A STATISTICAL INVESTIGATION OF FATIGUE BEHAVIOUR ACCORDING TO WEIBULL'S WEAKEST-LINK THEORY

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Abstract

The problem of deciding which statistical model is the "true one" in fatigue studies has occupied investigators for many years. Since the amount of data is generally not sufficient to discriminate conclusively between various models, a "defect-tolerant-approach" was used to postulate a suitable model. The two-parameter Weibull distribution was adopted since it can be deduced from a statistical distribution of crack initiating defects. Parameter estimation is carried out based on a large quantity of experimental high cycle fatigue data, which can be used in the context of reliability studies and quantification of the size effect. The Weibull shape parameter was estimated and found to be presented approximately by a Normal distribution. The Weibull scale parameter was suggested to be modelled as a deterministic value.

Introduction

One important problem, which was discovered at an early stage of the fatigue research, is the scatter in fatigue strength or fatigue life. Even if great effort has been made to create similar experiments, the results still exhibit significant scatter and it becomes desirable to use statistical theory. One of the first attempts in this direction was made by Weibull [1]. Based on weakest-link theory he stated the probability distribution which bears his name. The basic premise for the model is that all materials contain inhomogeneities which are distributed at random with a certain density per unit volume. Examples of such inhomogeneities are non-metallic inclusions, which in this paper are referred to as defects. When the defects become the fracture origin, it is found that fatigue failure is triggered by the largest defect present. In this framework, some other assumptions are also worth noting: (a) the largest flaw or the weakest-link of material provides the crack initiation site, (b) the size of defects is small compared with the distance between them (no interaction) and (c) failure is defined as the first failure of any element, i.e. a serial system is postulated. Further, the Weibull distribution enables to take into account the influence of load, component cross-section and component size on the fatigue strength. Viewed from this standpoint, the two-parameter Weibull distribution was chosen to describe the scatter observed in constant amplitude high cycle fatigue.

This study comprises to the application of the weakest link theory together with a presentation of estimated distribution parameters. The estimates are obtained from high cycle fatigue test specimens loaded in tension or rotating bending. The results obtained in this study allow a quantification of the size effect [2-4] and estimation of reliability when the loading can be regarded as deterministic.

Weakest-link theory

Probability of failure at a given number of cycles

The uniform gauge length of a standard smooth fatigue specimen has the volume V_0. It is subjected to a homogeneous stress of amplitude s. If the stress is multiaxial, all stress amplitudes have to be understood as equivalent stresses. Defects are considered to be randomly distributed within the specimen with a finite density per unit volume. The volume can be divided into many small volume elements dV, such that the probability of finding more than one defect is small. It is assumed that defects can be located in any of these volume elements, with the occurrence being independent of each other. Then the probability of finding a defect in dV is proportional to dV when the volume element is small, and may be written as

$$\mathrm{d}P = \lambda \mathrm{d}V, \tag{1}$$

where λ is a positive function. Let D denote the number of defects located in the volume V_0. The probability $P(D = d)$ is then a Poisson process with rate λ [5]. From the property of the Poisson distribution it follows that λ is the number of defects per unit volume, which causes failure at a stress equal to, or less than s, i.e. $\lambda(s)$. According to Weibull's weakest link theory, the probability of survival of the homogeneously stressed volume V_0 is given by the probability that all the $m = V_0/\mathrm{d}V$ volume elements survive, i.e.

$$Q_S = 1 - P = \prod_{i=1}^{m}(1 - \mathrm{d}P) = (1 - \lambda(s)\mathrm{d}V)^m. \tag{2}$$

