On Some 3-D Thermoelastic Problems of Periodically Layered Composites Containing Interface Cracks

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ABSTRACT: *In this contribution, the homogenized model with microlocal parameters of two-layered periodic elastic space is applied to analyse the three-dimensional problems of thermal stresses near interface cracks. Within the framework of this model, the method of solving value-boundary problems resulting from thermal means and the general formulation for an arbitrary interface crack in terms of integro-differential singular equations are outlined.*

INTRODUCTION

Increasing use of composite layered materials in situations involving both mechanical and thermal environments requires the study of different aspects of their failure behaviour. In particular, the interface fracture is a common encountered case (see, for example, the recent proceedings edited by Rossmanith [1]).

This paper is considered with the stationary thermoelastic problem of bimaterial periodically space weakened by an arbitrary plane crack with a smooth profile lying on one of the straight interfaces of layers. It is a sequel of our earlier investigations [2,3] in the two-dimensional case. To get the result an approximate method will be employed. The layered space is replaced with a homogenized model of the linear thermoelasticity with microlocal parameters, which was devised by Woźniak [4] and found wide application (see a survey paper [5]). Within this model the harmonic potentials are constructed from the thermal boundary conditions on the crack plane by solving the ordinary boundary-value problems related to a half-space analogously in isothermal situations (see a potential function method described by Kaczyński [6]). Further, the general problem is formulated in terms of integro-differential singular equations of Newton's potential type.

Problems close to that considered here will be found, for example, in [7-8] for a homogeneous infinite medium as well as in [9] for bonded dissimilar materials.

PROBLEM FORMULATION AND BASIC EQUATIONS

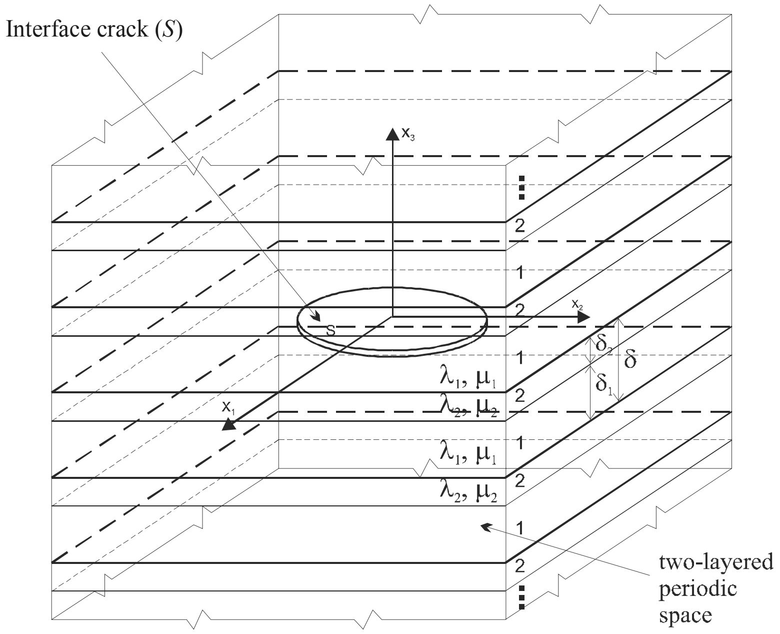

Figure 1: Two-layered periodic space with an interface crack

Consider a microperiodic-laminated space as shown in Figure 1. A repeated thin fundamental layer of thickness δ is composed of two homogeneous sublayers, denoted by 1 and 2, with thicknesses δ_1 and δ_2 $\left(\delta = \delta_1 + \delta_2\right)$, and with different thermo-mechanical properties. Let λ_l, μ_l be the Lamé constants, k_l the thermal conductivity, $\beta_l / \left(\lambda_l + \frac{2}{3}\mu_l\right)$ the coefficients of the volume expansion; here and in the sequel, all quantities (material constants, stresses, etc.) pertaining to these sublayers will be designated by the index l or (l) taking the values of 1 and 2, respectively. A Cartesian coordinate system $\left(x_1, x_2, x_3\right)$ is introduced with the x_3-axis directed normal to the layering and the $x_1 x_2$-plane being one of the interfaces of the materials.

