_0+h)\big)\right)d\zeta\right.$$
$$\left.+\frac{x}{x^2+(y-y_0)^2} - \frac{1}{2h}\right), \qquad 0 \le y \le d$$
$$\frac{Q_n^{2T}(x,y)}{\delta_T} = q_n^2(x,y)$$
$$= -\frac{Kn_x}{2\pi}\left(\int_0^\infty \frac{cos(\zeta x)}{\sinh \zeta h}\left(e^{-\zeta h}\sinh\big(\zeta(y_0-y)\big) + \sinh\big(\zeta(y+y_0-h)\big)\right)d\zeta + \frac{y-y_0}{x^2+(y-y_0)^2}\right)$$
$$-\frac{Kn_y}{2\pi}\left(\int_0^\infty \frac{sin(\zeta x)}{\sinh \zeta h}\left(\cosh\big(\zeta(y+y_0-h)\big) + \cosh\big(\zeta(y-y_0-h)\big)\right)d\zeta\right.$$
$$\left.+\frac{x}{x^2+(y-y_0)^2} - \frac{1}{2h}\right), \qquad d \le y \le h$$
(5)

Consequently, by virtue of Eq. 5 heat flux is Cauchy singular at dislocation position, i.e., $Q_n \sim 1/r$, as $r = \sqrt{x^2+(y-d)^2} \to 0$. The uncoupled theory of thermo-elasticity is the solution of the following equation

$$\nabla^4\varphi + \frac{8\mu}{\kappa+1}\alpha\nabla^2 T = 0$$
(6)

In the above equation, $\varphi$ is Airy stress function and define as $\sigma_{xx} = \partial^2\varphi(x,y)/\partial y^2$, $\sigma_{yy} = \partial^2\varphi(x,y)/\partial x^2, \sigma_{xy} = -\partial^2\varphi(x,y)/\partial x\,\partial y$, μ is the shear modulus of elasticity, for plane stress the Kolosov constant $\kappa = (3-\upsilon)/(1+\upsilon)$, and $\alpha = \alpha_T$, where υ is the Poisson's ratio of material and $\alpha_T$ is the coefficient of thermal expansion; these quantities for plane strain are $\kappa = 3-4\upsilon$ and $\alpha = \alpha_T/(1+\upsilon)$. The stress and displacement fields for the thermo-elastic dislocations in the layer are represented by

$$\sigma_{yy}(x,0) = \sigma_{xy}(x,0) = \sigma_{yy}(x,h) = \sigma_{xy}(x,h) = 0,$$
$$\sigma_{yy}(x,y_0^+) = \sigma_{yy}(x,y_0^-), \qquad \sigma_{xy}(x,y_0^+) = \sigma_{xy}(x,y_0^-)$$
$$u(x,y_0^+) - u(x,y_0^-) = b_x H(x), \qquad v(x,y_0^+) - v(x,y_0^-) = b_y H(x)$$
(7)

where $b_x$ and $b_y$ are the Burgers vectors for the climb and glide edge dislocations, respectively. The solution to Eq. 6 subjected to boundary conditions (7) is achieved using Fourier transform. The stress components by virtue of Airy stress function are expressed as

$$\sigma_{xx}(x,y) = \frac{2\mu}{\pi(\kappa+1)}\left[\left(\frac{x^2(y_0-y)-(y_0-y)^3}{(x^2+(y-y_0)^2)^2} + \frac{2(y-y_0)}{x^2+(y-y_0)^2}\right)b_x\right.$$
$$\left.+\left(\frac{2x(y_0-y)^2}{(x^2+(y-y_0)^2)^2} - \frac{x}{x^2+(y-y_0)^2}\right)b_y\right] + F_{xx1}^1(x,y)\delta_T + F_{xx2}^1(x,y)b_x$$
$$+ F_{xx3}^1(x,y)b_y$$

$$\sigma_{yy}(x,y)=\frac{2\mu}{\pi(\kappa+1)}\left[\frac{(y_0-y)^3-x^2(y_0-y)}{(x^2+(y_0-d)^2)^2}b_x+\left(\frac{2x(y_0-y)^2}{(x^2+(y-y_0)^2)^2}+\frac{x}{x^2+(y-y_0)^2}\right)b_y\right]$$
$$+F^1_{yy1}(x,y)\delta_T+F^1_{yy2}(x,y)b_x+F^1_{yy3}(x,y)b_y$$