Introducing $V_0 = m\mathrm{d}V$ into equation (2) yields

$$Q_S = \left(1 - \frac{\lambda(s)V_0}{m}\right)^{\frac{m}{\lambda(s)\mathrm{d}V}\lambda(s)\mathrm{d}V}, \tag{3}$$

and when m increases infinitely, while dV decreases at the same time, the expression becomes an exponential function given by

$$Q_S = \exp(-\lambda(s)V_0). \tag{4}$$

According to Weibull [1] the rate $\lambda(s)$ can be approximated by a power rule model written as

$$\lambda = \frac{1}{V_0}\left(\frac{s}{s_0^*}\right)^{b_s}, \tag{5}$$

where s_0^* and b_s are the Weibull scale and shape parameters, respectively. Introducing equation (5) into equation (4) enables the probability of survival to be written as a two-parameter Weibull distribution:

$$Q_S = \exp\left\{-\left(\frac{s}{s_0^*}\right)^{b_s}\right\}. \tag{6}$$

When $s = s_0^*$, the probability of survival is equal to 36.8 per cent. For an arbitrary component of homogenous material it may be shown that the component's probability of survival is found by integration of the stress over the volume V [1]:

$$Q_S = \exp\left\{ -\int_V \left(\frac{s}{s_0^*} \right)^{b_s} \frac{\mathrm{d}V}{V_0} \right\}. \tag{7}$$

Introducing the net nominal stress s_{net}, the component's probability can be rewritten as

$$Q_S = \exp\left\{ -\left(\frac{s_{\mathrm{net}}}{s_0^*} \right)^{b_s} \frac{V}{V_0} \frac{1}{V} \int_V \left(\frac{s}{s_{\mathrm{net}}} \right)^{b_s} \mathrm{d}V \right\}. \tag{8}$$

For convenience the Weibull stress factor K_{W} is introduced

$$K_{\mathrm{W}} = \left\{ \frac{1}{V} \int_V \left(\frac{s}{s_{\mathrm{net}}} \right)^{b_s} dV \right\}^{1/b_s}, \tag{9}$$

which allows equation (8) to be written as

$$Q_S = \exp\left\{ -\left(K_{\mathrm{W}} \frac{s_{\mathrm{net}}}{s_0^*} \right)^{b_s} \frac{V}{V_0} \right\}. \tag{10}$$

Therefore, the fatigue strength s_{net} of an arbitrary component of volume V at the same number of cycles and probability of survival as a standard specimen of volume V_0 and fatigue strength $s_{\mathrm{net,\,0}}$ is given by

$$s_{\mathrm{net}} = \frac{s_{\mathrm{net,\,0}}}{K_{\mathrm{W}} \left(V / V_0 \right)^{1/b_s}}. \tag{11}$$

The nominator of this expression may be interpreted as the fatigue notch factor of the component,

$$K_{\mathrm{f}} = K_{\mathrm{W}} \left(V / V_0 \right)^{1/b_s}. \tag{12}$$

From the definition of K_{W} follows that it is always less than the theoretical stress concentration factor K_{t}.

Probability of failure within n cycles

Consider now the fatigue life N as a random variable, evaluated at a given stress of amplitude s and mean stress s_{m}. When the distribution of fatigue strength $F_S = 1\text{-}Q_S$ is given and the S-N-curve can be described by a power relationship (Basquin's equation)

$$NS^m = C, \tag{13}$$

the cumulative distribution function of fatigue life F_N is implied. That is, one can transform F_S by the S-N equation to obtain F_N mathematically as

$$P\left[N\left(s_{\mathrm{a}}, s_m \right) \le n \right] = F_N\left(n \mid s_{\mathrm{a}}, s_{\mathrm{m}} \right) = 1 - \exp\left\{ -\left(\frac{n}{n_0^*\left(s_{\mathrm{a}}, s_{\mathrm{m}} \right)} \right)^{b_n} \frac{V}{V_0} \right\}, \tag{14}$$

where the shape parameter b_n and scale parameter n_0^* is related to b_s and s_0^* through the following equations:

$$b_n = \frac{b_s}{m}, \tag{15a}$$

$$n_0^* = C\left(s_0^*\right)^{-m}. \tag{15b}$$

The relationship between n and V is then obtained by equating the survival probabilities for varying volume, i.e.

$$\frac{n_1}{n_2} = \left(\frac{V_2}{V_1}\right)^{1/b_n}, \tag{16}$$

where b_n determines the extent to which the fatigue life varies with the volume.