We deal with the spatial thermoelastic problems involving this stratified medium weakened by an interface arbitrary crack and conducting heat under steady-state conditions. The perfect mechanical bonding and ideal thermal contact between the layers (excluding the region occupied by the crack) is assumed. Moreover, the crack surfaces are required to be free of tractions and certain thermal conditions on the crack faces (temperature-free crack or insulated crack) are taking into account. The closed solutions of the considered problems cannot be obtained because of a complicated geometry of the body and complex boundary conditions. Therefore, the homogenized model of this layered composite is applied to seek an approximate solution. Without going into details we present only the final basic equations of the homogenized model of the treated body (see [6] for a thorough derivation)[1]:

– the equation of heat conduction for a macro-temperature ϑ (in the absence of heat sources)

$$\vartheta_{,11} + \vartheta_{,22} + k_0^{-2}\,\vartheta_{,33} = 0 \tag{1}$$

– the macro-displacement w_i equations (in the absence of the body forces)

$$\begin{aligned} &0{,}5\left(c_{11}+c_{12}\right) w_{\gamma,\,\gamma\alpha} + 0{,}5\left(c_{11}-c_{12}\right) w_{\alpha,\,\gamma\gamma} + \\ &+ c_{44} w_{\alpha,\,33} + \left(c_{13}+c_{44}\right) w_{3,3\alpha} = K_1\,\vartheta_{,\,\alpha} \\ &\left(c_{13}+c_{44}\right) w_{\gamma,\gamma 3} + c_{44}\, w_{3,\,\gamma\gamma} + c_{33}\, w_{3,\,33} = K_3\,\vartheta_{,\,3} \end{aligned} \tag{2}$$

– the constitutive relations for the fluxes $q_i^{(l)}$ and the stresses $\sigma_{ij}^{(l)}$

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[1] Indices $i,\,j$ run over 1, 2, 3 and are related to the Cartesian coordinates whereas indices $\alpha,\,\gamma$ run over 1, 2. Summation convention holds unless otherwise stated. Subscripts preceded by a comma indicate partial differentiation with respect to the corresponding coordinates.

$$
\begin{aligned}
& q_\alpha^{(l)} = -k_l\,\vartheta_{,\alpha}\,, \quad q_3^{(l)} = -K\,\vartheta_{,3} \\
& \sigma_{\alpha 3}^{(l)} = c_{44}\left(w_{\alpha,3} + w_{3,\alpha}\right) \\
& \sigma_{33}^{(l)} = c_{13}\left(w_{1,1} + w_{2,2}\right) + c_{33}\,w_{3,3} - K_3\,\vartheta \\
& \sigma_{12}^{(l)} = \mu_l\left(w_{1,2} + w_{2,1}\right) \\
& \sigma_{11}^{(l)} = d_{11}^{(l)}\,w_{1,1} + d_{12}^{(l)}\,w_{2,2} + d_{13}^{(l)}\,w_{3,3} - K_2^{(l)}\,\vartheta \\
& \sigma_{22}^{(l)} = d_{12}^{(l)}\,w_{1,1} + d_{11}^{(l)}\,w_{2,2} + d_{13}^{(l)}\,w_{3,3} - K_2^{(l)}\,\vartheta
\end{aligned}
\tag{3}
$$

Positive coefficients appearing in the above equations, describing the material and geometric properties of the composite constituents, are given in the Appendix. Let us observe that the condition of perfect thermal and mechanical contact between the layers is satisfied. Assuming that two sublayers have the same thermo-mechanical properties we pass directly to the well-known equations of classical theory of uncoupled stationary thermoelasticity for a homogeneous isotropic space, given in [10].

BOUNDARY-VALUE PROBLEM AND METHOD OF SOLUTION

We are interested in determining the steady-state thermal stresses and deformations in a two-layered space weakened by a crack occupying the region *S* (of an arbitrary shape with a smooth boundary) in the $x_1\,x_2$ - plane (see Figure 1). Within the framework of the above-homogenized model we consider the boundary-value problem: find fields ϑ and w_i suitable smooth on $R^3 - S$ such that (1) and (2) hold and satisfy the prescribed thermal and mechanical boundary conditions on the crack surfaces resulting from a given external loading. Without loss of generality, it is assumed that the crack faces are taking to be free from mechanical tractions. According to the results given in [6], the method of solving consists of seeking the temperature potential ω related to the solution of (1) (with the prescribed thermal conditions) as follows

$$
\begin{aligned}
& \vartheta\left(x_1, x_2, x_3\right) = -\frac{\partial^2 \omega\left(x_1, x_2, z_0\right)}{\partial z_0^2}\,\Big|_{z_0 = k_0 x_3}\,, \\
& \vartheta_{,11} + \vartheta_{,22} + k_0^{-2}\,\vartheta_{,33} = 0 \;\Rightarrow\; \nabla^2\omega \equiv \omega_{,11} + \omega_{,22} + \frac{\partial^2 \omega}{\partial z_0^2} = 0
\end{aligned}
\tag{4}
$$

and of representing the macro-displacements by three harmonic potentials denoted by $\phi_i\left(x_1, x_2, z_i\right),\ z_i = t_i x_3,\ \nabla^2\phi_i = 0,\ \forall\ i \in \{1,2,3\}$ in the form

$$
\begin{aligned}
w_1 &= \left(\phi_1 + \phi_2 + c_1\,\omega\right)_{,\,1} - \phi_{3,\,2} \\
w_2 &= \left(\phi_1 + \phi_2 + c_1\,\omega\right)_{,\,2} + \phi_{3,\,1} \\
w_3 &= m_1\,t_1\,\frac{\partial\phi_1}{\partial z_1} + m_2\,t_2\,\frac{\partial\phi_2}{\partial z_2} - c_2\,k_0\,\frac{\partial\omega}{\partial z_0}
\end{aligned}
\tag{5}
$$