$$\sigma_{xy}(x,y)=\frac{2\mu}{\pi(\kappa+1)}\left[\left(\frac{2x(y_0-y)^2}{(x^2+(y-y_0)^2)^2}-\frac{x}{x^2+(y-y_0)^2}\right)b_x+\frac{(y-y_0)^3-x^2(y-y_0)}{(x^2+(y-y_0)^2)^2}b_y\right]$$
$$+F^1_{xy1}(x,y)\delta_T+F^1_{xy2}(x,y)b_x+F^1_{xy3}(x,y)b_y,\qquad 0\le y\le d$$

$$\sigma_{xx}(x,y)=\frac{2\mu}{\pi(\kappa+1)}\left[\left(\frac{(y-y_0)^3-x^2(y-y_0)}{(x^2+(y-y_0)^2)^2}-\frac{2(y-y_0)}{x^2+(y-y_0)^2}\right)b_x\right.$$
$$\left.-\left(\frac{2x(y-y_0)^2}{(x^2+(y-y_0)^2)^2}-\frac{x}{x^2+(y-y_0)^2}\right)b_y\right]+F^2_{xx1}(x,y)\delta_T+F^2_{xx2}(x,y)b_x$$
$$+F^2_{xx3}(x,y)b_y$$

$$\sigma_{yy}(x,y)=\frac{2\mu}{\pi(\kappa+1)}\left[\frac{x^2(y-y_0)-(y-y_0)^3}{(x^2+(y-y_0)^2)^2}b_x+\left(\frac{2x(y-y_0)^2}{(x^2+(y-y_0)^2)^2}+\frac{x}{x^2+(y-y_0)^2}\right)b_y\right]$$
$$+F^2_{yy1}(x,y)\delta_T+F^2_{yy2}(x,y)b_x+F^2_{yy3}(x,y)b_y$$

$$\sigma_{xy}(x,y)=\frac{2\mu}{\pi(\kappa+1)}\left[\left(\frac{2x(y-y_0)^2}{(x^2+(y-y_0)^2)^2}-\frac{x}{x^2+(y-y_0)^2}\right)b_x+\frac{(y-y_0)^3-x^2(y-y_0)}{(x^2+(y-y_0)^2)^2}b_y\right]$$
$$+F^2_{xy1}(x,y)\delta_T+F^2_{xy2}(x,y)b_x+F^2_{xy3}(x,y)b_y,$$

$$d\le y\le h \qquad (8)$$

where the non-singular functions $F^r_{...}, r\in\{1,2\}$, are defined in the Appendix. From Eq. 8, it is evident that stress components are Cauchy singular at the dislocation location.

**Solution of the crack problem.** The solution to thermo-elastic dislocation accomplished in the preceding section is employed to analyze layers with several arbitrarily oriented cracks. Utilizing Eqs 8 and 5 the stress and heat flux components caused by the climb, glide and thermal dislocations located at a point with coordinates $(x_0, y_0)$ where dislocation line is parallel to the x-axis read

$$\sigma_{ij}(x,y)=\begin{cases}k^{1t}_{ij}(x,y)\delta_t+k^{1x}_{ij}(x,y)b_x+k^{1y}_{ij}(x,y)b_y & 0<y<d\\ k^{2t}_{ij}(x,y)\delta_t+k^{2x}_{ij}(x,y)b_x+k^{2y}_{ij}(x,y)b_y & d<y<h\end{cases}$$

$$Q_i(x,y)=\begin{cases}q^1_i(x-x_0,y)\delta_t & 0<y<d\\ q^2_i(x-x_0,y)\delta_t & d<y<h\end{cases}\qquad \{i,j\}\in\{x,y\}\qquad (9)$$

The coefficients of $b_x$ , $b_y$ and $\delta_t$ in stress fields (9) are obtained from Eq. 8 and are presented in the Appendix. Let $N$ be the number of cracks in the strip. A crack configuration with respect to coordinate system *x*, *y* may be described in parametric form as

$$x_i=\alpha_i(\eta),\qquad y_i=\beta_i(\eta)\qquad i\in\{1,2,\dots,N\}\qquad -1\le\eta\le1 \qquad (10)$$

The moveable orthogonal coordinate system *n*-*s* is chosen such that the origin moves on the crack while s-axis remains tangent to the crack surface. Suppose climb, glide and thermal dislocations with unknown density functions $B_{xj}(t)$, $B_{yj}(t)$, and $B_{Tj}(t)$, respectively, are distributed on the infinitesimal segment $\sqrt{[\alpha_j'(t)]^2+[\beta_j'(t)]^2}dt$ on the surface of the *j-th* crack where $-1\le t\le 1$.

