Periodic Set Of The Interface Cracks With Contact Zones In An Anisotropic Bimaterial

Sergii Kozinov[a], Volodymyr Loboda[b]

Department of Theoretical and Applied Mechanics, Dniepropetrovsk National University,

Naukova 13, Dnipropetrovsk 49050, Ukraine

[a]kozinov@list.ru, [b]loboda@mail.dsu.dp.ua

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Abstract. A closed form solution to the plane problem of the theory of elasticity for an infinite anisotropic bimaterial space (plane) with a periodic set of the interface cracks with frictionless contact zones near its tips is obtained. By means of the complex function presentation the problem is reduced to the combined Dirichlet-Riemann boundary value problem for a sectionally-holomorphic function and solved exactly. The equations for the determination of the contact zone lengths as well as the closed form expressions for the stress intensity factors are carried out. The variation of the mentioned values with respect to the distance between the cracks is illustrated.

Introduction

In process of composite materials and adhesive or bonded joints manufacture and exploitation numerous defects occur at the material interface. In some cases these defects are distributed periodically or they can be approximately considered as periodical. As fracture is usually originated from such defects, the problem of the periodic set of the interface cracks is quite important for the applications.

There are two main models of the interface cracks. First model (oscillatory model) initially assumes that the crack is completely open. Such assumption leads to the oscillatory singularities in the mechanical fields near the crack tips and, consequently, to the complex valued stress intensity factors (SIF). Second model (contact model) [1] allows for the contact zones near the crack tips. This model leads to the square root singularities in the mechanical fields at the crack tips and the SIFs acquire the real values.

The contact model, which is physically more adequate, has been widely investigated in the literature and some important results were obtained for isotropic bimaterials: see e.g. [2,3]. Particularly, it was shown in these papers that the contact zone is usually extremely small for a pure tensile loading. However, appearance of the shear field leads to the decreasing or increasing of the contact zone length depending on the shear loading direction.

The problem of an interface crack between two different anisotropic materials is essentially complicated, and that is why it was not investigated so actively as for isotropic dissimilar materials. An analytical solution of the problem in question in the framework of the classical interface crack model has been obtained by Clements [4] and afterwards explicitly investigated [5,6].

For anisotropic materials the problem with contact zones was reduced to a singular integral equation [7] which was solved numerically. An analytical solution was found by Ni and Nemat-Nasser [8]. The problem for several interface cracks with contact zones between dissimilar anisotropic materials was solved by Kharun and Loboda [9].

The periodic set of the interface cracks between two isotropic materials under "contact" model assumption has been investigated recently by Kozinov et al. [10]. To the authors' knowledge for an anisotropic bimaterial such investigation has not yet been made. In the present paper an exact analytical solution for the periodic set of the interface cracks with contact zones in an anisotropic bimaterial has been found. Simple equations for the contact zone length determination and the associated SIFs are presented.

Statement of the problem

Consider an infinite bimaterial medium containing periodic set of the interface cracks, as shown in Fig.1. The materials are assumed to be anisotropic with the compliance constants $s_{ij}^{(k)}$, where $k=1$ is related to the "upper" material and $k=2$ to the "lower" one.

