About the Dependence between Exponent and Constant in the Paris Law for Fatigue Crack Growth Rate

Jan Kohout

University of Defence, Kounicova St 65, CZ-612 00 Brno, Czech Republic

jan.kohout@unob.cz

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Abstract. Very close relationship between the constant and the exponent of the Paris-Erdogan equation describing fatigue crack growth rate has been described in many papers. Mostly a linear relation between the exponent and the logarithm of the constant has been found which leads to an idea of a *crossover point* or a *pivot point*, in which all the Paris straight lines of one type of structural materials intersect. Deeper statistic studies show that the mentioned very close relationship is only the consequence of unsuitable notation of the Paris-Erdogan equation and in fact this relationship is only illusory. Consequently the location of this point can be determined with such high uncertainty that it is even impossible to tell about certain point of intersection. Hence it seems that the idea of the *crossover point* or the *pivot point* should be left.

Introduction

It is common knowledge that more than 90 per cent of machinery failures are caused due to fatigue of applied structural materials. Before final fracture appears, the stage of fatigue crack growth plays a significant role. The range of stable crack growth is described by the Paris-Erdogan equation [1]

$$v = C(\Delta K)^n \tag{1}$$

where v is fatigue crack growth rate defined as the increment of fatigue crack length during one loading cycle and ΔK is the range of stress intensity factor. The dependence (1) is usually drawn in the log-log fit where is presented by the Paris straight line with the slope of n. Most theoretical considerations lead to the value of $n = 2$ but also higher integers 3, 4 or 6 are mentioned. Regression calculations of experimental results give the real numbers from the range between 2 and 8. Indeed, as mentioned in very successful textbook of Kunz [2], the unit of constant C depending on n value is quite exotic: for the range of stress intensity factor in $\mathrm{MPa\,m^{1/2}}$ and fatigue crack growth rate in m/cycle it is $\mathrm{MPa^{-n}\,m^{1-n/2}}$/cycle. If mm/cycle is used for fatigue crack growth rate (as it will be used in the whole contribution) then the unit of constant C is even more exotic. From practical point of view the best approach is to consider C as a usual number whose value depends on the units used for the range of stress intensity factor and for the fatigue crack growth rate.

The problems connected with the unit of the constant C can be simply removed using the form of the Paris equation [2]

$$v = A\left(\frac{\Delta K}{\Delta K^*}\right)^n \tag{2}$$

when the unit of constant A is the same as the unit of fatigue crack growth rate. If in addition the value $\Delta K^* = 1\ \mathrm{MPa\,m^{1/2}}$ is used, then the numerical values of numbers A and C are equal. The problem consists in the fact that the value $\Delta K^* = 1\ \mathrm{MPa\,m^{1/2}}$ is far from the range of the Paris law validity, mostly even far below the threshold value of the range of stress intensity factor.

The results of experimental determination of fatigue crack growth rate can be fitted using Eq. (1) which contains two regression parameters. But also in Eq. (2) containing in total three parameters only two independent regression parameters can be calculated: they are the exponent n and only one of two constants A a ΔK^* because the other of them must be chosen a priori. The triplet of all parameters n, A and ΔK^* contained in Eq. (2) cannot be used as regression parameters from principal reasons: all straight lines including the Paris straight line are given by only two independent parameters. The more natural approach is to think of parameter ΔK^* as chosen parameter and parameter A as regression parameter.

On principle the parameter ΔK^* can be chosen arbitrary, but for practical reasons it should be chosen *somewhere in the middle* of the region of measured values of the range of stress intensity factor because only then correlation between parameters A and n will be very low. Even such value of ΔK^* can be found for which, when given experimental fatigue crack growth curve is considered, the correlation between mentioned parameters will be zero. This special value of parameter ΔK^* is very close to the geometrical average of fitted values of the range of stress intensity factor ΔK_i. On the other hand, the very high value of the correlation between parameters C and n if the Paris equation (1) is used as the regression function is the consequence of the fact that the case corresponds to the value $\Delta K^* = 1\ \mathrm{MPa\,m}^{1/2}$ which is usually very far from the region of the measured values of ΔK_i (all the more that logarithmic scale is considered).

Studying fatigue crack growth of certain groups of structural materials and describing their growth rate using the Paris equation (1), quite extensive sets of parameter couples C and n can be obtained, see e.g. the paper of Sinclair and Pieri [3] for two aluminium alloys 2024 T3 and 7075 T6. From the reasons discussed above the parameters C and n show very close mutual dependence, see Fig. 1 drawn for both mentioned alloys together (excluding two most distant couples of C and n).