The heat flux and traction components on the surface of the *i-th* crack at a point $\left(x(\eta), y(\eta)\right)$, due to the presence of the above mentioned distribution of dislocations on all *N* cracks' surfaces yield

$$\sigma_{ni}\left(x(\eta), y(\eta)\right) = \sum_{j=1}^{N} \int_{-1}^{1} \{K_{11ij}(\eta,t)B_{Tj}(t) + K_{12ij}(\eta,t)B_{sj}(t) + K_{13ij}(\eta,t)B_{nj}(t)\} \sqrt{\left[\alpha_j'(t)\right]^2 + \left[\beta_j'(t)\right]^2}\, dt$$

$$\sigma_{si}\left(x(\eta), y(\eta)\right) = \sum_{j=1}^{N} \int_{-1}^{1} \{K_{21ij}(\eta,t)B_{Tj}(t) + K_{22ij}(\eta,t)B_{sj}(t) + K_{23ij}(\eta,t)B_{nj}(t)\} \sqrt{\left[\alpha_j'(t)\right]^2 + \left[\beta_j'(t)\right]^2}\, dt$$

$$Q_{ni}\left(x(\eta), y(\eta)\right) = \sum_{j=1}^{N} \int_{-1}^{1} K^{t}(\eta,t)B_{Tj}(t) \sqrt{\left[\alpha_j'(t)\right]^2 + \left[\beta_j'(t)\right]^2}\, dt$$

$$i \in \{1,2,\dots,n\}, \qquad -1 \le \eta \le 1 \qquad (11)$$

Utilizing Eq. 9, after making the replacement $(x_0, y_0) \to \left(x_j(t), y_j(t)\right)$, leads to the kernels of integral Eq. 13 as

$$K_{11ij} = -\frac{1}{2}\left(k_{xx}^{r1} - k_{yy}^{r1}\right)\cos(2\theta_i) - k_{xy}^{r1}\sin(2\theta_i) + \frac{1}{2}\left(k_{xx}^{r1} + k_{yy}^{r1}\right)$$

$$K_{12ij} = \frac{1}{2}\left[\left(k_{yy}^{r2} - k_{xx}^{r2}\right)\cos(\theta_j) + \left(k_{yy}^{r3} - k_{xx}^{r3}\right)\sin(\theta_j)\right]\cos(2\theta_i) - \left[k_{xy}^{r2}\cos(\theta_j) + k_{xy}^{r3}\sin(\theta_j)\right]\sin(2\theta_i) + \frac{1}{2}\left(k_{xx}^{r2} + k_{yy}^{r2}\right)\cos(\theta_j) + \frac{1}{2}\left(k_{xx}^{r3} + k_{yy}^{r3}\right)\sin(\theta_j)$$

$$K_{13ij} = \frac{1}{2}\left[\left(k_{yy}^{r3} - k_{xx}^{r3}\right)\cos(\theta_j) + \left(k_{xx}^{r2} - k_{yy}^{r2}\right)\sin(\theta_j)\right]\cos(2\theta_i) - \left[k_{xy}^{r3}\cos(\theta_j) + k_{xy}^{r2}\sin(\theta_j)\right]\sin(2\theta_i) + \frac{1}{2}\left(k_{xx}^{r3} + k_{yy}^{r3}\right)\cos(\theta_j) - \frac{1}{2}\left(k_{xx}^{r3} + k_{yy}^{r3}\right)\sin(\theta_j)$$

$$K_{21ij} = \frac{1}{2}\left(k_{xx}^{r1} - k_{yy}^{r1}\right)\sin(2\theta_i) - k_{xy}^{r1}\cos(2\theta_i)$$

$$K_{22ij} = \frac{1}{2}\left[\left(k_{xx}^{r2} - k_{yy}^{r2}\right) cos(\theta_j) + \left(k_{xx}^{r3} - k_{yy}^{r3}\right) sin(\theta_j)\right] sin(2\theta_i) - \left[k_{xy}^{r2}\, cos(\theta_j) + k_{xy}^{r3}\, sin(\theta_j)\right] cos(2\theta_i)$$