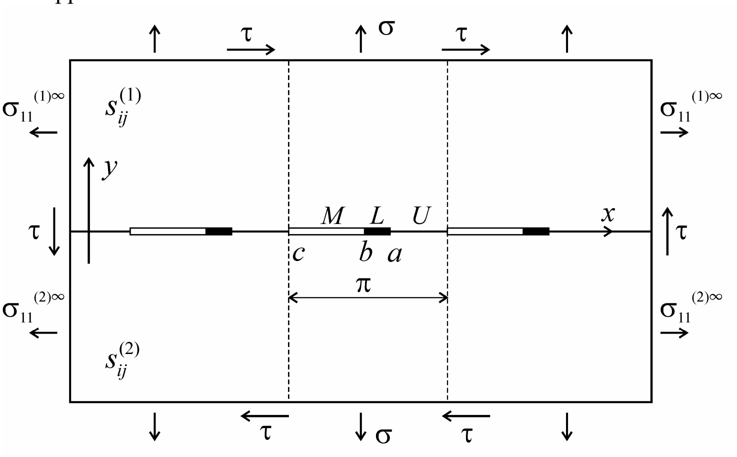


Fig.1. Geometry of the problem

The medium is subjected to a uniformly distributed tension-shear (σ–τ) loading at infinity. The following denotation is introduced further: the point between the contact zone and the open crack faces is denoted as *b*, the point between the bonded interface and the contact zone as *a* and the point between the bonded interface and the open crack faces as *c* (see Fig.1). Additionally, the union of the open crack faces will be denoted as *M*, the set of the contact zones as *L* and the bonded parts of the interface are denoted as *U*.

The fields of stresses and displacements for a plane problem of elasticity can be expressed by the formulas [11]

$$\mathbf{u}(x,y)=\mathbf{Af}(z)+\overline{\mathbf{Af}(z)},\ \mathbf{t}(x,y)=\mathbf{Bf}'(z)+\overline{\mathbf{Bf}'(z)}, \tag{1}$$

$$\sigma_{11}(x,y)=2\,\mathrm{Re}[\mu_1^2 f_1'(z_1)+\mu_2^2 f_2'(z_2)], \tag{2}$$

where $\mathbf{t}=\{\sigma_{12},\sigma_{22}\}^{\mathrm{T}}$ – stress vector; $\mathbf{u}=\{u_1,u_2\}^{\mathrm{T}}$ – displacement vector; $\mathbf{A},\mathbf{B}$ – 2x2 matrices; $\mathbf{f}(z)=\{f_1(z_1),f_2(z_2)\}^{\mathrm{T}}$, $f_j(z_j)$ – analytical functions of complex variable $z_j=x+\mu_j y$.

Assuming that the contact zones are frictionless and the open crack faces are unloaded, the continuity and boundary conditions can be written in Cartesian coordinates *xy*, depicted in Fig.1, as

$$[\mathbf{t}(x)]=0,\ \ x\in L+M+U;\quad [\mathbf{u}(x)]=0,\ \ x\in U; \tag{4}$$

$$\sigma_{12}^{(1)}(x,0) = -\tau, \left[u_2(x)\right] = 0, \ x \in L; \quad \mathbf{t}^{(1)}(x,0) = -\mathbf{t}^{\infty}, \ x \in M; \tag{5}$$

where $\mathbf{t}^{\infty} = \{\tau, \sigma\}$; $\sigma_{ij}^{(k)}$ and $u_i^{(k)}$ – stress and displacement components in the "upper" ($k=1$) and "lower" ($k=2$) materials;

$$[s_{ij}(x)] = s_{ij}^{(1)}(x,+0) - s_{ij}^{(2)}(x,-0), \quad [u_i(x)] = u_i^{(1)}(x,+0) - u_i^{(2)}(x,-0).$$

Complex-function representation for the stresses and displacement jump derivatives along the *x*-axis

Omitting some calculations and intermediate steps, the following expressions for the mechanical fields depending on the single vector-function are obtained

$$\begin{cases} \sigma_{22}^{(1)}(x,0) + n_* \sigma_{12}^{(1)}(x,0) = \Phi^+(x) + \gamma \Phi^-(x), \\ s_* \left([u_1'(x)] - n_* [u_2'(x)] \right) = \Phi^+(x) - \Phi^-(x), \end{cases} \tag{6}$$

where $s_* = S_{11}$, $n_* = N_{11}$, $\Phi(z) = \Phi_1(z)$.

