Determination of a Limit State of Elastic Solids with Thin-Walled Elastic Inclusions Using the J-integral

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Abstract. For determination of a limit state of elastic solids with thin-walled elastic inclusions of arbitrary rigidity the use of the *J*-integral is proposed. The relationship between the *J*-integral and GSIFs for all three modes of deformation is determined. It is shown with the help of a method of dominating GSIF that the *J*-integral (energy release rate) is positive for soft inclusions and negative for rigid ones. This means that there are energetically favorable conditions for the growth of soft inclusions and for the reduction of the size of rigid ones. Numerical algorithms for determination of separate GSIFs from the values of the *J*-integral are offered.

Introduction

Thin-walled defects, inclusions in particular, are one of the most widespread types of defects of inhomogeneous material structure. Those are cracks, oxide films, sulfide and carbon inclusions in metals, various cavities filled with gas, liquid or solid substance, and other heterogeneities, which arise during manufacture and processing of materials, in particular, structural materials. Besides, stiffeners, thermal and strain sensors, fibers in composite materials, phase transformation and chemical reaction products including those on materials interface can be modeled as thin-walled inclusions.

The bibliography concerning problems of modeling and determination of stress/strain and limit state of solids containing thin-walled inclusions is voluminous (e.g. see [1]). The pioneering works in this field were made by K. Chobanyan and A. Khachikyan [2], D. Grilitskyi and G. Sulym [3], Ya. Pidstrygach [4]. Various models, which describe soft or rigid thin-walled inclusions, were proposed in [5,6] etc. The solution of the problem of thin-walled elastic inclusion and the asymptotic distribution of stresses and displacements in the vicinity of its tip was obtained in [7]. Results of studies of thermoelastic equilibrium of deformable solids with thin-walled inclusions are presented in the monograph [1].

Nevertheless, there is a very little attention paid to energetic approaches, in particular, to *J*-integral and its application in the study of a limit state of solids containing thin-walled inclusions. To the authors' knowledge only two works [8,9] concerning this topic have been published. The first of them presents the relationship between generalized stress intensity factors (GSIFs) and the *J*-integral for the plane problem for elastic solids containing thin-walled inclusions, and the second studies prestressed plates with rigid line inclusions. Detailed study of this problem is necessary because of two principal reasons. Firstly, the application of direct numeric methods (FEM, BEM) to thin-walled inclusions problems for determination of GSIFs, when asymptotic relations are used gives very inaccurate results even for cracks [10], and not only for elastic inclusions. That is caused by the fact that these asymptotic relations for real cracks and thin-walled inclusions, which are specified by some thickness and end part shape, correspond to real stress state of solid only in some ring which surrounds the defect tip (the stress intensity zone) [1]. Therefore, the invariant *J*-integral that "takes into account" only the singularity of the stress field can become an effective tool for calculation of GSIFs near thin-walled elastic inclusions. Secondly, the application of energetic approaches, which have large physical sense, can play a key role in determination of the fracture criterion for solids containing thin-walled inclusions, or, at least, give the certain qualitative (and possibly a quantitative) assessment of this process.

The relationship between the *J*-integral and GSIFs. Energy release rate

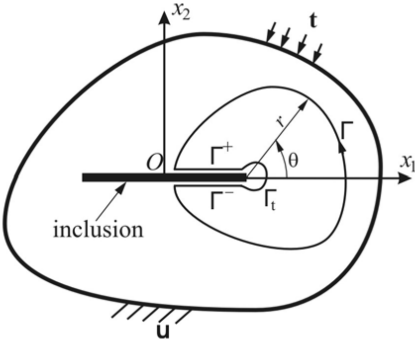

Fig. 1. The problem scheme

Let us consider a plane problem of elasticity for isotropic solid with thin elastic inclusion with the length of l and the width of h $(h \ll l)$. Some part of the boundary of a solid is loaded with surface tractions $\mathbf{t} = (t_1, t_2, 0)^{\mathbf{T}}$ and the displacement constraints $\mathbf{u} = (u_1, u_2, 0)^{\mathbf{T}}$ are set on the other (see Fig. 1). Let us direct the Ox_1 axis of right Cartesian coordinate systems Ox_1x_2 along the axis of the inclusion. Due to the thinness of inclusion, it is possible to model it by a mathematical cut with the interaction conditions set on its edges [1]. Distribution of stresses and displacements in the tip vicinity of such defect is determined within the asymptotic dependences [1]:

