Modelling crack growth by level sets

M. Stolarska [1] , D.L. Chopp [1] , N. Moës [2] , and T. Beltyschko [3]

[1] Department of Engineering Science and Applied Mathematics, Northwestern University, Evanston, IL 60208, U.S.A.
[2] Laboratoire de Mécanique et Matériaux, Ecole Centrale de Nantes, Nantes, France
[3] Department of Mechanical Engineering, Northwestern University, Evanston, IL 60208, U.S.A

ABSTRACT: An algorithm which couples the level set method with the extended finite element method to model crack growth is described. The level set method is used to represent the crack location, including the location of crack tips. The extended finite element method is used to compute the stress and displacement fields necessary for determining the rate of crack growth. This combined method requires no remeshing as the crack progresses, making the algorithm very efficient. The combination of these methods has a tremendous potential for a wide range of applications. A numerical example is presented to demonstrate the accuracy of the combined methods. In addition, a level set algorithm for modelling crack growth in three dimensions is described.

INTRODUCTION

In this paper, which is a summary of the work presented in [1], we describe an algorithm where the level set method (LSM) is coupled with the extended finite element method (X-FEM) to model crack growth. The LSM is a numerical scheme developed by Osher and Sethian [2] to model the motion of interfaces. In the LSM the interface is represented as the zero level set of a function of one higher dimension. The current formulation of the LSM has no provisions for modelling free moving endpoints on curves. Here, we present an extension of the LSM for modelling the evolution of an open curve segment and use this extension to model fatigue crack growth. We also present an extension of the level set formulation for crack growth in three dimensions.

The X-FEM [3] algorithm enables the modelling of crack growth without remeshing. Rather than adapting the mesh so that it coincides with the discontinuity of the crack, X-FEM allows for the crack to pass arbitrarily through elements by incorporating enrichment functions to handle the field discontinuities. In this manner the mesh can remain fixed throughout the evolution of the crack.

The LSM and X-FEM work well, offering complimentary capabilities. The level set representation of the crack simplifies the selection of the enriched nodes, as well as the definition of the enrichment functions. In addition to modelling the crack growth problem, the combined methods were also used to model holes and material inclusions in [4] and three-dimensional planar crack growth in [5]. The LSM and X-FEM, as described in this paper, provide a simple and efficient algorithm for modelling two-dimensional crack growth. Moreover, the LSM provides a simple and natural method for extending the crack growth model into three dimensions.

GOVERNING EQUATIONS

In this section we review the governing equations for the displacement field in an elasto-static analysis. The domain of the problem is Ω with boundary Γ. The boundary Γ is subdivided into two parts, Γ_u, where the displacement is prescribed, and Γ_t, where the traction is prescribed. In addition to the external boundary, the crack surface presents an additional boundary, Γ_c, inside Ω. The coincident crack surfaces are denoted by Γ_c^+ and Γ_c^- and are traction free.

The strong form of the equilibrium equations and boundary conditions is

$$\nabla \cdot \sigma + b = 0 \text{ in } \Omega \tag{1}$$

$$\sigma \cdot n = \mathrm{T} \text{ on } \Gamma_t \tag{2}$$

$$\sigma \cdot n = 0 \text{ on } \Gamma_{c+} \text{ and } \Gamma_{c-} \tag{3}$$

$$u = \mathrm{U} \text{ on } \Gamma_u \tag{4}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{u}$ is the displacement, $\boldsymbol{b}$ is the body force per unit volume, and $\boldsymbol{n}$ is the unit outward normal. The prescribed traction and displacement are respectively **T** and **U**.

The weak form of the equilibrium equation is

$$\int_{\Omega} \varepsilon(u) : C : \varepsilon(v)\, d\Omega = \int_{\Omega} b \cdot v \, d\Omega + \int_{\Gamma_t} \mathrm{T} \cdot v \, d\Gamma \quad \forall\, v \in U_0 \tag{5}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor, $\boldsymbol{C}$ is the Hooke tensor, and $\boldsymbol{v}$ is a test function in the space of test functions U_0. For details of the transition from strong form to weak form see [6].

THE LEVEL SET METHOD

The LSM is a numerical technique for tracking the motion of interfaces. In this method, the interface of interest is represented as the zero level set of a function $\phi(\boldsymbol{x}(t),t)$. This function is one dimension higher than the dimension of the interface. The evolution equation for the interface can then be expressed as an equation for the evolution of ϕ. For our purposes, cracks will be considered as one-dimensional manifolds in two-dimensional space.