$$K_{23ij} = \frac{1}{2}\left[\left(k_{xx}^{r3} - k_{yy}^{r3}\right)\cos(\theta_j) + \left(k_{yy}^{r2} - k_{xx}^{r2}\right)\sin(\theta_j)\right]\sin(2\theta_i) + \left[k_{xy}^{r3}\cos(\theta_j) + k_{xy}^{r2}\sin(\theta_j)\right]\cos(2\theta_i)$$

$$K^{t} = q_y^r\cos(\theta_i) - q_x^r\sin(\theta_i) \qquad (12)$$

Where $r = 1$ is for $y_i < y_j$, $r = 2$ is for $y_i > y_j$ and θ is the crack angle with x-axis. The system of integral equations (13) is Cauchy singular for $i = j$ as $t \to \eta$. By virtue of the Bueckner's superposition theorem the left side of first two Eq. 13 are zero; because un-crack layer is free of traction. The left side of third Eq. 13 is the heat flux with opposite sign obtained from un-crack layer under thermal load on its boundaries. For a partially heat permeable crack Eq. 17 should be multiplied by the coefficient of heat permeability $0 < \beta \le 1$, where $\beta = 1$, is for a thermally insolated crack. The single-valued property of displacement and temperature fields out of a crack surface yields the following closure conditions

$$\int_{-1}^{1} B_{pi}(\eta)\sqrt{\left[\alpha_i'(\eta)\right]^2+\left[\beta_i'(\eta)\right]^2}d\eta=0, \qquad i\in\{1,N\}, p=\{n,s,t\} \tag{13}$$

For embedded cracks, Eqs. 13 and 15 should be solved simultaneously.

**Results and Discussion.** The geometry and material properties of the layer in the ensuing examples, wherein only full opening of cracks occur, are identified as: layer thickness, $h=0.2(m)$, coefficients of thermal conductivity and expansion $k=51.9\,(W/m.K)$ and $\alpha_T=1.96\times 10^{-5}\,(°C^{-1})$, respectively, Poisson's ratio $\nu=0.29$ and plain strain condition prevails. At the ambient temperature $T=0$ layer is stress free. Then temperature on the boundaries of layer changes to

$$T(x,0)=0, \qquad T(x,h)=T_0e^{-\gamma|x|} \tag{14}$$

where $T_0=\mp 100(°C)$ and $\gamma=20(m^{-1})$. In a partially heat permeable crack, the coefficient of permeability is taken as $\beta=0.2$. The deviser for making SIFs dimensionless for straight cracks is $K_0=\mu\alpha|T_0|\sqrt{l}$, where $l$ is half of a crack length. In the first example, an oblique crack with variable distance from y-axis is analyzed, Figs (2a-b). The crack is horizontal and SIFs increase once it approaches upper layer which is attributed to the higher temperature gradient at this region where thermal load is applied. The crack is completely opened from middle surface of strip, since simple bending occurs under thermal loading.

(a) (b)

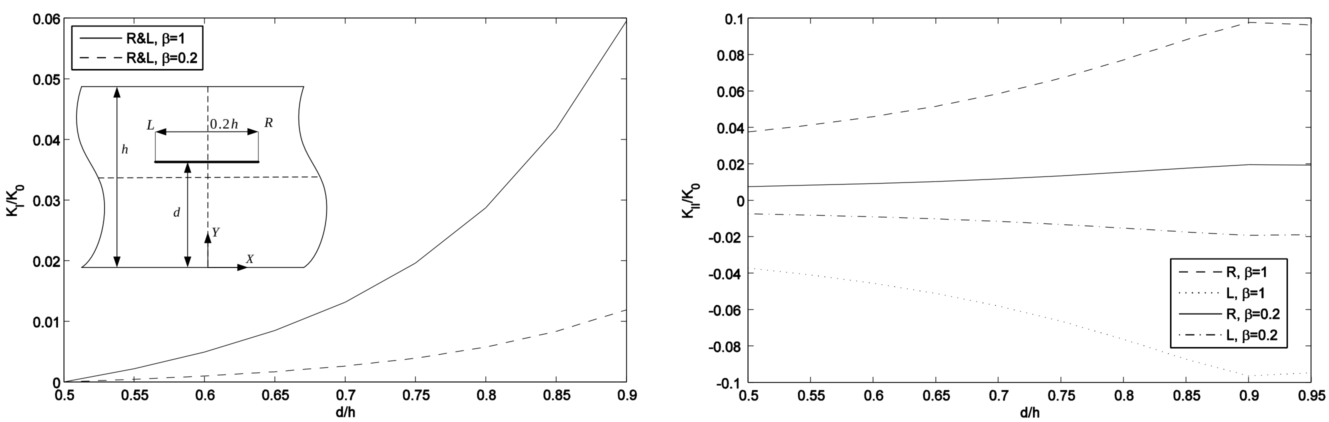


Fig. 2. SIFs of a crack moving vertically, (a-mode I), (b-mode II).