$$\begin{pmatrix}\sigma_{22}\\ \sigma_{11}\\ \sigma_{12}\end{pmatrix} = \frac{1}{4\sqrt{2\pi r}}\left[K_{11}\begin{pmatrix}5\cos\theta_1 - \cos\theta_5\\ 3\cos\theta_1 + \cos\theta_5\\ -\sin\theta_1 + \sin\theta_5\end{pmatrix} + K_{21}\begin{pmatrix}-\sin\theta_1 + \sin\theta_5\\ -7\sin\theta_1 - \sin\theta_5\\ 3\cos\theta_1 + \cos\theta_5\end{pmatrix} + \right.$$
$$\left. + \kappa_* K_{12}\begin{pmatrix}(2\kappa-3)\cos\theta_1 + \cos\theta_5\\ -(2\kappa+5)\cos\theta_1 - \cos\theta_5\\ -(2\kappa+1)\sin\theta_1 - \sin\theta_5\end{pmatrix} + \kappa_* K_{22}\begin{pmatrix}(2\kappa+3)\sin\theta_1 - \sin\theta_5\\ -(2\kappa-5)\sin\theta_1 + \sin\theta_5\\ (2\kappa-1)\cos\theta_1 - \cos\theta_5\end{pmatrix}\right] + O(1), \tag{1}$$
$$\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \frac{\sqrt{r}}{4\mu\sqrt{2\pi}}\left[K_{11}\begin{pmatrix}(2\kappa-1)\cos\theta_1 - \cos\theta_3\\ (2\kappa+1)\sin\theta_1 - \sin\theta_3\end{pmatrix} + K_{21}\begin{pmatrix}(2\kappa+3)\sin\theta_1 + \sin\theta_3\\ -(2\kappa-3)\cos\theta_1 - \cos\theta_3\end{pmatrix} + \right.$$
$$\left. + \kappa_* K_{12}\begin{pmatrix}-(4\kappa+1)\cos\theta_1 + \cos\theta_3\\ \sin\theta_1 + \sin\theta_3\end{pmatrix} + \kappa_* K_{22}\begin{pmatrix}-\sin\theta_1 - \sin\theta_3\\ (4\kappa-1)\cos\theta_1 + \cos\theta_3\end{pmatrix}\right] + O\left(r^{3/2}\right),$$

where r is a distance to inclusion's tip; K_{ij} are GSIFs; κ is a Muskhelishvili constant; μ is the shear module of the material of a solid; $\kappa_* = 2/(\kappa - 1)$; $\theta_j = j\theta/2$; θ is the polar angle. Let us note that in the case of a crack $K_{12} = 0$, $K_{22} = 0$, $K_{11} \sim K_1$, $K_{21} \sim K_2$, where K_1, K_2 are classic SIFs of the crack theory. Let us consider the expression

$$J^{(j)} = \int_L \left(W\delta_{ij} - \sigma_{ik}u_{k,j}\right)n_i dL = \int_L b_{ij}n_i dL\,, \tag{2}$$

that is known as Eshelby *J*-integral [11]. Here and further, the rule of summation by repeated index is used; $W = \sigma_{ij}\left(u_{i,j} + u_{j,i}\right)/4$ – strain energy density. Eshelby tensor $\hat{b}$ is divergence-free ($\operatorname{div}\hat{b} = b_{ij,i} = 0$) in the absence of inhomogeneities [11], therefore, according to the divergence theorem the integral $J^{(j)}$ taken along a closed contour (or a closed surface for 3D problems), that does not enclose singularities of stress field, is equal to zero. Further on only the component $J = J^{(1)}$ of integral in Eq. 2 is considered.