Experimental investigation

Above, distribution functions were presented for the fatigue strength and fatigue life by means of theoretical considerations. Each of these distribution functions contains two parameters, usually denoted as the scale parameter and the shape parameter. In the following, estimates of the shape parameters are presented, obtained from a large quantity of high cycle fatigue data. Only data with fatigue lives greater than fifty thousand cycles were considered. The parameters associated with the distribution of fatigue life were estimated only from complete samples, while it was decided to consider both complete and censored samples for the fatigue strength distribution.

Fatigue test data were given in one of the following categories: (a) fatigue data by using several specimens at one or more stress levels and (b) fatigue data obtained by the staircase method [6-8]. Smooth and notched standard specimens as well as large smooth and notched specimens have been considered. The data have been grouped by loading types, specimen forms, type of material and references to the literature in Table 1. All specimens considered were tested in a standard laboratory environment.

Moment estimators for Weibull parameters

A data set consisted of nominally identical specimens tested at the same load level or by means of the staircase method. For each data set of size k, the parameters of a two-parameter Weibull distribution were determined by means of the classical methods of moments [9]. When several identical specimens were tested at two different stress levels, but close to each other, Weibull quantiles [9] were used to obtain estimates of the shape parameter associated with the fatigue strength distribution. Each estimate was also checked from the observation traced on the Weibull probability paper [9].

Estimation of the shape parameter

The representation in subsequent figures is worth noting. The ordinate is the estimated mean probability, obtained from order statistic by sorting the estimated values according to increasing magnitudes. The cumulative probability of the i^{th} estimated shape parameter can then be calculated as

$$P_{(i)} = \frac{i}{k+1}, \qquad (17)$$

where k is the sample size. The abscissa is the estimated shape parameter b_s or b_n as defined in equations (6) and (14), respectively.

TABLE 1 Summary of fatigue specimens.

Material	Designation	Specimen shape[1]	Loading[2]	Specimens	Ref.
Forged steel, Fig. 1a and Fig. 2a	30CrNiMo8, 36CrNiMo6, SS 1650-01, JIS SF 50, X2CrNi19 9, 0.21%C	SM & RD	AX & RB	950	[8, 10-16]
	30CrNiMo8, SS 1650-01	NT	AX & RB	530	[10-11, 16]
Cast steel, Fig. 1b	JIS ScMn 2A, 500-7/ISO 1083, 300/ISO 185, 250/ISO 185	SM & NT	AX & RB	128	[8, 12]
Aluminium alloys, Fig. 2b	AlZnMgCu1.5, 6061-T6	SM & NT	AX	415	[15, 17]

[1]SM, smooth specimen; RD, hourglass specimen; NT, notched specimen

[2]AX, axial loading; RB, rotating bending

The data sets for the forged steel specimens were analysed first. To access the dependence on the estimated shape parameter on specimen geometry, the data sets were divided into two groups. The first group consisted of smooth and hourglass specimens, while the other group consisted only of notched specimens.

The distribution of the estimated shape parameter b_s is shown in Fig. 1a for a variety of forged steels subjected to either axial loading or rotating bending, as indicated in Table 1. The total number of points in Fig. 1a is 53, equal to the number of data sets. Each data set consists of 6 to 50 nominally identical specimens. For both groups it was observed that the estimated values could be approximated by a Normal distribution [9] with parameters μ and σ, i.e. mean and standard deviation. The sample mean value for the smooth and hourglass specimens was found to be slightly larger than the corresponding value for the notched specimens, as has also been observed by Tanaka *et al*. [18]. The sample mean value was determined as the average of the estimated values. Fig. 1a shows that the estimated mean value was 27 for smooth and hourglass specimens, while 21 was found for the notched specimens. This implies that the scatter in fatigue strength is less for smooth and hourglass specimens than for notched specimens.