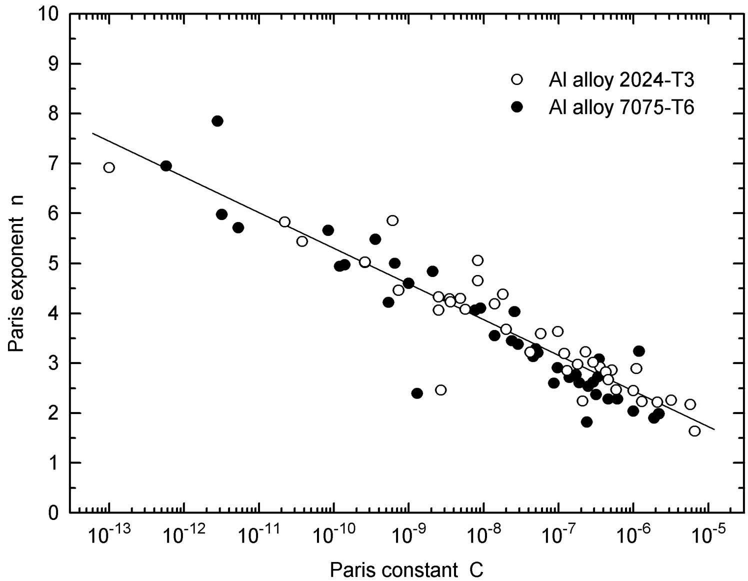

Fig. 1. Dependence between the logarithm of constant C and exponent n for both mentioned alloys together (constant C corresponds to the units $[\Delta K] = \mathrm{MPa\,m}^{1/2}$ and $[v]$ = mm/cycle).

The dependence in Fig. 1 can be classified as linear dependence between log C and n. From both possibilities the possibility chosen in textbook [2] is accepted

$$\log C = a\,n + b \tag{3}$$

which is suitable for further considerations. Substituting this relation into the Paris equation (1) and using relations proposed in [2]

$$a = -\log \Delta K^*\,, \quad b = \log A \tag{4}$$

the equation formally equivalent to Eq. (2) is obtained but here the values of parameters A and ΔK^* are determined by parameters a and b and only one parameter, i.e. exponent n, can change its value. In this case Eq. (2) describes the family of the Paris straight lines intersecting in a common point of intersection with coordinates

$$\Delta K = \Delta K^*\,, \quad v = A \tag{5}$$

which can differ only in slope values n. This point of intersection is usually called as *crossover point* [4, 5] or *pivot point* [6].

Calculation of the position of *pivot point*

Considering parameter n as the dependent variable, the regression function has the form derived from Eq. (3)

$$n = \frac{\log C - b}{a} \tag{6}$$

Regression calculation led to the values of regression parameters and their standard deviations $a = (-1.3896 \pm 0.0763)$ and $b = (-2.6142 \pm 0.3125)$. The value of correlation coefficient of dependence (6) $r = -0.9167$ means that the dependence is relatively quite close. Due to high number of points in Fig. 1 (79) the value of *corrected* correlation coefficient decreases (in magnitude) with respect to *usual* correlation coefficient only slightly: $r_{corr} = -0.9153$. The value of correlation coefficient between regression parameters is c = –0.9588, which will play substantial role in following considerations. Using relations

$$\Delta K^* = 10^{-a}\,, \quad A = 10^{b} \tag{7}$$

inverse to relations (4), the position of the intersection point of all Paris straight lines can be determined: $\Delta K^* = 24.53$ MPa m$^{1/2}$ and $A = 2.431 \cdot 10^{-3}$ mm/cycle. In Fig. 2 the position of the intersection point is drawn together with the areas of reliability in which the point can be found with the probabilities of 68.3 and 90 per cent. Fig. 2 shows that any idea dealing with the intersection point of the Paris straight lines has no sense because the accuracy of its position is catastrophically low: the area covering 90 per cent of intersection points needs nearly one order on the scale of the range of stress intensity factor and nearly three orders on the scale of fatigue crack growth rate. Moreover, the remaining 10 per cent of intersection points are situated even out of that area. Nevertheless, the variances (the ratios of standard deviations and mean values) of both parameters a and b are quite low: 5 and 12 per cent, respectively. Extremely large area of possible localizations of intersection point is the consequence of two facts: (i) relations (7) determining the coordinates of the intersection points are of power type and (ii) the value of correlation coefficient between the parameters a and b is too close to 1 in magnitude. Just that value causes extreme elongation of the reliability area in diagonal direction. Negative value of this coefficient means that the main half-axis of the ellipse representing the reliability area lies on straight line with negative slope.

