CORRELATION BETWEEN PARIS' LAW PARAMETERS BASED ON SELF-SIMILARITY AND CRITICALITY CONDITION

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ABSTRACT

The question about the existence of a correlation between the parameters *C* and *m* of the Paris' law is re-examined in this paper. According to dimensional analysis and incomplete self-similarity concepts applied to the linear range of fatigue crack growth, a power-law asymptotic representation relating the parameter *C* to *m* and to the governing variables of the fatigue phenomenon is derived. Then, from the observation that the Griffith-Irwin instability must coincide with the Paris' instability at the onset of rapid crack growth, the exponents entering this correlation are determined. A fair good agreement is found between the proposed correlation and the experimental data concerning steels and Aluminium alloys.

Introduction

Fatigue crack growth data for ductile materials are usually presented in terms of the crack growth rate, d*a*/d*N*, and the stress-intensity factor range, $\Delta K = (K_{max} - K_{min})$. At present, it is a common practice to describe the process of fatigue crack growth by a logarithmic d*a*/d*N* vs. ΔK diagram (see e.g. Fig. 1).

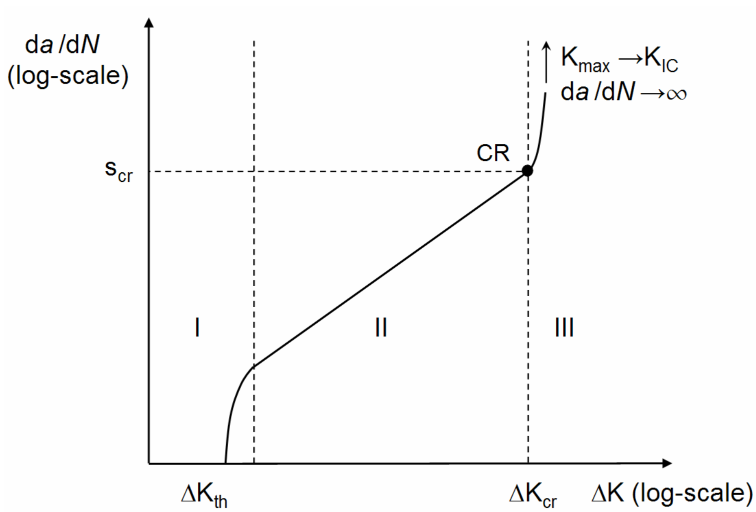

Figure 1. Scheme of the typical fatigue crack propagation curve

Three regions are generally recognized on this diagram for a wide collection of experimental results [1]. The first region corresponds to stress-intensity factor ranges near a lower threshold value, ΔK_{th}, below which no crack propagation takes place. This region of the diagram is usually referred to as *Region I*, or the near-threshold region [2]. The second linear portion of the diagram defines a power-law relationship between the crack growth rate and the stress-intensity factor range and is usually referred to as *Region II* [3]. Finally, when K_{max} tends to the critical stress-intensity factor, K_{IC}, rapid crack propagation takes place and crack growth instability occurs (*Region III*) [4]. In Region II the Paris' equation [5,6] provides a good approximation to the majority of experimental data:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C(\Delta K)^m, \tag{1}$$

where *C* and *m* are empirical constants usually referred to as Paris' law parameters.

From the early 60's, research studies have been focused on the nature of the Paris' law parameters, demonstrating that *C* and *m* cannot be considered as material constants. In fact, they depend on the testing conditions, such as the loading ratio $R = \sigma_{min}/\sigma_{max} = K_{min}/K_{max}$ [7], on the geometry and size of the specimen [8, 9] and, as pointed out very recently, on the initial crack length [10]. However, an important question regarding the Paris' law parameters still remains to be answered: are *C* and *m* independent of each other or is it possible to find a correlation between them based on theoretical considerations? Concerning this point, it is important to take note of the controversy in the Literature about the existence of a correlation between *C* and *m*. For instance, Cortie [11] stated that the correlation is formal with a little physical relevance, and the high coefficient of correlation between *C* and *m* is due to the logarithmic data representation. Similar arguments were proposed in [12], where a correlation-free representation was presented. On the other hand, a very consistent empirical relationship between the Paris' law parameters was found by several Authors [13, 14] and supported by experimental results [3, 13, 15].

In this paper, the correlation existing between the Paris' law parameters is derived on the basis of theoretical arguments. To this aim, both self-similarity concepts [9] and the condition that the Paris' law instability corresponds to the Griffith-Irwin instability at the onset of rapid crack growth are profitably used. Comparing the functional expressions derived according to these two independent approaches, a relation between the Paris' law parameters *C* and *m* is proposed. As a result, it is shown that only one macroscopic parameter is needed for the characterization of damage during fatigue crack growth.