Substitution of Eqs. (6) into the boundary conditions (5) yields the following boundary value problem:

$$\begin{aligned} &F^+(x) + \gamma F^-(x) = 0, \quad x \in M, \\ &\operatorname{Im} F^{\pm}(x) = 0, \quad x \in L. \end{aligned} \tag{7}$$

Here one introduced the function

$$F(z) = \Phi(z) + p \exp(i\beta^*), \quad p = \frac{p}{1+\gamma}, \tag{8}$$

in which $p = \sqrt{(\sigma + n'\tau)^2 + (n''\tau)^2}$, $\beta^* = \arctan\left(\dfrac{n''\tau}{\sigma + n'\tau}\right)$, $n' = \operatorname{Re} n_*$, $n'' = \operatorname{Im} n_*$.

Thus the problem has been reduced to the determination of the function $F(z)$. It is analytical in the entire plane except for segments $L \cup M$ where the conditions (7) should be satisfied.

Solution of the boundary value problem

The problem (7), obtained in the previous section, is a periodic combined homogeneous Dirichlet-Riemann boundary value problem. A solution to (7) one presents in the following form [12]:

$$S(z) = Z(z) e^{i\psi(z)} \sin^{-\alpha}(z-a), \tag{9}$$

where

$$Z(z) = \left(\frac{\sin(z-b)}{\sin(z-c)} \right)^{1/2 - i\varepsilon}$$ is the canonical solution of the periodic Riemann problem,

α is an integer value, $\psi(z)$ is a solution of the periodic Dirichlet problem

$$\mathrm{Re}\,\psi^{\pm}(x) = h^{\pm}(x), \quad x \in L, \tag{10}$$

$$h^{\pm}(x) = \pi n^{\pm} - \arg Z^{\pm}(x) + \alpha\left[\arg\sin(x-a)\right]^{\pm}, \tag{11}$$

which is finitesimal in the nodes and at infinity, $n^{\pm}$ are integer values.
The solution to (10) can be written in the following form [13]:

$$\psi(z) = \frac{Y(z)}{4\pi i}\int_L \frac{h^{+}(t)+h^{-}(t)}{Y^{+}(t)\sin(t-z)}dt + \frac{1}{4\pi i}\int_L \left(h^{+}(t)-h^{-}(t)\right)\mathrm{ctg}(t-z)dt,$$

where $Y(z) = \sqrt{\sin(z-b)\sin(z-a)}$.

Imparting to the $n^{\pm}$ and α various values, one obtains the set of the canonical solutions, two of which

$$S_1(z) = \frac{e^{i\varphi(z)}}{\sqrt{\sin(z-c)\sin(z-b)}}, \quad S_2(z) = \frac{ie^{i\varphi(z)}}{\sqrt{\sin(z-c)\sin(z-a)}}, \tag{12}$$

are linearly independent, where

$$\varphi(z) = -\frac{\varepsilon Y(z)}{2}\int_M \frac{dt}{Y(t)\sin(t-z)}.$$

A general solution of the problem (7) can be written in the following form:

$$F(z) = S_1(z)P(e^{iz}) + S_2(z)Q(e^{iz}), \tag{13}$$

where $P(e^{iz}), Q(e^{iz})$ are arbitrary polynomials. In order to satisfy the conditions at infinity they must have the form

$$P(z) = C_1\cos(z-a_*) + C_2\sin(z-a_*), \quad a_* = (c+b)/2,$$
$$Q(z) = D_1\cos(z-b_*) + D_2\sin(z-b_*), \quad b_* = (c+a)/2,$$

where C_1, C_2, D_1, D_2 are arbitrary real constants.

Substituting (12) into (13), the general solution of the homogeneous mixed Dirichlet-Riemann boundary problem (7) can be written as

$$F(z) = \frac{\exp(i\varphi(z))}{\sqrt{\sin(z-c)}}\left(\frac{P(z)}{\sqrt{\sin(z-b)}} + i\frac{Q(z)}{\sqrt{\sin(z-a)}}\right), \tag{14}$$

where $\varphi(z) = 2\varepsilon\ln\left(\frac{\sqrt{\sin(a-b)\sin(z-c)}}{\sqrt{\sin(a-c)\sin(z-b)}+\sqrt{\sin(b-c)\sin(z-a)}}\right)$.