Let us choose the curve $L = \Gamma \cup \Gamma^{+} \cup \Gamma_{\mathrm{t}} \cup \Gamma^{-}$ (see Fig. 1) as an integration path. It does not enclose singularities; therefore, the *J*-integral, taken along L, is equal to zero. From Eq. 1 it follows that the sum of integrals along Γ^{+} and Γ^{-} is zero. Thus, the *J*-integral taken along the arbitrary "closed" contour Γ (starting from a point $\left(x_1, -h/2\right)$ and with the end at $\left(x_1, h/2\right)$) equals

$$J = \int_{\Gamma} b_{i1}n_i d\Gamma = -\int_{\Gamma_{\mathrm{t}}} b_{i1}n_i d\Gamma_{\mathrm{t}}\,,$$

and the change of the Γ_{t} path tracing to counter-clockwise gives

$$J = \int_{\Gamma} b_{i1}n_i d\Gamma = \int_{\Gamma_{\mathrm{t}}} b_{i1}n_i d\Gamma_{\mathrm{t}}\,. \tag{3}$$

Here Γ_{t} is a small contour enclosing the stress field singularity at inclusion's tip.

As for the rigid line inclusion, which cannot rotate, as well as for a crack, the Γ contour should not be necessarily closed, only with both end points on inclusion edges. It is obvious that in this case on Γ^{+} and Γ^{-} the following conditions are satisfied $n_1 = 0$, $u_{i,1} = 0$, therefore, $b_{i1}n_i \equiv 0$ on Γ^{+} and Γ^{-}.

It is proved (e.g. see [11]) that the *J*-integral is equal to the potential energy release rate of the deformed solid corresponding to the infinitesimal translation of point singularity along Ox_1 axis:

$$J = -\partial\Pi/\partial l\,. \tag{4}$$

Inclusion's "growth" can be described as a translation (phase transformation of a material) of its tip, which induces stress field singularity, along its axis. That means that Eq. 4 fulfils not only for cracks but also for thin-walled elastic inclusions.

The relationship between the *J*-integral and GSIFs near inclusion's tip can be obtained directly from Eq. 3. Considering the asymptotic distributions of stresses and displacements Eq. 1 and taking into account the relationship between derivatives on Cartesian and polar coordinates we obtain

$$J = \int_{-\pi}^{\pi} b_{i1}n_i r d\theta = (\kappa+1)/(8\mu)\left[K_{11}^2 + K_{21}^2 - \kappa\cdot\kappa_*^2\left(K_{12}^2 + K_{22}^2\right) + 2\left(K_{11}K_{12} + K_{21}K_{22}\right)\right]. \tag{5}$$

As well as in the case of a crack, by means of a method of H. Kitagawa et al. [12] the *J*-integral for the inclusion can be successfully decoupled on symmetric J_1 and asymmetric J_2 components

$$J_i = (\kappa+1)/(8\mu)\left[K_{i1}^2 + 2K_{i1}K_{i2} - \kappa\cdot\kappa_*^2 K_{i2}^2\right] \quad (J = J_1 + J_2). \tag{6}$$

Similarly to the plane problem let us find the relationship between the J-integral and GSIFs for antiplane shear of an elastic cylindrical solid with ribbon-like linear elastic inclusion. Considering that the boundary conditions are set in the form $\mathbf{u} = (0,0,u_3)^{\mathbf{T}}$, $\mathbf{t} = (0,0,t_3)^{\mathbf{T}}$ and taking into account the asymptotic distribution of stresses [1]

$$\begin{pmatrix}\sigma_{13}\\ \sigma_{23}\end{pmatrix} = \frac{K_{31}}{\sqrt{2\pi r}}\begin{pmatrix}-\sin\theta_1\\ \cos\theta_1\end{pmatrix} + \frac{K_{32}}{\sqrt{2\pi r}}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} + O(1), \tag{7}$$

where in the case of a crack GSIF K_{31} coincides with a classic SIF K_3 and $K_{32} = 0$, directly from Eq. 7 and Eq. 2 using a circle with small radius r as an integration path it follows

$$J_3 = \int_{-\pi}^{\pi} b_{i1} n_i r d\theta = \left(K_{31}^2 - K_{32}^2\right)/(2\mu). \tag{8}$$