In order to be able to study the effect on material on the shape parameter it was further decided to estimate the distribution parameters for cast steels. For cast steels too few data sets were found in the literature to make a distinction between smooth and notched specimens. Thus, both smooth and notched specimens were assumed to belong to a single population. As for the forged steels, it was observed that the estimated values could be approximated by a Normal distribution as shown in Fig. 1b. The estimated sample mean value for the cast steels was found to be $b_s \approx 14.8$, a lower estimated value than for the forged steels shown in Fig. 1a. This is supported by Beretta and Murakami [19], who found that the probability of finding a critical defect in a given volume is larger for cast steels compared with forged steels. It should further be mentioned that the specimen volume and the value of the shape parameter was found to be uncorrelated.

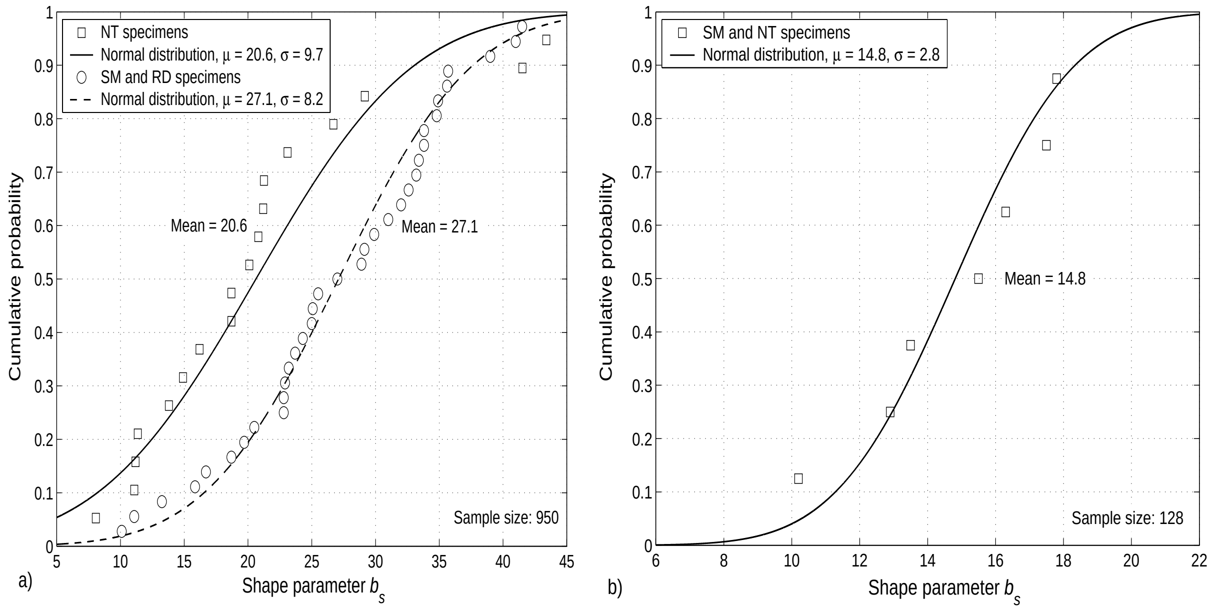

FIGURE 1. Distribution of the estimated shape parameter b_s for (a) forged steels and (b) cast steels.

Consider now the distribution of fatigue life as expressed in equation (15) with the associated shape parameter denoted by b_n. The distribution of the estimated shape parameter is shown in Fig. 2a for forged steels. Due to limited data sets for the notched specimens it was assumed that smooth, hourglass and notched specimens belong to a single population. As for the fatigue strength distribution, it was observed that the estimated values could be approximated by a Normal distribution. The estimated mean sample value was found to be $b_n \approx 5.2$. Fatigue data for aluminium alloys have also been collected to be able to establish estimates of the shape parameter b_n as shown in Fig. 2b. From this figure the value of the estimated mean sample value is 4.6.

