Critical Plane Approach for Fatigue Life Estimation under Multiaxial Random Loading

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ABSTRACT*: A multiaxial fatigue criterion for metals subjected to random loading is presented. Accordingly, the orientation of the critical plane, where fatigue life estimation is carried out, is determined from the weighted mean position of the principal stress directions; a nonlinear combination of stress components acting on the critical plane is used to define an equivalent normal stress. A specific cyclic counting method and a general damage model are employed to process such an equivalent stress and to determine fatigue endurance. Finally, an application of the criterion to some relevant random fatigue tests (proportional bending and torsion) is presented.*

INTRODUCTION

Many multiaxial high-cycle fatigue criteria related to constant amplitude loading are aimed at reducing a given multiaxial stress state to an equivalently effective uniaxial stress condition (see, for instance, Ref. 1 for a critical assessment of these criteria). Criteria for variable amplitude loading (e.g. random loading) are usually proposed as a generalisation of their counterparts for constant amplitude loading, by introducing a cycle counting method (e.g. rainflow method) and a damage model (e.g. the Miner rule). Some criteria present specific cycle counting methods to resolve multiaxial loading histories into individual cycles (e.g. see Refs 2,3).

A multixial high-cycle fatigue criterion has recently been proposed by the first two authors [4] for constant amplitude loading. Such a criterion, based on the critical plane approach, considers (for fatigue failure assessment) an equivalent stress based on a nonlinear combination of the maximum normal stress and the amplitude of shear stress, acting on the critical plane. In a recent paper [5], the authors have attempted to extend the original criterion of Ref. 4 to random loading, by cycle counting the equivalent stress and employing a damage model.

In the present paper, the equivalent stress is cycle counted in a different manner with respect to that of Ref. 5, in order to account for the time-varying direction of the shear stress acting on the critical plane. Finally, such a proposal is discussed by considering experimental data related to combined bending and torsion random loading [6].

ORIENTATION OF THE CRITICAL PLANE

Let $s(t)$ be the stress tensor at point P of a body subjected to a given fatigue loading. At each time instant, the principal stresses, s_1, s_2, s_3 ($s_1 \geq s_2 \geq s_3$), and the principal stress directions can be calculated. The instantaneous orientation of the orthogonal coordinate system P123 is determined by using the principal Euler angles $f(t), q(t)$ and $y(t)$ [7-9]. Then, the mean directions $\hat{1}, \hat{2}$ and $\hat{3}$ of the principal stress axes can be obtained from averaging the instantaneous values of the principal Euler angles. The averaging procedure is carried out by employing a weight function which accounts for the effect of the maximum principal stress. Such a function depends on two parameters deduced from the S-N curve for uniaxial tension-compression : the normal stress fatigue limit, s_{af}, and the coefficient $m_s = -1/m$, where m is the negative slope of the S-N curve considered.

A correlation between the weighted mean direction $\hat{1}$ of the maximum principal stress and the normal **w** to the critical plane has been proposed in Ref. 4:

$$d = 45 \; \frac{3}{2} \left[1 - \left(\frac{t_{af}}{s_{af}} \right)^2 \right] \qquad (1)$$

where d is the angle, expressed in degrees, between $\hat{1}$ and **w** (Fig. 1a), and t_{af} is the shear stress fatigue limit for fully reversed torsion.

DEFINITION OF AN EQUIVALENT STRESS

Consider the critical plane Δ, passing through point P, and the related orthogonal coordinate system Puvw (Fig. 1b). The direction cosines of u-, v- and w-axis, with respect to the PXYZ frame, can be computed as a function of the two angles φ and ϑ [4]. The stress vector $\mathbf{S}_w$ acting at point P of the

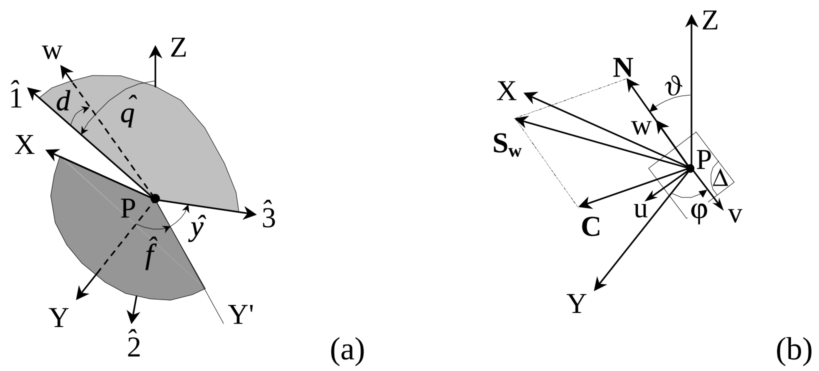

Figure 1: (a) Correlation between weighted mean principal stress directions ($\hat{1},\hat{2}$ and $\hat{3}$) and normal **w** to the critical plane; (b) Puvw and PXYZ coordinate systems, with the w-axis normal to the critical plane Δ