The subscripts in J-integral notation correspond to three modes of deformation of a solid relatively to the inclusion's position.

Analysis of received dependences

Let us consider limit cases of relative rigidity of the thin elastic inclusion. The soft inclusion, which relative rigidity tends to zero, in the limit, even if h does not tend to zero but is small comparing with l, turns into the elongated cavity which is described by the same equation as the mathematical cut (crack). It is known, that for cracks (e.g. see [1,13]), only three of six GSIFs are nonzero ones: $K_{i2} = 0\ \left(i = \overline{1,3}\right)$. Thus from Eq. 6 and Eq. 8 it follows

$$J_1 = K_{11}^2(\kappa+1)/(8\mu),\ J_2 = K_{21}^2(\kappa+1)/(8\mu),\ J_3 = K_{31}^2/(2\mu). \tag{9}$$

Eq. 9 are the known [13] relationships between the J-integral and SIFs for a crack.

For the rigid line inclusion $K_{i1} = 0\ \left(i = \overline{1,3}\right)$ [1] and from Eq. 6 and Eq. 8 it follows

$$J_1 = -K_{12}^2\kappa(\kappa+1)\kappa_*^2/(8\mu),\ J_2 = -K_{22}^2\kappa(\kappa+1)\kappa_*^2/(8\mu),\ J_3 = -K_{32}^2/(2\mu). \tag{10}$$

First two formulas in Eq. 10 accurate with the constant multiplier, which relates GSIFs K_{i2} and the SIFs $K_{(\in)i}$ from [9], correspond to the relationship between the J-integral and SIFs obtained in [9] basing on the solution for energy release rate for infinitesimal rigid line inclusion "growth". The third formula in Eq. 10 to the authors' knowledge is presented here for the first time.

Therefore, for a crack from Eq. 9 based on Eq. 4 the known relation follows: $\partial\Pi/\partial l < 0$ – potential energy of the deformed solid decreases during the crack "growth" (releases and in particular takes part in formation of a new surface). Contrary to this basing on Eq. 10 it is obvious that for the rigid line inclusion $J < 0$ that is $\partial\Pi/\partial l > 0$ – energy of deformation during its "growth" increases. The phenomenon similar to the "growth" of the rigid inclusion can be described as indurations of a small zone ahead it (solidification). Solidification tightens a material ahead the inclusion, therefore, $\partial\Pi/\partial l > 0$. It is impossible to describe this phenomenon within the framework of the classical theory of elasticity as there can be involved various physical and chemical processes, for example, polymerization, phase transformation, etc., that release or absorb additional energy. The released one will increase the potential energy of the deformed solid. In the absence of such factors, only the reduction of inclusion's length (for example, its delaminating or softening) is energetically favorable. The thought about the reduction of inclusion's size is formulated also in [8,9] for the plane problem. The phenomenon of solidification along with the usage of *J*-integral and GSIFs as the criterion parameters for fracture of solids with rigid inclusions (see [14]) demand additional study beyond this work.