So far only the shape parameter has been considered. This is due to the fact that the scale parameter can be found when the shape parameter is known. Although some uncertainty may be associated with this simplification, it is practical to consider both n_0^* and s_0^* as deterministic values. The relation between the scale and shape factor can be obtained when the mean fatigue life or fatigue strength is known. By equating the sample mean value with the expected value of the Weibull distribution, the relation between the shape parameter b_n and scale parameter n_0^* can be expressed as

$$n_0^* = \frac{\frac{1}{k}\sum_{i=1}^{k} n_i}{\Gamma\left(\frac{1}{b_n}+1\right)}, \tag{19}$$

where Γ(.) denotes the gamma function. The same expression is also valid for the relation between s_0^* and b_s.

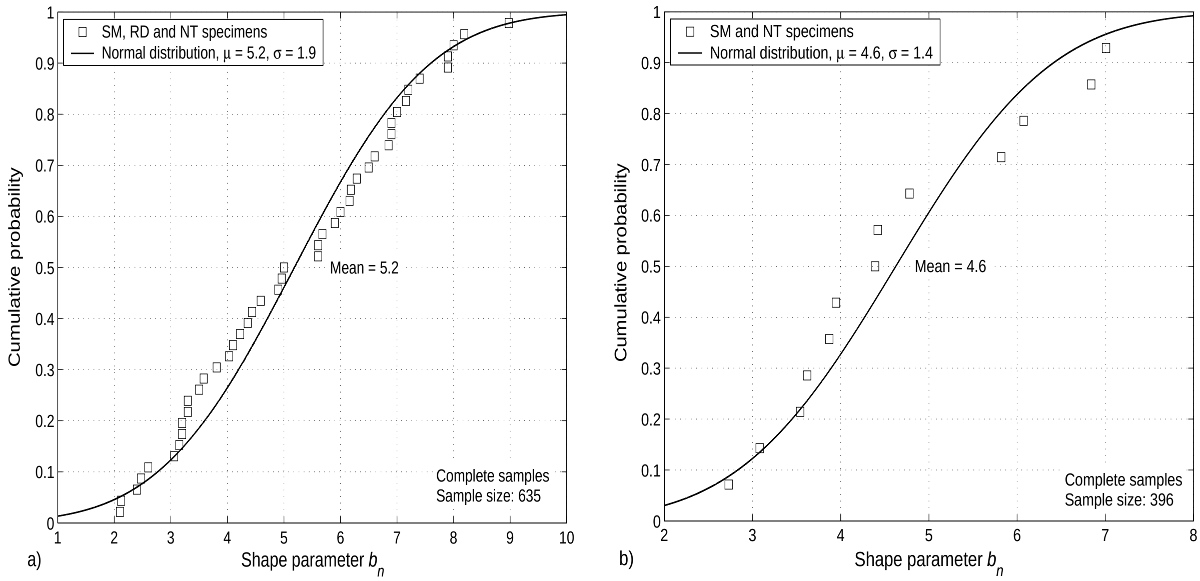

FIGURE 2. Distribution of the estimated shape parameter b_n for (a) forged steels and (b) aluminium alloys.

Conclusions

Based on the literature data reviewed, and on the analyses presented in this paper, the following conclusions are drawn.

1. The basic assumption of the Weibull model is the existence of a statistical distribution of defects in the volume.
2. Based on the Weibull distribution a statistical quantification of the size effect can be deduced.
3. The weakest-link approach can be used to estimate the probability of failure for a generic component from a standard smooth fatigue specimen.
4. The estimated shape parameter b_s for cast steels is less than for forged steels, due to the greater probability of finding a larger defect in a cast steel.
5. It is found that the shape parameter for the fatigue strength distribution is grater than the shape parameter for the fatigue life distribution.

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