In general, an $(n-1)$-dimensional interface $\gamma(t) \in \mathrm{R}^n$ can be formulated as the level set curve of a function $\phi(\boldsymbol{x},t)$: $\mathrm{R}^n \times \mathrm{R} \to \mathrm{R}$, where $\gamma(t) = \{\boldsymbol{x} \in \mathrm{R}^n : \phi(\boldsymbol{x},t) = 0\}$. The motion of $\gamma(t)$ is translated into an evolution equation for ϕ. An important property of the function ϕ is that it is defined to be the signed-distance to the interface, i.e. $\phi(x,t) = \pm \min_{x_\gamma \in \gamma(t)} \left\| x - x_\gamma \right\|$. Therefore, at a given point in the domain we know where we are in relation to the interface. We will make use of this feature in modelling crack growth.

The LSM has typically been used to track interfaces which are either closed curves or curves that extend to the boundary of the domain. To represent interfaces that are open curves the level set method needs to be extended. A crack is represented as the zero level set of a function $\psi(\boldsymbol{x},t)$, and a crack tip is represented as the intersection of the zero level sets of $\psi(\boldsymbol{x},t)$ and $\phi_i(\boldsymbol{x},t)$. Here, in the case of more than one crack tip, the subscript *i* corresponds to the *i*th tip.

THE EXTENDED FINITE ELEMENT METHOD

Modelling crack growth in a traditional finite element framework is cumbersome due to the need for the mesh to match the geometry of the discontinuity. Many methods require remeshing of the domain at each time step. In X-FEM the need for remeshing is eliminated. The mesh does not change as the crack grows and is completely independent of the location and geometry of the crack. The discontinuities across the crack are modelled by enrichment functions.

To illustrate this, consider the X-FEM displacement approximation for a vector valued function $\boldsymbol{u}(\boldsymbol{x})$: $\mathrm{R}^2 \to \mathrm{R}^2$ given by

$$u^h(x,t)=\sum_{i\in I} u_i(t)N_i(x)+\sum_{j\in J} b_j(t)N_j(x)H(\psi(x,t))+\sum_{k\in K} N_k(x)\left(\sum_{l=1}^{4} a_k^l(t)B_l(r,\theta)\right) \quad (6)$$

where N_i $(\boldsymbol{x})$ is the shape function associated with node i and t is the time. In Eq. 6 J is the set of all nodes whose support is bisected by the crack. The set K contains all the nodes of the elements containing the crack tip. The nodal degrees of freedom corresponding to the displacement are $\boldsymbol{u}_i$, $\boldsymbol{b}_j$, and $\boldsymbol{a}_k$.

The second important and distinguishing factor to note in Eq. 6 is the enrichment functions $H(\psi(\boldsymbol{x},t))$ and $B_l(r,\theta)$.The function $H(y)$ is defined as

$$H(y)=\begin{cases} 1 \;\text{ for } y>0 \\ -1 \;\text{ for } y<0 \end{cases} \quad (7)$$

where $y=0$ is defined to be along the crack. This implies that the discontinuity occurs at the location of the crack. The branch function B_l is defined by

$$B_l(r,\theta)=\left\{\sqrt{r}\sin\frac{\theta}{2},\ \sqrt{r}\cos\frac{\theta}{2},\ \sqrt{r}\sin\frac{\theta}{2}\sin\theta,\ \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\} \quad (8)$$

where (r,θ) is a polar coordinate system with its origin at the crack tip and $\theta=0$ tangent to the crack at its tip. The above functions span the asymptotic crack tip solution of elasto-statics, and $\sqrt{r}\sin{\theta}/{2}$ takes into account the discontinuity across the crack face.

The introduction of the discrete approximation in Eq. 6 into the principle of virtual work given by Eq. 5 leads to a system of linear equations. The stress intensity factors are computed using the domain form of the J –integral as described in [7]. The direction in which the crack will propagate from its current tip, θ_c, is obtained using the maximum hoop stress criteria [3].