Numerical methods for GSIFs evaluation from *J*-integral values

Basing on the results of analytical studies of thin-walled inclusions by jump function method [1] it can be noticed that for soft inclusions dominating are GSIFs K_{i1} that is $K_{i2} \approx 0$ $\left(i = \overline{1,3}\right)$. Therefore, basing on Eq. 6 and Eq. 8 we can write that

$$K_{11} \approx \pm\sqrt{8\mu J_1/(\kappa+1)}\,,\; K_{21} \approx \pm\sqrt{8\mu J_2/(\kappa+1)}\,,\; K_{31} \approx \pm\sqrt{2\mu J_3}\,. \tag{11}$$

On the other hand for sufficiently rigid elastic inclusions GSIFs K_{i2} prevail: $K_{i1} \approx 0$ $\left(i = \overline{1,3}\right)$, so

$$K_{12} \approx \pm\sqrt{-8\mu J_1\Big/\left[(\kappa+1)\kappa\cdot\kappa_*^2\right]}\,,\; K_{22} \approx \pm\sqrt{-8\mu J_2\Big/\left[(\kappa+1)\kappa\cdot\kappa_*^2\right]}\,,\; K_{32} \approx \pm\sqrt{-2\mu J_3}\,. \tag{12}$$

The sign of GSIFs is determined with the reference to numerical solution of the problem in the stress intensity zone and asymptotic dependences given by Eq. 1, Eq. 7.

The other approach for separation of GSIFs is to use the known solution, e.g. asymptotic one, and its imposition (for cracks such approach is considered in [13,15]). Let us consider the two independent elastic states of solid with thin-walled inclusion: the target problem field denoted with superscript „t" and the auxiliary field denoted by „a". Let us denote the superposition of these fields by „s". Obviously, one can mention that $J_i^{(s)} = J_i^{(t)} + J_i^{(a)} + J_i^{(t,a)}$ $\left(i = \overline{1,3}\right)$ thus Eq. 6 gives

$$J_i^{(t,a)} = J_i^{(s)} - J_i^{(t)} - J_i^{(a)} = \frac{\kappa+1}{4\mu}\left[K_{i1}^{(a)}\left(K_{i1}^{(t)} + K_{i2}^{(t)}\right) + K_{i2}^{(a)}\left(K_{i1}^{(t)} - \kappa\kappa_*^2 K_{i2}^{(t)}\right)\right] \;\left(i = \overline{1,2}\right), \tag{13}$$

and from Eq. 8 it follows

$$J_3^{(t,a)} = \left[K_{31}^{(a)}K_{31}^{(t)} - K_{32}^{(a)}K_{32}^{(t)}\right]\Big/\mu\,. \tag{14}$$

Similarly to that approach for cracks [13,15] let us name $J_i^{(t,a)}$ $\left(i = \overline{1,3}\right)$ the mutual integrals. Setting in Eq. 13, Eq. 14 the linearly independent combinations of GSIFs $K_{ij}^{(a)}$ of an auxiliary problem we will receive enough equations for determination of GSIFs $K_{ij}^{(t)}$ of a target one.

Numerical examples

1. Plane elasticity. The numerical analysis of GSIFs near thin-walled inclusion for uniform load at infinity was made in [8]. Here the results of GSIFs study using the *J*-integral are presented for the loading of the basic material of a solid by the concentrated forces, which induce essentially inhomogeneous stress field. The problem scheme is presented on the insertion to Fig. 2. Numerical calculations were made based on boundary element analysis of the problem.

The shape of inclusion's boundary was described as an elongated ellipse with the ratio of semiaxes of $l/h = 100$. The Muskhelishvili constants of inclusion and matrix materials were $\kappa = \kappa_i = \kappa_m = 2$ and the relative rigidity of inclusion was defined by the parameter $k = \mu_i/\mu_m$.