The real constants C_1, C_2, D_1, D_2 are to be determined from the conditions at infinity $F(z)_{z\to\pm i\infty} = p\exp(i\beta^{*})$.

The expressions for the main characteristics at the interface

Using of Eqs. (6), (8) and the solution (14) lead to the following expressions for the stresses and the displacement jump derivatives at the interface:

$$\sigma_{22}^{(1)}(x)=\frac{2\exp(\pi\varepsilon)}{\sqrt{\sin(x-c)}}\left(\frac{P(x)}{\sqrt{\sin(x-b)}}\operatorname{ch}\left(\varphi(x)-\pi\varepsilon\right)+\frac{Q(x)}{\sqrt{\sin(a-x)}}\operatorname{sh}\left(\varphi(x)-\pi\varepsilon\right)\right),x\in L, \tag{15}$$

$$\sigma_{22}^{(1)}(x)+n_*\sigma_{12}^{(1)}(x)=\frac{(1+\gamma)\exp(i\varphi(x))}{\sqrt{\sin(x-c)}}\left(\frac{P(x)}{\sqrt{\sin(x-b)}}+i\frac{Q(x)}{\sqrt{\sin(x-a)}}\right),x\in U, \tag{16}$$

$$[u_2'(x)]=\frac{2\operatorname{ch}(\pi\varepsilon)}{\sqrt{\sin(x-c)}}\left(\frac{P(x)}{\sqrt{\sin(b-x)}}\cos\left(\varphi^*(x)\right)-\frac{Q(x)}{\sqrt{\sin(a-x)}}\sin\left(\varphi^*(x)\right)\right),x\in M, \tag{17}$$

where $\varphi(x)=2\varepsilon\arctan\sqrt{\dfrac{\sin(b-c)\sin(a-x)}{\sin(a-c)\sin(x-b)}},\quad x\in L,$

$$\varphi^*(x)=2\varepsilon\ln\frac{\sqrt{\sin(a-b)\sin(x-c)}}{\sqrt{\sin(a-c)\sin(b-x)}+\sqrt{\sin(b-c)\sin(a-x)}},\quad x\in M.$$

Contact zone length and the SIFs determination

Eqs. (15) – (17) can be used at any position of the point b, but in order for the obtained solution corresponds the formulated mechanical problem the following conditions should be satisfied:

$$[u_2'(b)]=0;\quad \sigma_{22}^{(1)}(x,0)\le 0,\, x\in L;\quad [u_2(x)]\ge 0,\, x\in M. \tag{18}$$

These conditions mean that the crack faces close smoothly, the normal stress in the contact region is compressive and there is no overlapping of the crack faces.

Making use of Eq. (18_1) and expanding Eq. (15) or (17) in Taylor's series at the point b one can derive the following transcendental equation for the contact zone length determination:

$$P(b)=0. \tag{19}$$

The choice of the required root is provided by the conditions (18_2) and (18_3) satisfying. Eq. (19) can be written as follows:

$$\tan\left(x-\frac{c+b}{2}\right)=-\frac{n''\tau\cos\varsigma+\sigma\sin\varsigma}{\sigma\cos\varsigma-n''\tau\sin\varsigma}\tanh\chi. \tag{20}$$

where

$$\chi=2\varepsilon\arctan\frac{\sqrt{\sin(a-c)}\sin\frac{b-c}{2}+\sqrt{\sin(b-c)}\sin\frac{a-c}{2}}{\sqrt{\sin(a-c)}\cos\frac{b-c}{2}+\sqrt{\sin(b-c)}\cos\frac{a-c}{2}},\qquad \varsigma=\varepsilon\ln\frac{\sin\frac{a+b-2c}{2}+\sqrt{\sin(a-c)\sin(b-c)}}{\sin\frac{a-b}{2}};$$