The interaction of two identical parallel cracks A and B with length $0.2h$ where crack A is fixed and crack B is changing location in horizontal direction is investigated, in second example. Figures 3 show the SIFs of crack B where the both of cracks are completely open. The mode I SIF is symmetric with respect to the y-axis. The variation of SIFs of crack A is much smaller than that of crack B. At far distance from the y-axis, heat flow and stress fields attenuate leading to the decrease of SIFs.

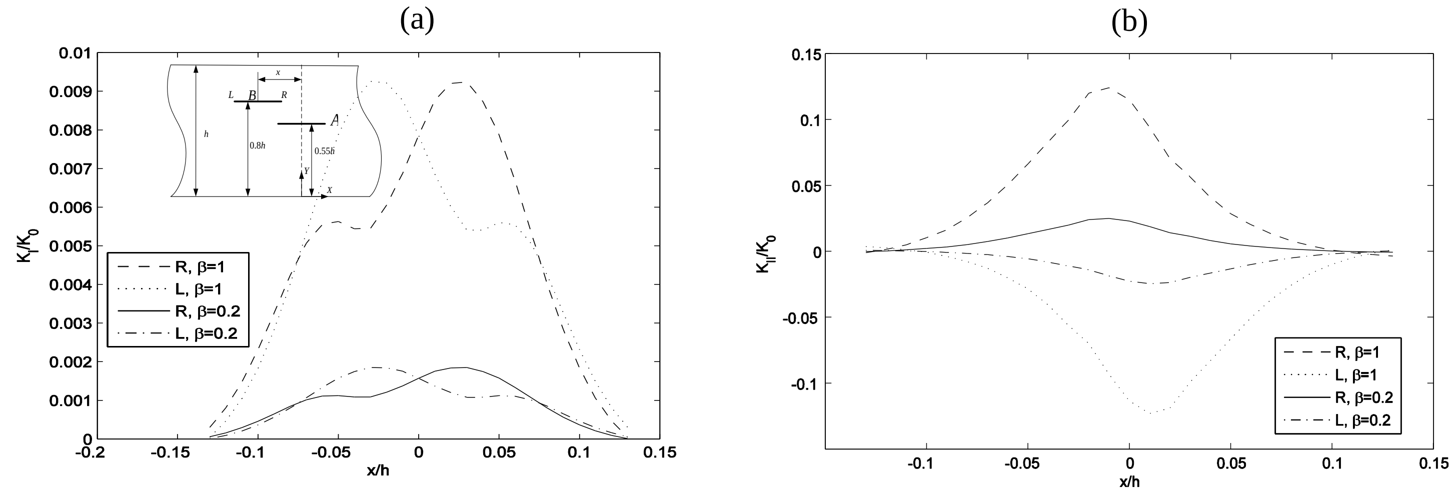


Fig. 3. Interaction of two identical parallel cracks moving horizantaly, (a-mode I SIF), (b-mode II SIF).

In the above examples, the increase in heat permeability of the crack diminishes the maximum values of SIFs.

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**Appendix**

The coefficients in Eq. (8) are as follows

$$L_t^{*r} = \frac{\mu(\pi\zeta\delta(\zeta) - i)e^{-\zeta(y+y_0)}}{(e^{4\zeta h}k - 4e^{2\zeta h}h^2\zeta^2 - 2e^{2\zeta h} + 1)(k+1)\zeta^2}\left[A_1^{*r}e^{2\zeta(y_0+2h)} + A_2^{*r}e^{2\zeta(y+y_0+h)} + A_3^{*r}e^{2\zeta(h+y_0)} + A_4^{*r}e^{2\zeta(y+h)} + A_5^{*r}e^{2\zeta(y+y_0)} + A_6^{*r}e^{2\zeta h} + A_7^{*r}e^{4\zeta h} + A_8^{*r}e^{2\zeta y}\right]$$