LEVEL SET ALGORITHM FOR MODELLING CRACK GROWTH IN 2D

We model one-dimensional crack growth in a level set framework by representing the crack as the zero level set of a function $\psi(\boldsymbol{x},t)$. An endpoint of the crack is represented as the intersection of the zero level set of ψ with an orthogonal zero level set of the function $\phi_i(\boldsymbol{x},t)$,where i is the number of tips on a given crack. The values of the level set functions are stored only at the nodes. The functions are interpolated over the mesh by the same shape functions as the displacement.

The level set function representing the initial crack is constructed by computing the signed-distance function for the crack. A difficulty in doing this arises from the fact that, although the crack tip lies within the domain, the level set function representing the crack must initially be constructed on the entire domain. To circumvent this problem, the initial crack is extended tangentially from its tip and the signed-distance function is constructed from this extended crack. The level set functions that represent the crack tip are initially defined by $\phi_i(x,0) = (x - x_i)\cdot \bar{t}$,where $\bar{t}$ is a unit vector tangent to the crack at its tip and $\boldsymbol{x}_i$ is the location of the ith crack tip. The function ϕ_i is planar and has a zero level set which is orthogonal to ψ at the crack tip. The initial level set functions, ψ and ϕ_i, and the representation of the crack are shown in Figure 1.

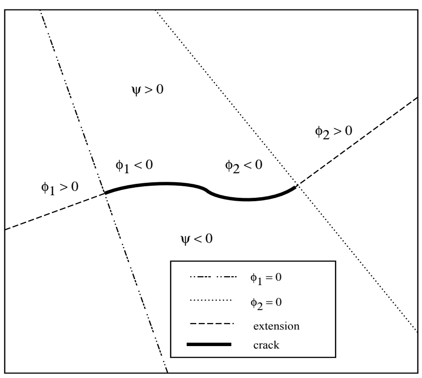

Figure 1: Construction of initial level set
ψ is functions.

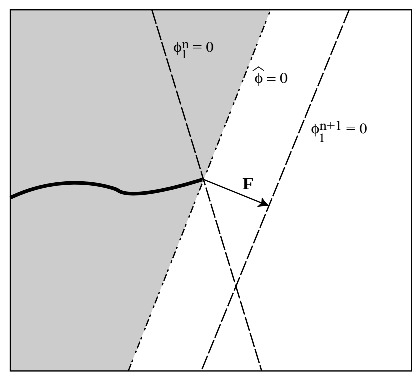

Figure 2: Level set function update.
recomputed only in white region.

An important consideration is that, although the actual crack is embedded inside the domain, the zero level set of ψ cuts through the entire domain. In the level set framework, the crack is considered to be the zero level set of ψ where $\phi_i \leq 0$ for all ϕ_i associated with the given crack. This is consistent with the initial conditions and will continue to be so as the level set functions are updated.

The evolution of ϕ_i and ψ for each crack is determined by the crack growth direction for the given crack, θ_c. In each step, the displacement of the crack tip is given by the prescribed vector $\boldsymbol{F} = (F_x, F_y)$. The magnitude of crack extension depends on the crack growth law. The current location of the crack tip, $\boldsymbol{x}_i = (x_i, y_i)$, is also used in the equations of evolution.

Let the current values of ϕ_i and ψ at step n be ϕ_i^n and ψ^n. The algorithm for the evolution of the level set functions ϕ_i and ψ is as follows.

1. We first rotate ϕ_i^n so that $\boldsymbol{F}$ is orthogonal to the zero level set of ϕ. ϕ_i^n after rotation is referred to as $\overline{\phi_i}$ and given by $\overline{\phi_i} = (x - x_i) \cdot F / \|F\|$. This gives the signed distance to $\phi=0$.

2. The crack is extended by computing new values of ψ^{n+1} only where $\overline{\phi_i} > 0$. In this region the newly computed value of ψ is $\psi^{n+1} = \pm \left\| (x - x_i) \times F / \|F\| \right\|$. Again, this is the signed-distance to $\psi=0$. The sign of ψ^{n+1} is chosen so that it is consistent with the current sign on a given side of the crack.

3. ϕ_i^{n+1} is computed so that it represents the updated location of the crack tip. Eq. 9 is in the form of the general iterative equation of evolution for level set functions as given in [2].

$$\phi_i^{n+1} = \overline{\phi_i} - \Delta t \|F\| \tag{9}$$

   The rotated level set function $\overline{\phi_i}$ is calculated exactly. Since ϕ_i^{n+1} is calculated from $\overline{\phi_i}$, it is important to note that rather than being updated by an iterative process, ϕ_i^{n+1} is also explicitly recalculated at each step. The recalculation of ϕ_i^n to ϕ_i^{n+1} is illustrated in Figure 2.