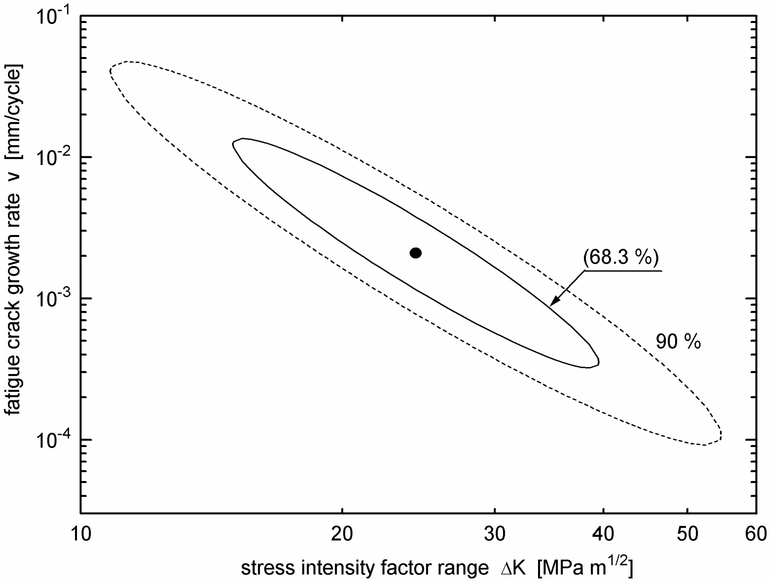

Fig. 2. The position of pivot point common for both aluminium alloys together with the areas of reliability containing presented percentages of pivot points.

In Figs 1 and 2 both aluminium alloys are considered together. Now one can ask if the same results will be obtained in the case that each of the alloys is considered separately. In Fig. 3 the dependences between parameters C and n are drawn separately and for comparison also together, in Fig. 4 the reliability areas are drawn also separately for each of the alloys and also together. Both the figures are evidence that the result of the localization of pivot point is not quantitatively dependent on the fact if both alloys are studied together or separately. The main result of all considerations is the same: the location of the pivot point is extremely vague and, therefore, the idea of pivot point should be left.

Brief description of reliability areas

The aim of the contribution is to present evidence to experts in materials science and engineering that the idea dealing with the common intersection point of the family of the Paris straight lines is very interesting theoretical speculation but, with respect to catastrophic uncertainty in the determination of its position even in the case when the results of many fatigue crack growth curves are available and the dependence between the parameters C and n seems to be very close, the idea remains only the speculation with no practical importance. Therefore only very brief qualitative description of the determination of reliability areas containing defined part of the intersection points of the Paris straight lines will be presented, without deep mathematical derivation.

In general cases of m parameters the reliability areas are described by m-dimensional concentric ellipsoids. In presented case of two parameters the areas are presented by concentric ellipses. Analytically they are described by quadratic forms containing the elements of reciprocal Hessian of the system of regression equations and as statistical criterions the fractiles (quantiles) of χ^2 distribution are used. Because the numbers of degrees of freedom are relatively very high (38 and 37 for individual alloys and 77 if considered together), χ^2 distribution can be approximately expressed using the normal distribution (for this approximation at least 30 degrees of freedom are asked).

Using directly the standard deviations, the 68.3 per cent reliability is obtained. Higher percentage is obtained by multiplying with suitable coefficients which can be determined using fractiles of the Student distribution for given percentage and given number of degrees of freedom.

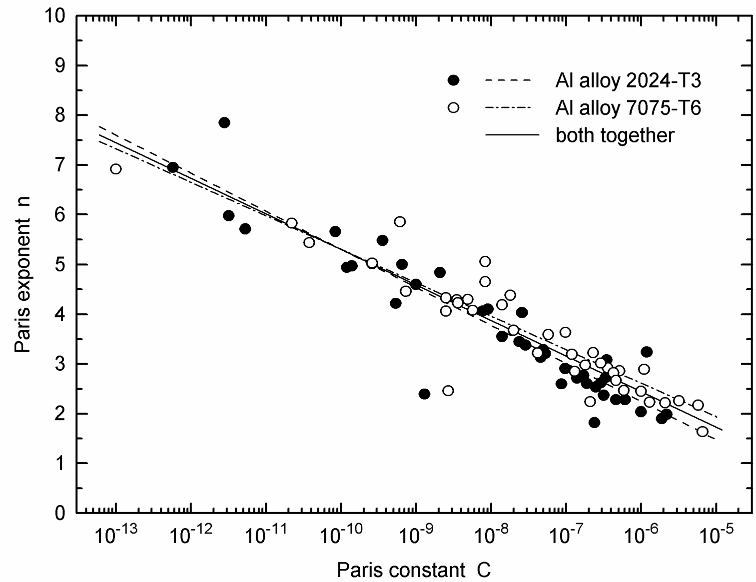

Fig. 3. Dependence between the logarithm of constant C and exponent n for both alloys together and separately (constant C corresponds to the units [ΔK] = MPa m$^{1/2}$ and [v] = mm/cycle).