This general representation is given under the assumption that $\mu_1 \neq \mu_2$ and $t_\alpha \neq k_0$ (the other cases are detailed in [6] and the constants $t_i,\ c_\alpha,\ m_\alpha$ appearing here are given also in [6]).

The classical procedure for obtaining the solution is used. The macro-temperature ϑ or the thermal potential ω is first found to determine the induced thermal stresses by using (5) and (6). Making use of the appropriate boundary conditions on the crack surface S and the principle of superposition, the problem under study may be reduced to some mixed boundary-value problem related to a half-space $x_3 \geq 0$. It is convenient to resolve the general problem into symmetric part A associated with the prescribed temperature T_0 (or temperature gradient q_0) and skew-symmetric part B arising from the insulation of the crack as follows:

$$
\text{A:}\quad
\begin{cases}
\vartheta\big|_{S^+} = T_0 \\
\vartheta_{,\,3}\big|_{\left(R^2 - S^+\right)} = 0
\end{cases}
\quad \text{or} \quad
\begin{cases}
\vartheta_{,\,3}\big|_{S^+} = q_0 \\
\vartheta_{,\,3}\big|_{\left(R^2 - S^+\right)} = 0
\end{cases}
$$

$$
\begin{cases}
\sigma_{31}\big|_{R^2} = \sigma_{32}\big|_{R^2} = 0 \\
\sigma_{33}\big|_{S^+} = 0 \\
w_3\big|_{\left(R^2 - S^+\right)} = 0
\end{cases}
\tag{6}
$$

$$\text{B:} \quad \begin{cases} \vartheta_{,3}\Big|_{S^+} = Q_0 \\ \vartheta_{,3}\Big|_{(R^2-S^+)} = 0 \end{cases}, \quad \begin{cases} \sigma_{33}\Big|_{R^2} = 0 \\ \sigma_{3\alpha}\Big|_{S^+} = 0 \\ w_\alpha\Big|_{(R^2-S^+)} = 0 \end{cases} \tag{7}$$

Invoking a direct analogy between the thermal crack problems in hand and their mechanical counterparts, exploited in [6], both cases are reduced to classical mixed problems of potential theory. Thus, the conditions (6) and (7) involve the reduction of the thermoelastic crack problems to that of finding one harmonic potential *f* in case A and two harmonic potential *g*, *h* in case B in a half-space $x_3 \geq 0$ as follows (the constants $a_\alpha, \hat{v}, \hat{\beta}$ are defined in the cited paper [6]):

$$\text{A:} \begin{cases} f_{,33}\Big|_{x_3=0^+} \overset{S}{=} -\dfrac{t_1 t_2 (a_2 - a_1 + \alpha_0)}{c_{44}\, t_-} \vartheta\Big|_{x_3=0^+}, \\ f_{,3}\Big|_{x_3=0^+} \overset{R^2-S}{=} 0 \end{cases} \tag{8}$$

$$\text{B:} \begin{cases} g_{,33} + \hat{v}\left(g_{,22} - h_{,12}\right) \overset{S}{=} \hat{\beta}\,\omega_{,31}\Big|_{x_3=0^+} \\ h_{,33} + \hat{v}\left(h_{,11} - g_{,12}\right) \overset{S}{=} \hat{\beta}\,\omega_{,32}\Big|_{x_3=0^+} \end{cases}, \quad g_{,3} = h_{,3} \overset{R^2-S}{=} 0 \tag{9}$$