The location of the new crack tip i can now be determined by finding the intersection of the zero level sets of ϕ_i^{n+1} and the newly extended ψ^{n+1}.

COUPLING THE LSM AND X-FEM

The LSM and X-FEM couple naturally to model crack growth. The values of ψ and ϕ_i are stored at nodes. Any information needed for crack growth, such as the location of the crack tip, can be obtained from these nodal values, making it unnecessary to store any other information pertaining to the crack. The X-FEM algorithm is an efficient finite element scheme that solves the elliptical problem that determines the evolution of a crack on a mesh. The mesh is unchanged throughout the computation of the evolution of the crack. For these reasons, the LSM and X-FEM work well together.

Moreover, the level set representation of the crack facilitates the computation of the enrichment. The Heaviside enrichment function is defined so that the discontinuity is coincident with the crack, $\psi = 0$. Therefore, to determine the location of a point relative to the crack one merely has to determine the sign of ψ at that point. The branch enrichment functions are defined in coordinates local to the crack tip. Because $\psi = 0$ is orthogonal to $\phi = 0$, these two level sets create this needed coordinate system naturally. Finally, the nodes chosen for enrichment can be determined by the nodal values of ψ and ϕ.

EXAMPLE - CRACK GROWTH FROM A FILLET

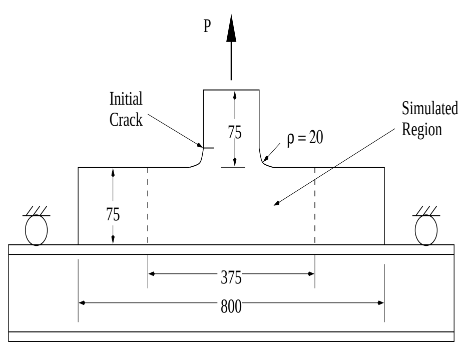

Figure 3: Experimental configuration of crack crack) and growth from a fillet taken from [8].

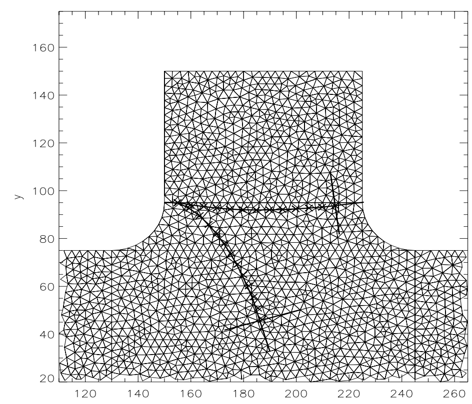

Figure 4: Crack paths for rigid (upper and flexible (lower crack) constraint.

This example shows the effect of I-beam thickness of the growth of a crack from a fillet in a structural member. The configuration of the problem is taken from experimental work found in [8] and is shown in Figure 3. The computational domain is outlined by the dashed line. Only the limiting cases of a very thick, rigid I-beam and a very thin, flexible I-beam are discussed. The effects of the thickness are incorporated into the problem through the boundary conditions.

The structure is loaded at the top boundary with a load of P =*20 kN*. The initial crack is 5 *mm* in length. Crack growth was simulated for a total of 12 steps, with each step size of length 5 *mm*. Figure 4 is a close-up of the mesh in the vicinity of the fillet. The level set representation of the crack (solid line) is compared to a piecewise-linear segment representation [9] (shown by X 's).

LEVEL SET FORMULATION FOR CRACK GROWTH IN 3D

We are currently investigating the three dimensional extension of the level set algorithm described above. Here we present a formulation that will be coupled with a boundary element method to model crack growth in three dimensions. Three-dimensional crack growth has also been modelled by X-FEM and the LSM in [5] and [10]. The difference between the following algorithm and [5] is that this allows for crack to grow out of plane, i.e. the cracks are not required to stay flat. In [10], this out of plane growth is modelled by level set methods. However, the following scheme will prove to be much more efficient than the iterative process that is used in [10].