$$L_x^{*r} = \frac{\mu(i\pi\zeta\delta(\zeta) + 1)e^{-\zeta(y+y_0)}}{(e^{4\zeta h}k - 4e^{2\zeta h}h^2\zeta^2 - 2e^{2\zeta h} + 1)(k+1)\zeta}\left[B_1^{*r}e^{2\zeta(y_0+2h)} + B_2^{*r}e^{2\zeta(y+y_0+h)} + B_3^{*r}e^{2\zeta(h+y_0)} + B_4^{*r}e^{2\zeta(y+h)} + B_5^{*r}e^{2\zeta(y+y_0)} + B_6^{*r}e^{2\zeta h} + B_7^{*r}e^{4\zeta h} + B_8^{*r}e^{2\zeta y}\right]$$

$$L_y^{*r} = \frac{\mu(\pi\zeta\delta(\zeta) - i)e^{-\zeta(y+y_0)}}{(e^{4\zeta h}k - 4e^{2\zeta h}h^2\zeta^2 - 2e^{2\zeta h} + 1)(k+1)\zeta^2}\left[C_1^{*r}e^{2\zeta(y_0+2h)} + C_2^{*r}e^{2\zeta(y+y_0+h)} + C_3^{*r}e^{2\zeta(h+y_0)} + C_4^{*r}e^{2\zeta(y+h)} + C_5^{*r}e^{2\zeta(y+y_0)} + C_6^{*r}e^{2\zeta h} + C_7^{*r}e^{4\zeta h} + C_8^{*r}e^{2\zeta y}\right]$$

(A.1)

where $r \in \{1,2\}$ and coefficients are

$A_1^{*1} = -A_8^{*2} = 2\alpha(y - y_0)$
$A_2^{*1} = \alpha[4(h-y)(y_0-h)\zeta + 2(y_0 - y)]$
$A_3^{*1} = \alpha[8yh(h-y_0)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y-y_0)]$
$A_4^{*1} = \alpha[8yh(y_0-h)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y-y_0)]$
$A_5^{*1} = 2\alpha(y + 2yy_0\zeta - y_0)$
$A_6^{*1} = \alpha[4(h-y)(y_0-h)\zeta + 2(y-y_0)]$
$A_7^{*1} = 2\alpha(y_0 + 2yy_0\zeta - y)$
$B_1^{*1} = -B_8^{*2} = 2(y - y_0)$
$B_2^{*1} = 4(h-y)(y_0-h)\zeta + 2(y_0-y)$
$B_3^{*1} = 8yh(h-y_0)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y-y_0)$
$B_4^{*1} = 8yh(y_0-h)\zeta^2 - 4h(y-h+y_0)\zeta - 2(y-y_0)$
$B_5^{*1} = 2(y + 2yy_0\zeta - y_0)$
$B_6^{*1} = 4(h-y)(y_0-h)\zeta + 2(y-y_0)$
$B_7^{*1} = 2(y_0 + 2yy_0\zeta - y)$
$C_1^{*1} = 2(1 - (y-y_0)\zeta)$
$C_2^{*1} = -2(y-h)(y_0-h)\zeta^2 + (y_0+y-2h)\zeta - 2$
$C_3^{*1} = 8yh(h-y_0)\zeta^3 + 4h(y+h-y_0)\zeta^2 + 2(y+2h-y_0)\zeta + 2$
$C_4^{*1} = 8yh(h-y_0)\zeta^3 - 4h(y+h-y_0)\zeta^2 + 2(y-y_0+2h)\zeta - 2$
$C_5^{*1} = 4yy_0\zeta^2 - 2(y_0+y)\zeta + 2$
$C_6^{*1} = 2[2(y-h)(y_0-h)\zeta^2 + (y_0+y-2h)\zeta + 2]$
$C_7^{*1} = -4yy_0\zeta^2 - 2(y_0+y)\zeta - 2$
$C_8^{*1} = -2(1 - (y-y_0)\zeta)$