Correlation derived according to self-similarity concepts

According to dimensional analysis, the physical phenomenon under observation can be regarded as a *black box* connecting the external variables (called input or governing parameters) with the mechanical response (output parameters). In case of fatigue crack growth in Region II, we assume that the mechanical response of the system is fully represented by the crack growth rate, q_0=d*a*/d*N*, which is the parameter to be determined. This output parameter is a function of a number of variables:

$$q_0 = F\left(q_1, q_2, \ldots, q_n; s_1, s_2, \ldots, s_m; r_1, r_2, \ldots, r_k\right), \tag{2}$$

where q_i are quantities with independent physical dimensions, i.e. none of these quantities has a dimension that can be represented in terms of a product of powers of the dimensions of the remaining quantities. Parameters s_i are such that their dimensions can be expressed as products of powers of the dimensions of the parameters q_i. Finally, parameters r_i are nondimensional quantities.

As regards the phenomenon of fatigue crack growth, it is possible to consider the following functional dependence:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = F\left(\sigma_y, K_{IC}, \varpi; \Delta K, D, h, a_0; 1-R\right), \tag{3}$$

where the governing variables are summarized in Tab. 1, along with their physical dimensions expressed in the Length-Force-Time class (LFT). From this list it is possible to distinguish between three main categories of parameters. The first category regards the material parameters, such as the yield stress, σ_y, and the fracture toughness, K_{IC}. The second category comprises the variables governing the testing conditions, such as the stress-intensity factor range, ΔK, the loading ratio, *R*, and the frequency of the loading cycle, ω. Concerning environmental conditions and chemical phenomena, they are not considered as primary variables in this formulation and their influence on fatigue crack growth can be taken into account as a degradation of the material properties. Finally, the last category includes geometric parameters related to the material microstructure, such as the internal characteristic length, *h*, and to the tested geometry, such as the characteristic structural size, *D*, and the initial crack length, a_0.

Considering a state with no explicit time dependence, it is possible to apply the Buckingham's Π Theorem [16] to reduce by *n* the number of parameters involved in the problem (see e.g. [8,17-23] for some relevant applications of this method in Solid Mechanics). As a result, we have:

$$\frac{\mathrm{d}a}{\mathrm{d}N}=\left(\frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{y}}}\right)^2\Phi\left(\frac{\Delta K}{K_{\mathrm{IC}}},\frac{\sigma_y{}^2}{K_{\mathrm{IC}}{}^2}D,\frac{\sigma_y{}^2}{K_{\mathrm{IC}}{}^2}h,\frac{\sigma_y{}^2}{K_{\mathrm{IC}}{}^2}a_0;1-R\right)=\left(\frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{y}}}\right)^2\Phi\left(\Pi_1,\Pi_2,\Pi_3,\Pi_4,\Pi_5\right) \tag{4}$$

Variable	Definition	Symbol	Dimensions
q_1	Tensile yield stress of the material	σ_y	FL^{-2}
q_2	Material fracture toughness	K_{IC}	$\mathrm{FL}^{-3/2}$
q_3	Frequency of the loading cycle	ω	T^{-1}
s_1	Stress-intensity range	$\Delta K = K_{\max} - K_{\min}$	$\mathrm{FL}^{-3/2}$
s_2	Characteristic structural size	D	L
s_3	Characteristic internal length	h	L
s_4	Initial crack length	a_0	L
r_1	Loading ratio	$R = \dfrac{K_{\min}}{K_{\max}}$	–

Table 1. Governing variables of the fatigue crack growth phenomenon

At this point, we want to see if the number of the quantities involved in the relationship (4) can be reduced further from five. Considering the nondimensional parameter $\Delta K/K_{IC}$, it has to be noticed that this is usually small in the Region II of fatigue crack growth. However, since it is well-known that the fatigue crack growth phenomenon is strongly dependent on this variable (see e.g. the Paris' law in Eq. (1)), a *complete self-similarity* in this parameter cannot be accepted. Hence, assuming an *incomplete self-similarity* in Π_1, we have:

$$\frac{\mathrm{d}a}{\mathrm{d}N}=\left(\frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{y}}}\right)^2\left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^{\beta_1}\Phi_1\left(\Pi_2,\Pi_3,\Pi_4,\Pi_5\right) \tag{5}$$

where the exponent β_1 and, consequently, the nondimensional parameter Φ_1, cannot be determined from considerations of dimensional analysis alone. Moreover, the exponent β_1 may depend on the nondimensional parameters Π_i. It has to be noticed that Π_2 takes into account the effect of the specimen size and it corresponds to the square of the nondimensional number Z defined in [8], and to the inverse of the square of the *brittleness number s* introduced in [17, 18, 24]. Moreover, the parameter Π_4 is responsible for the dependence of the fatigue phenomenon on the initial crack length, as recently pointed out in [10].