The configuration for the level set functions in three dimensions is shown in Figure 5. As in the 2D case, the crack face corresponds to $\psi(\boldsymbol{x},t) = 0$, and the crack tip, which is now a one-dimensional manifold, is represented by the intersection of an additional level set function, $\phi(\boldsymbol{x},t) = 0$, and $\psi(\boldsymbol{x},t) = 0$. Again, these two function are the signed-distance to their zero level set, $\phi = 0 \perp \psi = 0$, and the actual crack is represented as the set $\{x : \psi = 0 \text{ where } \phi \le 0\}$.

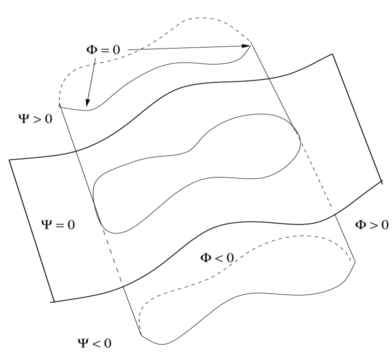

Figure 5: Configuration of level set functions in 3 dimensions.

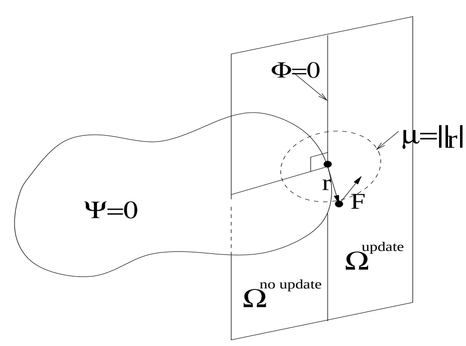

Figure 6: Geometry of update in 3 dimensions.

The update of level sets in three dimensions follows the same general procedure as for the two dimensional case and is described below. The extensions arise in the fact that the two dimensional calculations are in essence done on a series of planes perpendicular to the crack. In addition, because the velocity at the crack tip depends on position, it must be extended to the entire computational domain in order for the level set functions to be updated everywhere. Finally, in order to find the position of a point in the computational domain relative to the crack tip an additional level set function must be introduced. This function is defined as the perpendicular distance to the crack tip, and is denoted as $\mu(\boldsymbol{x},t)$. The update geometry along with $\mu(\boldsymbol{x},t)$ is shown in Figure 6. Assume that the values of ϕ, ψ, and μ at time n are ϕ^n, ψ^n, and μ^n.

1. $\phi = 0 \cap \psi = 0$, i.e. the crack tip, is found using methods described in [11], and μ^{n+1} is computed using a Fast Marching Method (FMM), a method based on level sets the details of which are given in [12]. The position relative to the crack tip is given by $\boldsymbol{r} = \|\mu\|\nabla\mu$.

2. We then extend the velocity vector given on the crack tip into the entire computational domain by an extension of the FMM.

3. ϕ^n is rotated so that it is orthogonal to the velocity field on the crack tip. Given a velocity field $\boldsymbol{F}$, this is done geometrically and given by $\bar{\phi} = \boldsymbol{r} \cdot \boldsymbol{F}/\|\boldsymbol{F}\|$. This and the following step are taken from the 2D case.

4. The crack is extended by computing new values of ϕ^{n+1} where $\bar{\phi} > 0$ (Ω^{update}). This is also done geometrically and given by $\psi^{n+1} = \pm\|\boldsymbol{r} \times (\boldsymbol{F}/\|\boldsymbol{F}\|)\|$.

5. ϕ^{n+1} is updated using Eq. 9.

A new crack tip velocity is now calculated using a boundary element method, or any other method which solves the necessary equilibrium equations, and the process is repeated. Because the level sets are updated geometrically, it is simple to implement and efficient.

CONCLUSIONS

The LSM and X-FEM couple naturally to solve the elasto-static fatigue crack problem. The level set formulation is used to model the crack and update the crack tip at each iteration. The geometry of the crack is easily represented by two signed-distance functions whose zero level sets are orthogonal to one another at the crack tip. These two properties facilitate the calculation of the enrichment functions. We use X-FEM to solve the elasto-static problem and determine the direction of crack growth. The example shows that the results obtained by the level set formulation are comparable to those obtained with a piecewise-linear segment representation of the crack. The advantages of the LSM and X-FEM in two dimensions are simple and useful. Moreover, the method presented here can also be extended to three dimensions.

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