$e^{\chi}\in\mathrm{Real},\quad \varsigma\in\mathrm{Real}.$

The SIFs at the crack tip a can be defined as

$$K_1 - iK_2 = \lim_{x\to a+0}\left(\sigma_{22}(x,0) - i\sigma_{12}(x,0)\right)\sqrt{\sin(x-a)}\,.$$

Making use of Eq. (23), the following formulas for the SIFs yield:

$$K_1 = -n'K_2,\quad K_2 = \frac{-(\sigma\cos\varsigma - n''\tau\sin\varsigma)\sinh\chi\cos\frac{a-c}{2} + (n''\tau\cos\varsigma + \sigma\sin\varsigma)\cosh\chi\sin\frac{a-c}{2}}{n''\sqrt{\sin(a-c)}}. \quad (21)$$

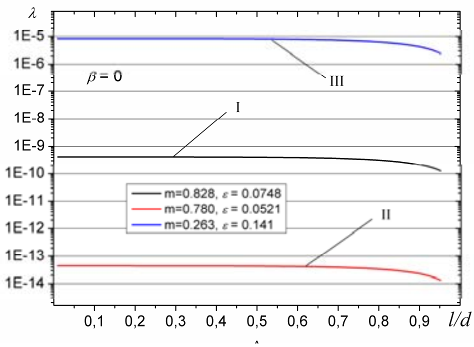
It is necessary to note that $n' \neq 0$ for a general case of anisotropy. When the directions of orthotropy coincide with the coordinate axes $n' = 0$ and, consequently, $K_1 = 0$.

Numerical results and discussion

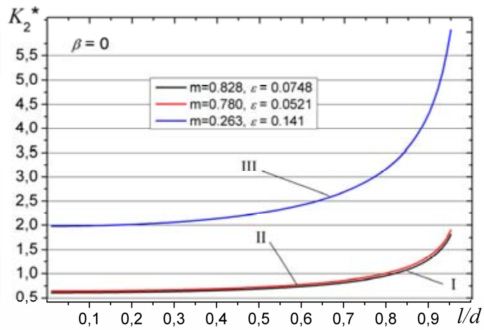
In this section some numerical results concerning the periodic set of the interface cracks are presented and discussed. The length of the crack is denoted as $l = a - c$. In order to clarify the interaction of the cracks it is sufficient to investigate the influence of the length of the crack l and the angle β between y-axis and the direction of the resultant load $\sqrt{\sigma^2+\tau^2}$ ($\tan\beta = \tau/\sigma$) on the relative contact zone lengths $(\lambda = (a-b)/l)$ and the SIFs at the tip a of the crack. Parameters β and ε (the relative stiffness of the materials) vary in the ranges: $-\pi/2 \le \beta \le \pi/2$, $-(\log 3)/2\pi \le \varepsilon \le (\log 3)/2\pi$. Positive (negative) values of the parameter ε mean that the "lower" ("upper") material is stiffer then "upper" ("lower") one.

For the numerical analysis we consider orthotropic bimaterials which principal directions of orthotropy coincide with the coordinate axes direction. Corresponding graphs for the bimaterial composed of silicon and reinforced fiberglass (orthogonally reinforced 5:1) denote as I ($n_* = -0.828i; \varepsilon = 0.0748$), carbon-filled plastic AS/4397 and plywood/DX210 as II ($n_* = -0.78i; \varepsilon = 0.0521$), carbon-filled plastic HMS/DX209 and fir as III ($n_* = -0.263i; \varepsilon = 0.141$).

In the following graphics $m = -\operatorname{Im} n_*$.