The values of GSIFs received using the *J*-integral were compared with the exact solutions for the crack [16] and for rigid line inclusion [17]

$$\begin{aligned} &K_{11} = P\left[\lambda^2(3+\kappa)+\kappa+1\right]\Big/\left[(\kappa+1)\sqrt{\pi l}\left(\lambda^2+1\right)^{3/2}\right], K_{12}=0 \quad \text{for } k \to 0, \\ &K_{12} = P\lambda^2(\kappa-1)\Big/\left[\kappa(\kappa+1)\sqrt{\pi l}\left(\lambda^2+1\right)^{3/2}\right], K_{11}=0, \lambda = d/l, \ \text{for } k \to \infty \end{aligned} \tag{15}$$

and with asymptotic dependences relating GSIFs and stresses at the inclusion's tip:

$$\begin{aligned} &\sigma_{22} = \sigma_{22}^{a} + \frac{2K_{11}}{\sqrt{\pi\rho}} \quad \text{for} \quad 0 < k \ll \frac{4(\kappa_i - 1)}{(\kappa_i+1)(\kappa_m+1)}\sqrt{\frac{\rho}{l}}, \\ &\sigma_{11} = \sigma_{11}^{a} - \frac{(\kappa+1)K_{12}}{(\kappa-1)\sqrt{\pi\rho}} \quad \text{for} \quad k \gg \frac{(\kappa_i+1)(\kappa_m+1)}{8\kappa_m}\sqrt{\frac{l}{\rho}}. \end{aligned} \tag{16}$$

In contrast to [18], where the conformable dependences were received, Eq. 16 considers the non-singular parts σ_{11}^{a}, σ_{22}^{a} of asymptotic expansions, which basing on the analytical solution for the elliptic inclusion [19], are

$$\sigma_{22}^{a} = \sigma_{22}^{0} - \sigma_{11}^{0}, \ \sigma_{11}^{a} = \left[\left(\kappa^2+4\kappa+3\right)\sigma_{11}^{0} + \left(\kappa^2-1\right)\sigma_{22}^{0}\right]\Big/[8\kappa]. \tag{17}$$

The superscript „0" here denotes components of stress tensor at inclusion's tip when $k = 1$ (inclusion and matrix materials are identical).

The functional dependence of normalized GSIFs $K_{ij}^{*} = K_{ij}\sqrt{l}\Big/\left(P\sqrt{\pi}\right)$ on relative rigidity of inclusion k and distance to the concentrated force λ is shown on Fig. 2. Continuous curves correspond to the mutual integral method and dashed-line to a method of dominating GSIF. It can be seen from these plots that for inclusions with essentially small or big enough relative rigidity the deviation between the data received by methods of dominating GSIF and of mutual integral is insignificant and does not exceed 0,4 % for very soft ($k \le 10^{-4}$) and 1,9 % for very rigid ($k \ge 10^{4}$) inclusions. It is necessary to add, that the maximum deviation of GSIFs values received for relative rigidities of inclusion $k = 10^{-4}$ and $k = 10^{4}$ from corresponding values for a crack and rigid line inclusion do not exceed 3 % (for K_{11}) and 4 % (for K_{12}) accordingly. Therefore, for limit cases of inclusion's rigidity the results received by different approaches are in a good agreement.

The curvature radius of elliptic inclusion's tip equals $\rho = h^2/l$. Therefore, for $l/h = 100$ Eq. 16 are fulfilled when $0 < k \ll 4\cdot 10^{-3}$ and $k \gg 60$. In this range of k the error of the offered methods does not exceed 5-7 %. Deviations of the values received basing on a method of interaction integral are smaller, because the last one considers all GSIFs, and not just dominating. For relative rigidities, near to 1, relative deviations become significant, because Eq. 16 are not correct any more for such values of k. Considering that received GSIFs are comparably small, when k limits to 1, and also basing on a good agreement of results for very soft and rigid inclusions it may be contended that with the help of *J*-integral the GSIFs can be determined even in the case of comparable rigidities. In the engineering view, the greatest stress concentrators are very soft or very rigid inclusions, therefore, the main interest possesses the determination of GSIFs near them, so dominating GSIF and mutual integral methods can be successfully applied to these problems.