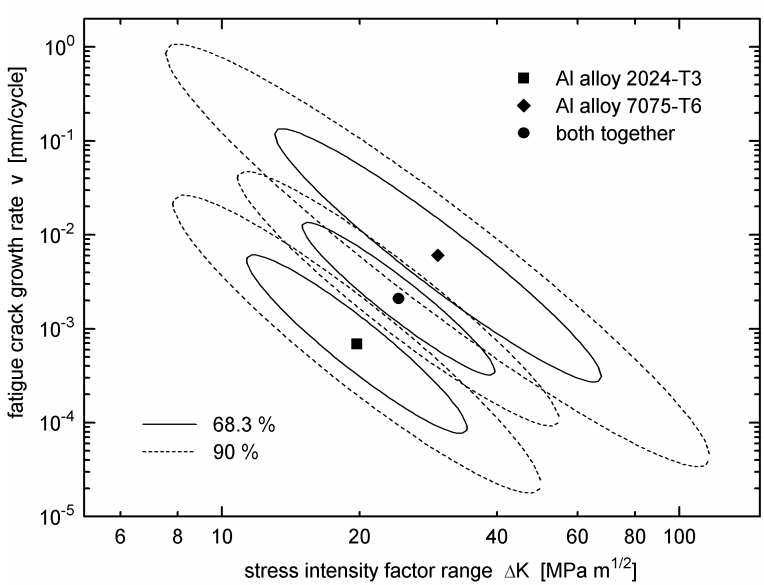

Fig. 4. The position of pivot points for both aluminium alloys together and separately, with the areas of reliability containing presented percentages of pivot points.

Discussion

The Paris-Erdogan equation (1) was published before more than 45 years [1]. The dependence between the Paris constant and the Paris exponent started to be studied early after the publication, e.g. textbook [2] presents nearly 20 publications from years 1972 to 1990 dealing with this dependence. The idea of common intersection point of the Paris straight lines was published before 35 years (e.g. [3] which may not be the first of them) but the author of this contribution has not found up to now any paper dealing with the accuracy of determination of that intersection point. The reason is quite clear: materials scientists are usually not top experts in statistics (neither the author is!) and for professional statisticians it is not the problem interesting enough. Moreover, most regression procedures do not give standard deviations of regression parameters as standard output and the correlation coefficients between the regression parameters which are necessary for the determination of reliability areas are available only very rarely (the author modified regression procedure for this reason).

The idea of pivot point is usually considered for groups of similar structural materials because it is not usual situation to keep at disposal many experimental fatigue crack growth curves of one certain structural material. Therefore at first both the aluminium alloys were considered together. However, as it was shown later, neither separate construction of reliability areas containing defined part of the pivot points led to qualitatively different results.

Conclusions

1. The idea of common intersection point of the Paris straight lines (usually called as *crossover point* or *pivot point*) seems to have no sense because the accuracy of the determination of its location is extremely low (its reliability area can exceed even several orders).
2. Fundamental role in extremely low accuracy in pivot point determination is played by the value of the coefficient of correlation between regression parameters whose magnitude is very close to 1. Standard deviations of regression parameters play only relatively marginal role.
3. The extremely low accuracy in pivot point determination is in principle very similar independently of the fact if both studied alloys are considered together or each of them is considered separately.

Acknowledgement

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References

[1] P.C. PARIS and F. ERDOGAN: J. Basic Engng Vol. 85 (1963), p. 528.

[2] J. KUNZ: *Fundamentals of Fracture Mechanics* (Czech Technical University Publishing House, Prague 2005) (in Czech).

[3] G.B. SINCLAIR and R.V. PIERI: Int. J. Fatigue Vol. 12 (1990), p. 57.

[4] P. HICKERSON and R.W. HERTZBERG: Met. Trans. Vol. 3 (1972), p. 179.

[5] A. IOST and J. LESAGE: Engng Fract. Mech. Vol. 36 (1990), p. 585.

[6] K. TANAKA and S. MATSUOKA: Int. J. Fracture Vol. 13 (1977), p. 563.