$A_1^{*2} = -A_8^{*1} = 2\alpha(y - y_0)$
$A_2^{*2} = \alpha[4(h-y)(y_0-h)\zeta + 2(y_0 - y)]$
$A_3^{*2} = \alpha[8y_0h(-y+h)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y_0-y)]$
$A_4^{*2} = \alpha[-8y_0h(h-y)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y-y_0)]$
$A_5^{*2} = 2\alpha(y + 2yy_0\zeta - y_0)$
$A_6^{*2} = \alpha[4(h-y)(y_0-h)\zeta + 2(y-y_0)]$
$A_7^{*2} = 2\alpha(y_0 + 2yy_0\zeta - y)$
$B_1^{*2} = -B_8^{*1} = 2(y - y_0)$
$B_2^{*2} = 4(h-y)(y_0-h)\zeta + 2(y_0-y)$
$B_3^{*2} = 8y_0h(-y+h)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y_0-y)$
$B_4^{*2} = -8y_0h(h-y)\zeta^2 - 4h(y-h+y_0)\zeta + 2(y-y_0)$
$B_5^{*2} = 2(y + 2yy_0\zeta - y_0)$
$B_6^{*2} = 4(h-y)(y_0-h)\zeta + 2(y-y_0)$
$B_7^{*2} = 2(y_0 + 2yy_0\zeta - y)$
$C_1^{*2} = 2(1 + (y-y_0)\zeta)$
$C_2^{*2} = -4(y-h)(y_0-h)\zeta^2 + 2(y_0+y-2h)\zeta - 2$
$C_3^{*2} = 8y_0h(h-y)\zeta^3 + 4h(y-h-y_0)\zeta^2 + 2(y_0+2h-y)\zeta - 2$
$C_4^{*2} = 8y_0h(h-y)\zeta^3 - 4h(y-h-y_0)\zeta^2 + 2(y_0+2h-y)\zeta + 2$
$C_5^{*2} = 4yy_0\zeta^2 - 2(y_0+y)\zeta + 2$
$C_6^{*2} = 2[2(y-h)(y_0-h)\zeta^2 + (y_0+y-2h)\zeta + 2]$
$C_7^{*2} = -4yy_0\zeta^2 - 2(y_0+y)\zeta - 2$
$C_8^{*2} = -2(1 - (y-y_0)\zeta)$

(A.2)

and

$$F^1{}_{xx1}(x,y) = 4\frac{\mu\alpha(y_0-h)(3hy_0 - 6yy_0 - h^2)}{h^3(k+1)} + \int_0^\infty f_{xx11}d\zeta \qquad F^r{}_{xy1}(x,y) = \int_0^\infty f_{xyr1}d\zeta$$

$$F^2{}_{xx1}(x,y) = 4\frac{\mu\alpha y_0(-4h^2 + 3(2y+y_0) - 6yy_0)}{h^3(k+1)} + \int_0^\infty f_{xx21}d\zeta \qquad F^r{}_{yy1}(x,y) = \int_0^\infty f_{yyr1}d\zeta$$

$$F^r{}_{ijk}(x,y) = \int_0^\infty \left[f_{ijrk} - f_{ijrk}^\infty\right]d\zeta \qquad \{i,j\} \in \{x,y\}, k \in \{2,3\} \tag{A.3}$$

where $r \in \{1,2\}$. In the above equations

$$f_{xxr1} = \frac{\partial^2 L_t^r(\zeta,y)}{\partial y^2} \qquad f_{yy12}^\infty = -\frac{2\mu}{\pi(k+1)}(y-y_0)\zeta e^{-\zeta(y_0-y)}\cos(\zeta x)$$