Repeating this reasoning for the parameter (1–*R*), which is a small number comprised between zero and unity, a complete self-similarity in Π_5 would imply that fatigue crack growth is independent of the loading ratio. However, this behavior is in contrast with some experimental results indicating an increase in the response d*a*/d*N* when increasing the parameter *R* [25]. Therefore, assuming again an incomplete self-similarity in Π_5, we have:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}N}&=\left(\frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{y}}}\right)^2\left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^{\beta_1}(1-R)^{\beta_2}\Phi_2\left(\Pi_2,\Pi_3,\Pi_4\right)=\\&=\left(K_{\mathrm{IC}}{}^{2-\beta_1}\sigma_{\mathrm{y}}{}^{-2}\right)(1-R)^{\beta_2}\Delta K^{\beta_1}\Phi_2\left(\Pi_2,\Pi_3,\Pi_4\right)\end{aligned} \tag{6}$$

Comparing Eq. (6) with the expression of the Paris' law, we find that our proposed formulation encompasses Eq. (1) as a limit case when:

$$
\begin{aligned}
& m = \beta_1, \\
& C = \left(K_{\mathrm{IC}}{}^{2-m} \sigma_{\mathrm{y}}{}^{-2}\right)(1-R)^{\beta_2} \Phi_2\left(\Pi_2, \Pi_3, \Pi_4\right).
\end{aligned} \tag{7}
$$

As a consequence, from Eq. (7) it is possible to notice that the parameter *C* is dependent on two material parameters, such as the fracture toughness, K_{IC}, and the yield stress, σ_y, as well as on the loading ratio, *R*, and on the nondimensional parameters Π_2, Π_3, and Π_4. Moreover, Eq. (7) demonstrates, from the theoretical standpoint, the existence of a relationship between the parameters *C* and *m*.

Correlation derived according to the crack growth instability condition

In this section we derive a correlation between the Paris' law parameters similar to that in Eq. (7) on the basis of the condition of crack growth instability. In fact, as firstly pointed out by Forman et al. [4], the crack propagation rate, d*a*/d*N*, is not only a function of the stress-intensity factor range, ΔK, but also on the condition of instability of the crack growth when the maximum stress-intensity factor approaches its critical value for the material.

Focusing our attention on this dependence, Forman et al. [4] observed that the crack propagation rate must tend to infinity when $K_{max} \rightarrow K_{IC}$, i.e.

$$
\lim_{K_{\max} \rightarrow K_{\mathrm{IC}}} \frac{\mathrm{d}a}{\mathrm{d}N} = \infty . \tag{8}
$$

This rapid increase in the crack propagation rate is then responsible for the fast deviation from the linear part of the Region II in the fatigue plot (see e.g. Fig. 1). Considering the transition point labelled CR in Fig. 1 between Region II and Region III, the following relationship between the crack growth rate and the stress-intensity factor range can be derived according to the Paris' law:

$$
\left(\frac{\mathrm{d}a}{\mathrm{d}N}\right)_{CR} = v_{CR} = C\left(\Delta K_{CR}\right)^m , \tag{9}
$$

where ΔK_{CR} denotes the value of the stress-intensity factor range at the point CR. Due to the fact that a rapid variation in the crack propagation rate takes place when the onset of crack instability is reached, it is a reasonable assumption to consider $K_{max}(CR) \cong K_{IC}$. As a consequence, it is possible to correlate the value of ΔK_{CR} with the material fracture toughness:

$$
\Delta K_{CR} = (1-R) K_{\mathrm{IC}} . \tag{10}
$$

Hence, introducing Eq. (10) into Eq. (9), an approximate relationship between the Paris' constants is derived according to the condition that the onset of the Paris' instability corresponds to the Griffith-Irwin instability:

$$
C \cong v_{CR} \left[\frac{1}{(1-R) K_{\mathrm{IC}}}\right]^m . \tag{11}
$$

Moreover, as regards the parameters v_{CR} and K_{IC} entering Eq. (11), it has to be remarked that they are almost constant for each class of material. The dependence on the loading ratio is also put into evidence in Eq. (11).