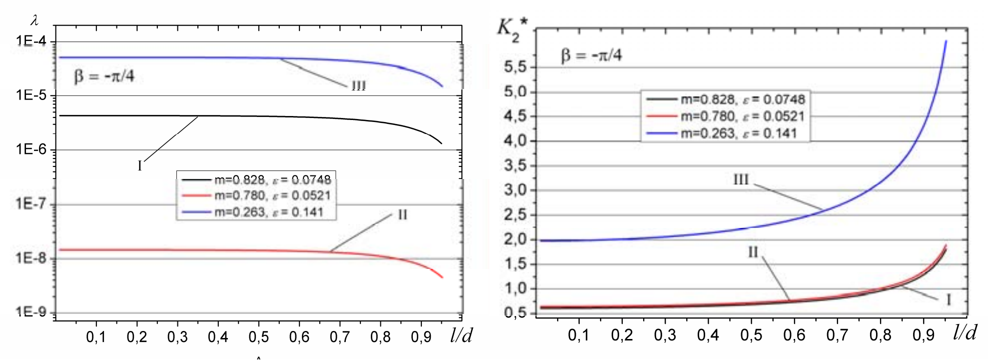


(a)



(b)

Fig.2. Dependency of the relative contact zone length and the SIFs on the distance between the cracks under tension loading.



(a) (b)

Fig.3. Dependency of the relative contact zone length and the SIFs on the distance between the cracks under combined loading with $\tau = -\sigma$.

In Figures 2, 3 the variations of the contact zone length and the dimensionless SIF $K_2^* = K_2/(p\sqrt{l})$ are shown for the cases of different types of loading uniformly distributed at infinity. In Figures 2a, 3a the logarithmic scale was used for convenience to illustrate the variation of λ for the different materials. As it was expected decreasing of the distance between the cracks lead to variation of the contact zone length at the crack tips as well as the SIFs. For the bimaterial III consisting of the most different half-planes the contact zone value is essentially larger than that for the bimaterials I and II which are composed of less different materials.

Results for the periodic set of the interface cracks between isotropic materials display similar dependency of the relative contact zone length and the SIFs on the distance between the cracks [10].

Moreover, for the isotropic bimaterial under "open crack" assumption (without contact zones) the expression for the SIFs can be reduced to the following form [14]:

$$K_1^{os} - iK_2^{os} = \left[\sin(a-c)\right]^{-1/2+i\varepsilon} \sin\left[(a-c)(1/2-i\varepsilon)\right]\operatorname{sech}(\pi\varepsilon)(\sigma - i\tau),$$

and for a homogeneous material it can be reduced to the formula:

$$K_1 - iK_2 = \frac{\sin\left((a-c)/2\right)}{\sqrt{\sin(a-c)}}(\sigma - i\tau).$$

It is important to note that the values of the relative contact zone length at small l/π ratio coincide with the correspondent results for a single interface crack. The following transcendental equation for the relative contact zone length λ determination was obtained in case of a single crack [15]:

$$\tan^{-1}\left(\frac{\sqrt{1-\lambda}\sigma + 2\varepsilon m\tau}{2\varepsilon\sigma - \sqrt{1-\lambda}m\tau}\right) = \varepsilon \log\left(\frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}\right), \tag{22}$$

where $m = n''$.

It can be easily seen that for different bimaterial properties and angle β the obtained data coincide with high accuracy.

Conclusion

The new exact analytical solution of the boundary value problem to a plane problem for an anisotropic bimaterial space (plane) with a periodic set of the interface cracks under remote mixed-mode loading has been found. The solution method is based on the theory of analytical functions leading to the formulation of the combined Dirichlet-Riemann problem (7). The stresses, SIFs and displacement jumps along the material interface are presented by means of closed-form analytical formulas. The transcendental equation (20) for the determination of the relative contact zone length λ has been found and solved for various material combinations. A comparison of the obtained results with the solution of the similar problem for a single crack has been performed.

The numerical illustrations of the obtained solutions are presented in Figures 2,3. As a result of the numerical analysis of Eqs. (20) and (21) it is shown that the relative length of the contact zone as well as the SIFs essentially depends on the distance between the cracks.

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