The above problems can be formulated by the integro-differential equations by using the representations of unknown functions $f_{,3}$, $g_{,3}$, $h_{,3}$ through the potentials of the simple layer and making use of the well-known their harmonic properties. Similarly to the derivation, given by Fabrikant [11], it is obtained the governing equations corresponding to opening mode of crack extension in case A and to the sliding and tearing modes in case B:

$$\text{A:}\, \Delta \iint\limits_S \frac{\omega_3(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2 + (x_2-y)^2}} = 2\pi A \sqrt{\frac{c_{11}}{c_{33}}} \left[-\frac{t_1 t_2 (a_2 - a_1 + \alpha_0)}{c_{44}\, t_-} \vartheta\Big|_{x_3=0^+} \right] \tag{10}$$

$$
\mathrm{B}:\Delta\iint_S \frac{\omega_1(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2+(x_2-y)^2}}+
$$

$$
+\ \hat{\nu}\left(\partial_{12}\iint_S \frac{\omega_2(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2+(x_2-y)^2}}-\partial_{22}\iint_S \frac{\omega_1(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2+(x_2-y)^2}}\right)=
$$

$$
=2\pi A\,\hat{\beta}\,\omega_{,31}\Big|_{x_3=0^+}\ ,
$$

$$
\Delta\iint_S \frac{\omega_2(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2+(x_2-y)^2}}+
$$

$$
+\ \hat{\nu}\left(\partial_{12}\iint_S \frac{\omega_1(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2+(x_2-y)^2}}-\partial_{11}\iint_S \frac{\omega_2(x,y)\,dx\,dy}{\sqrt{(x_1-x)^2+(x_2-y)^2}}\right)=
$$

$$
=2\pi A\,\hat{\beta}\,\omega_{,32}\Big|_{x_3=0^+} \tag{11}
$$

Here the following differential operators were used:

$$
\partial_{11}=\frac{\partial^2}{\partial x_1^2}\ ,\ \partial_{12}=\frac{\partial^2}{\partial x_1\partial x_2}\ ,\ \partial_{22}=\frac{\partial^2}{\partial x_2^2}\ ,\ \Delta=\partial_{11}+\partial_{22}\,.
$$

Moreover, $A=\dfrac{t_+\,c_{33}}{c_{11}\,c_{33}-c_{13}^2}$ and the unknowns functions are denoted by $\omega_i(x_1,x_2)=w_i\left(x_1,x_2,0^+\right)$.

Some closed-form solutions of the above integral-differential equations for a specific thermal and mechanical loading are known if the crack *S* has a circular shape. They lead to the typical non-oscillating inverse square-root stress singularities on the contrary to the classical solutions of interface crack problems (see, for example, [9]). Thus the intensification of local thermal stresses in the neighbourhood of the crack border may be measured by the stress intensity factors (SIF) governing the onset of crack propagation in linear fracture mechanics (see the expressions for SIF in [6]). In general, the results may be obtained by using numerical methods.

APPENDIX

Denoting by $B_l = \lambda_l + 2\mu_l \quad (l = 1,2)$, $\eta = \delta_1/\delta$, $\bar{B} = (1-\eta)B_1 + \eta B_2$, $\bar{K} = (1-\eta)\, k_1 + \eta\, k_2$, the positive coefficients in Eqs (1-3) are given by the following formulae:

$$k_0 = \left[\left(\eta\, k_1 \bar{K} + (1-\eta)\; k_2 \bar{K}\right)/k_1\, k_2\right]^{1/2} \;\; ;$$

$$c_{33} = B_1 B_2/\bar{B}, \quad c_{44} = \mu_1\mu_2/\left[(1-\eta)\,\mu_1 + \eta\,\mu_2\right]$$

$$c_{11} = c_{33} + \frac{4\eta\;(1-\eta)(\mu_1-\mu_2)(\lambda_1-\lambda_2+\mu_1-\mu_2)}{\bar{B}} \;\; ,$$

$$c_{12} = \frac{\lambda_1\lambda_2 + 2\left[\eta\,\mu_2 + (1-\eta)\,\mu_1\right]\left[\eta\,\lambda_1 + (1-\eta)\,\lambda_2\right]}{\bar{B}} \;\; ,$$

$$c_{13} = \left[(1-\eta)\,\lambda_2\; B_1 + \eta\,\lambda_1\; B_2\right]/\bar{B} \;\; ;$$

$$K = k_1\, k_2/\bar{K}, \quad K_3 = \left[(1-\eta)\,\beta_2\; B_1 + \eta\,\beta_1 B_2\right]/\bar{B}, \quad K_2^{(l)} = \left(2\mu_l\beta_l + \lambda_l K_3\right)/B_l$$

$$K_1 = \left[\eta\,\beta_1\;\lambda_2 + (1-\eta)\,\beta_2\;\lambda_1\right]/\bar{B} + 2\left[(1-\eta)\,\mu_1 + \eta\mu_2\right]\left[\eta\,\beta_1 + (1-\eta)\,\beta_2\right]/\bar{B}\,;$$

$$d_{11}^{(l)} = \left[4\mu_l\left(\lambda_l+\mu_l\right) + \lambda_l\; c_{13}\right]/B_l\,, \quad d_{12}^{(l)} = \left(2\mu_l\;\lambda_l + \lambda_l\; c_{13}\right)/B_l\,, \quad d_{13}^{(l)} = \lambda_l\; c_{33}/B_l$$

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