$$f_{xxr2} = \frac{\partial^2 L_x^r(\zeta, y)}{\partial y^2}$$
$$f_{xxr3} = \frac{\partial^2 L_y^r(\zeta, y)}{\partial y^2}$$
$$f_{xyr1} = -\zeta \frac{\partial L_t^r(\zeta, y)}{\partial y}$$
$$f_{xyr2} = -\zeta \frac{\partial L_x^r(\zeta, y)}{\partial y}$$
$$f_{xyr3} = -\zeta \frac{\partial L_y^r(\zeta, y)}{\partial y}$$
$$f_{yyr1} = -\zeta^2 L_t^r(\zeta, y)$$
$$f_{yyr2} = -\zeta^2 L_x^r(\zeta, y)$$
$$f_{yyr3} = -\zeta^2 L_y^r(\zeta, y)$$
$$f_{xx12}^{\infty} = \frac{2\mu}{\pi(k+1)} \big((y - y_0)\zeta + 2\big) e^{-\zeta(y_0 - y)} \cos(\zeta x)$$
$$f_{xx13}^{\infty} = \frac{2\mu}{\pi(k+1)} \big((y - y_0)\zeta + 1\big) e^{-\zeta(y_0 - y)} \sin(\zeta x)$$
$$f_{yy13}^{\infty} = -\frac{2\mu}{\pi(k+1)} \big((y - y_0)\zeta - 1\big) e^{-\zeta(y_0 - y)} \sin(\zeta x)$$
$$f_{xy12}^{\infty} = -\frac{2\mu}{\pi(k+1)} (\zeta(y - y_0) + 1) e^{-\zeta(y_0 - y)} \sin(\zeta x)$$
$$f_{xy13}^{\infty} = \frac{2\mu}{\pi(k+1)} \zeta(y - y_0) e^{\zeta(y_0 - y)} \cos(\zeta x)$$
$$f_{xx22}^{\infty} = \frac{2\mu}{\pi(k+1)} \big((y - y_0)\zeta - 2\big) e^{-\zeta(y - y_0)} \cos(\zeta x)$$
$$f_{xx23}^{\infty} = -\frac{2\mu}{\pi(k+1)} \big((y - y_0)\zeta - 1\big) e^{-\zeta(y - y_0)} \sin(\zeta x)$$
$$f_{yy22}^{\infty} = -\frac{2\mu}{\pi(k+1)} \zeta(y - y_0) e^{-\zeta(y - y_0)} \cos(\zeta x)$$
$$f_{yy23}^{\infty} = \frac{2\mu}{\pi(k+1)} (\zeta(y - y_0) + 1) e^{-\zeta(y - y_0)} \sin(\zeta x)$$
$$f_{xy22}^{\infty} = \frac{2\mu}{\pi(k+1)} (\zeta(y - y_0) - 1) e^{-\zeta(y - y_0)} \sin(\zeta x)$$
$$f_{xy23}^{\infty} = \frac{2\mu}{\pi(k+1)} \zeta(y - y_0) e^{-\zeta(y - y_0)} \cos(\zeta x)$$

(A.4)

where

$$L_k^r = \frac{1}{2\pi} \big( L_k^{*r}(\zeta, y)(\cos(\zeta x) - i \sin(\zeta x)) + L_k^{*r}(-\zeta, y)(\cos(\zeta x) + i \sin(\zeta x)) \big), \quad k \in \{t, x, y\} \tag{A.5}$$

The coefficients in Eqs (12) in integral form are

$$k_{xx}^{r1}(x, y) = F^r{}_{xx1}(x - x_0, y)$$
$$k_{yy}^{r1}(x, y) = F^r{}_{xy1}(x - x_0, y)$$
$$k_{yy}^{r2}(x, y) = F^r{}_{xy2}(x - x_0, y) + \frac{2\mu(y - y_0)}{\pi(\kappa + 1)} \frac{(x - x_0)^2 - (y - y_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2}$$
$$k_{yy}^{r3}(x, y) = F^r{}_{xy3}(x - x_0, y) + \frac{2\mu}{\pi(\kappa + 1)} \left( \frac{2(x - x_0)(y - y_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2} + \frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \right)$$
$$k_{xy}^{r1}(x, y) = F^r{}_{xy1}(x - x_0, y)$$
$$k_{xy}^{r2}(x, y) = F^r{}_{xy2}(x - x_0, y) + \frac{2\mu}{\pi(\kappa + 1)} \left( \frac{2(x - x_0)(y - y_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2} - \frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \right)$$
$$k_{xy}^{r3}(x, y) = F^r{}_{xy3}(x - x_0, y) + \frac{2\mu(y - y_0)}{\pi(\kappa + 1)} \frac{(y - y_0)^2 - (x - x_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2}$$
$$k_{xx}^{12}(x, y) = F^1{}_{xx2}(x - x_0, y) + \frac{2\mu}{\pi(\kappa + 1)} \left( \frac{(y_0 - y)^3 - (y_0 - y)(x - x_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2} + \frac{2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} \right)$$
$$k_{xx}^{13}(x, y) = F^1{}_{xx3}(x - x_0, y) + \frac{2\mu}{\pi(\kappa + 1)} \left( \frac{2(x - x_0)(y_0 - y)^2}{((x - x_0)^2 + (y - y_0)^2)^2} - \frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \right)$$
$$k_{xx}^{22}(x, y) = F^2{}_{xx2}(x - x_0, y) + \frac{2\mu}{\pi(\kappa + 1)} \left( \frac{(y - y_0)^3 - (y - y_0)(x - x_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2} - \frac{2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} \right)$$
$$k_{xx}^{23}(x, y) = F^2{}_{xx3}(x - x_0, y) + \frac{2\mu}{\pi(\kappa + 1)} + \left( \frac{-2(x - x_0)(y - y_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2} + \frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \right) \tag{A.6}$$

where $r \in \{1,2\}$.