A closer comparison between Eq. (11) and Eq. (7) permits to clarify the role played by v_{CR}. In fact, Eq. (11) corresponds to the correlation derived according to self-similarity concepts when:

$$m = \beta_1 = -\beta_2,$$
$$v_{CR} = \left(\frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{y}}}\right)^2 \Phi_2\left(\Pi_2, \Pi_3, \Pi_4\right), \tag{12}$$

confirming the experimental observation reported in [3] that v_{CR} should depend on the material properties, on the geometry of the tested specimen, and on the material microstructure. Therefore, considering the same testing conditions, this conventional crack growth rate is almost constant for each class of material and Eq. (11) establishes a one-to-one correspondence between the *C* and *m* values.

Experimental assessment of the proposed correlation

Parameters *C* and *m* entering the Paris' law are usually impossible to estimate according to theoretical considerations and fatigue tests have to be performed. However, many Authors [3, 13, 26] experimentally observed a very stable relationship between the parameters *C* and *m*, which is usually represented by the following empirical formula:

$$C = AB^m, \tag{13}$$

usually written in a logarithmic form:

$$\log C = \log A + m \log B. \tag{14}$$

Taking the logarithm of both sides of the theoretically based relationship between *C* and *m* in Eq. (11), we obtain

$$\log C = \log v_{CR} + m \log\left[\frac{1}{(1-R)K_{\mathrm{IC}}}\right] \tag{15}$$

which corresponds to Eq. (14) if

$$A = v_{CR},$$
$$B = \frac{1}{(1-R)K_{\mathrm{IC}}}. \tag{16}$$

In order to check the validity of the proposed correlation derived according to the instability condition of the crack growth, an experimental assessment is performed by comparing the experimentally determined values of *B* for steels and Aluminium alloys, with those theoretically predicted according to Eq. (16).

Concerning steels and Aluminium alloys, Radhakrishnan [13] collected a number of data from various sources and proposed the following least square fit relationships (ΔK being in MPa$\sqrt{}$m and d*a*/d*N* in m/cycle):

$$\log C = \log(7.6\times10^{-7}) + m\log(1.81\times10^{-2}) \quad \text{for steels,}$$
$$\log C = \log(2.5\times10^{-6}) + m\log(4.26\times10^{-2}) \quad \text{for Al alloys.} \tag{17}$$

In order to compare the prediction of our proposed correlation with the experimentally determined values of *B*, parameters *m* and K_{IC} have to be known in advance. However, only in a few studies both the values of the fatigue parameters and of the fracture toughness are experimentally determined and reported. Therefore, to avoid experimental tests, the values of the material fracture toughness are taken from selected handbooks.

Concerning steels, we assume $A = v_{\mathrm{CR}} = 7.6\mathrm{x}10^{-7}$ m/cycle, as experimentally determined by Radhakrishnan, *R*=0, and we try to estimate the parameter *B* on the basis of the values of the fracture toughness proposed in the ASM handbook [27]. This book provides a collection of values in a diagram K_{IC} vs. both the prior austenite grain size, and the temperature test. Over a large range of temperatures (T from –269°C to 27°C) and grain sizes (*d* from 1 µm to 16 µm), K_{IC} varies from 20 MPa$\sqrt{}$m to 100 MPa$\sqrt{}$m with an average value of K_{IC} = 60 MPa$\sqrt{}$m. Using these data we find:

$$\log C \cong \log(7.6\times10^{-7}) + m\log(1.67\times10^{-2} + \Delta B), \tag{18}$$

where ΔB is related to the fracture toughness excursion, ΔK_{IC} = 40 MPa√m, as:

$$\Delta B = \frac{\mp\Delta K_{\mathrm{IC}}}{\left(\overline{K_{\mathrm{IC}}} \pm \Delta K_{\mathrm{IC}}\right)\overline{K_{\mathrm{IC}}}} = \begin{cases} +3.3\times10^{-2} \\ -6.7\times10^{-3} \end{cases} \tag{19}$$

A good agreement between the proposed estimation based on an average value of the critical stress-intensity factor and the experimental relationship in Eq. (17) for steels is achieved, as clearly shown in Fig. 2(a) (see the dashed-dotted line compared with the solid line). The curves log *C* vs. log *m* obtained using K_{IC}(max) and K_{IC}(min) values are also reported in Fig. 2(a) with dashed lines.