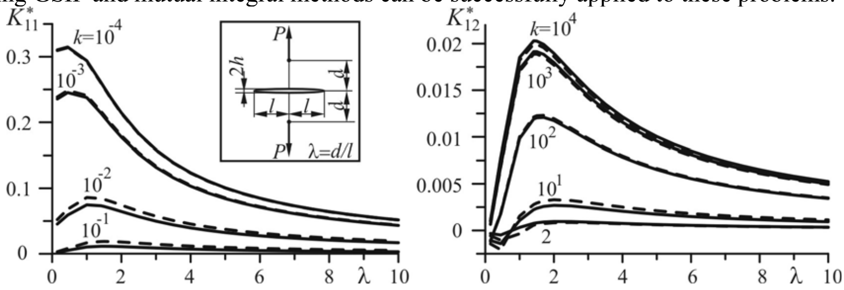

Fig. 2. GSIFs for plane problem with concentrated forces

2. Antiplane elasticity. As well as in the first example the form of inclusion's boundary was described by elongated ellipse with the semiaxes ratio of $l/h = 100$. The load schemes of a matrix with soft $(k<1)$ and rigid $(k>1)$ inclusions are shown on Fig. 3. The data obtained via numerical analysis were compared with the exact solutions for the crack [16] and ribbon like inclusion [1]

$$K_{31} = P\Big/\sqrt{\pi l\left(1+\lambda^2\right)},\quad K_{32}=0 \quad \text{for} \quad k\to 0;$$
$$K_{32} = P\Big/\sqrt{\pi l\left(\lambda^2-1\right)},\quad K_{31}=0 \quad \text{for} \quad k\to \infty, \tag{18}$$

and with asymptotic dependences relating GSIFs and stresses at elliptic inclusion's tip:

$$\sigma_{23} = \sigma^0_{23} + K_{31}\big/\sqrt{\pi\rho} \quad \text{for} \quad 0<k\ll\sqrt{\rho/l},$$
$$\sigma_{13} = \sigma^0_{13} + K_{32}\big/\sqrt{\pi\rho} \quad \text{for} \quad k\gg\sqrt{l/\rho}. \tag{19}$$

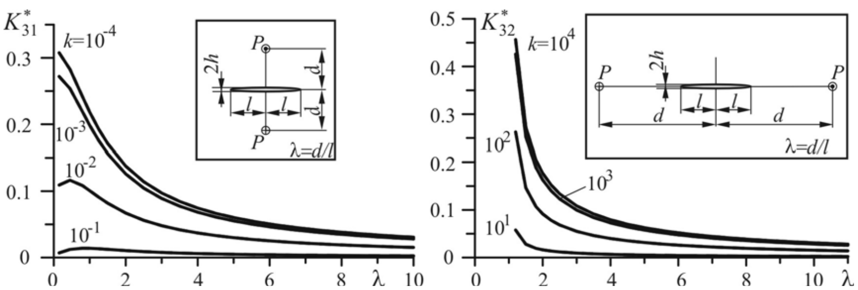

Fig. 3. GSIFs for antiplane shear of a solid with ribbon-like inclusion

The maximum deviation between numerical values of GSIFs for $k=10^{-5}$ and the analytical solution for a crack does not exceed 0.6 % for mutual integral method and 0.8 % for the method of dominating GSIF. For inclusions with relative rigidity of $k=10^5$ the deviation between GSIFs, obtained numerically and the analytical solution does not exceed 1.1 % and 1.4 % respectively. The deviation of stresses at the inclusion's tip obtained directly and by Eq. 19, in the estimated range ($0 < k \ll 10^{-2} \cup k \gg 10^{2}$) does not exceed 5 %.

Summary

This paper concerns the application of *J*-integral to the analysis of thin-walled elastic inclusions. The relationships between the *J*-integral (energy release rate) and generalized stress intensity factors are obtained for all three modes of deformation. With account of these relations, it is shown that rigid inclusions can increase in size only when there are some additional non-mechanical processes that releases energy. In the other case, the reduction of inclusion's size is only possible. Two numerical approaches for determination of GSIFs based on *J*-integral are offered. They have shown high efficiency in numerical examples.

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