The comparison can also be extended to Aluminium alloys. According to the same procedure discussed above, the estimated average value of the critical stress-intensity factor from handbooks [27–30] is equal to K_{IC} = 35 MPa√m with minimum and maximum values equal to 15 MPa√m and 49 MPa√m, respectively. According to these data we find:

$$\log C \cong \log(2.5\times10^{-6}) + m\log(2.86\times10^{-2} + \Delta B), \tag{20}$$

where ΔB is equal to:

$$\Delta B = \frac{\mp\Delta K_{\mathrm{IC}}}{\left(\overline{K_{\mathrm{IC}}} \pm \Delta K_{\mathrm{IC}}\right)\overline{K_{\mathrm{IC}}}} = \begin{cases} +3.8\times10^{-2} \\ -8.2\times10^{-3} \end{cases} \tag{21}$$

Also in this case, a good agreement between the proposed estimation based on an average value of the critical stress-intensity factor and the experimental relationship in Eq. (17) for Al alloys is achieved (see Fig. 2(b)).

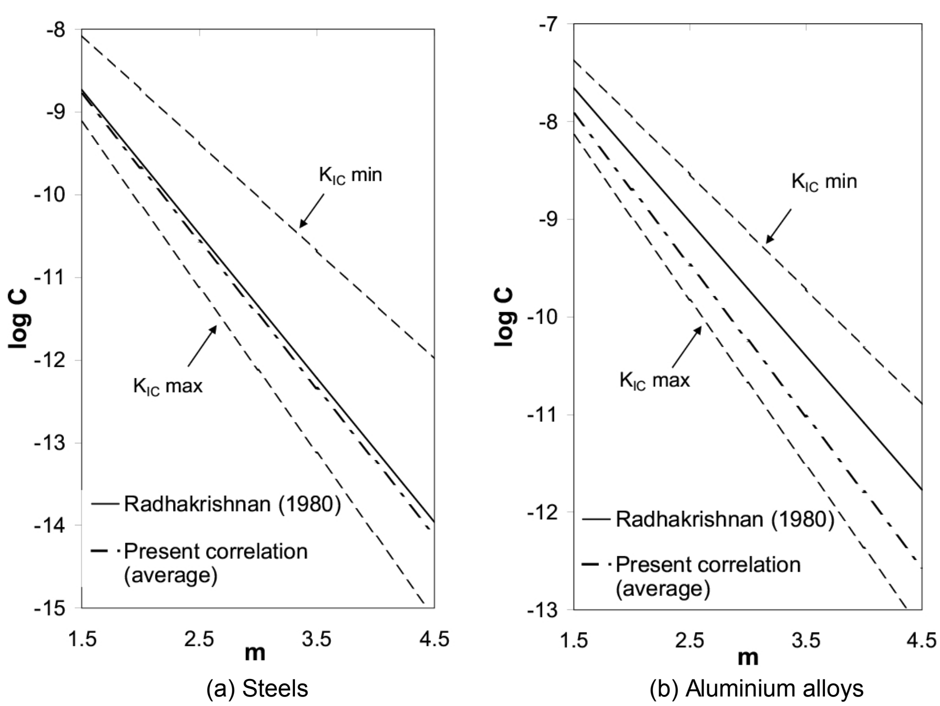

Figure 2. Proposed correlation between Paris' law parameters *C* and *m*.

Conclusion

To shed light on the controversy about the existence of a correlation between the Paris' constants, both self-similarity concepts and the condition that the Paris' law instability corresponds to the Griffith-Irwin instability at the onset of rapid crack growth have been profitably used. Comparing the functional expressions derived from these two independent approaches, an approximate relationship between C and m has been proposed. According to this theory, the parameter C is also dependent on the fracture toughness of the material, on the crack growth rate at the onset of crack instability, and on the loading ratio. The main consequence of this correlation is that only one macroscopic parameter is needed for the characterization of damage during fatigue crack growth. A good agreement between the theoretical predictions obtained using this correlations and experimental data has been achieved.

From the engineering standpoint, it has to be emphasized that our proposed correlation constitutes a useful tool for design purposes. In fact, in case of a lack of experimental fatigue data for a new material to characterize, one could, as a first approximation, determine the parameter C as a function of the exponent m according to Eq. (11). Then, a parametric analysis by varying the exponent m in its usual range of variation can be performed and numerical simulations of fatigue crack growth can be put forward. Parameters v_{CR} and K_{IC} entering the correlation can be either known in advance, or estimated from materials with similar composition, thermal treatment and mechanical properties (see e.g. [31] where this correlation was profitably used for the analysis of the fatigue behavior of structures composed of advanced materials).

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