critical plane Δ (Fig. 1b) can be expressed as follows, being **w** the unit vector normal to Δ:

$$\mathbf{S}_{\mathrm{w}} = \mathrm{s} \cdot \mathbf{w} \tag{2}$$

and the normal stress vector **N** is obtained from Eq. 2 :

$$\mathbf{N} = \left(\mathbf{w} \cdot \mathbf{S}_{\mathrm{w}}\right) \mathbf{w} \tag{3}$$

By recalling Eqs 2 and 3, the shear stress vector **C** lying on the critical plane Δ is computed through the following expression:

$$\mathbf{C} = \mathbf{S}_{\mathrm{w}} - \mathbf{N} \tag{4}$$

For *multiaxial constant amplitude cyclic loading*, the vectors **N** and **C** are periodic functions of time. Hence, it has been proposed [4] to consider the maximum value $N_{\max}$ of **N** and the amplitude C_a of **C** to determine the amplitude of the equivalent stress $s_{eq,a}$, namely:

$$s_{eq,a} = \sqrt{N_{\max}{}^2 + \left(\frac{s_{af}}{t_{af}}\right)^2 C_a{}^2} \tag{5}$$

Note that the amplitude $s_{eq,a}$ has to be compared with the fatigue limit s_{af} to perform fatigue limit assessment [4].

The definition of N_{max} is trivial, while the definition of the amplitude of **C** is a complex problem owing to its time-varying direction. The procedure proposed by Papadopoulos [10] has been adopted in Ref. 4 to determine the mean value C_m and the amplitude C_a of the shear stress vector **C** :

$$C_m = \min_{\mathbf{C}'} \left\{ \max_{0 \le t \le T} \left\| \mathbf{C}(t) - \mathbf{C}' \right\| \right\}; \quad C_a = \max_{0 \le t \le T} \left\| \mathbf{C}(t) - C_m \right\| \tag{6}$$

where the symbol $\|\cdot\|$ indicates the norm of a vector. In the case of syncronous sinusoidal loading, the curve described by the tip of the shear stress vector **C** becomes an ellipse, whose major semi-axis coincides with the amplitude C_a of **C** [10].

In the case of *random loading*, an equivalent normal stress s_{eq} can be defined by considering the norms $\|\mathbf{N}(t)\|$ and $\|\mathbf{C}(t)\|$ of the vectors **N** and **C**, that is [5] :

$$s_{eq}(t) = \sqrt{\|\mathbf{N}(t)\|^2 + \left(\frac{s_{af}}{t_{af}} \right)^2 \|\mathbf{C}(t)\|^2} \tag{7}$$

Such an equivalent stress $s_{eq}(t)$ represents a nonnegative one-dimensional random process.

CYCLE COUNTING AND FATIGUE LIFE ESTIMATION FOR RANDOM LOADING

Instead of directly cycle counting the variable $s_{eq}(t)$ as in Ref. 5 (in this way, changes in direction of the vector **C** are neglected), a new procedure is hereafter proposed.

Thescalar value of the vector $\mathbf{N}(t)$ is taken as the cycle counting variable, since the direction of such a vector is fixed with respect to time (e.g. see Ref. 11). For the sake of simplicity, $\mathbf{N}(t)$ and $\mathbf{C}(t)$ (defined through its components $C_u(t)$ and $C_v(t)$) are treated as discrete variables. Firstly, the sequence N_i is reduced by eliminating the time instants corresponding to non-extreme values (a peak/valley sequence N_j^* is obtained).

The same number of time instants is also eliminated in the sequence $\mathbf{C}_i$ to obtain the new sequence $\mathbf{C}_j^*$. In order to preserve maximum amplitudes of the shear stress during the above reduction procedure, the sequence $\mathbf{C}_i$ is treated as follows. Let *i* and *i* + *K* be the generic time instants corresponding to two successive extreme values of *N*. The mean value $C_m^{(i,i+k)}$ and the amplitude $C_a^{(i,i+k)}$ of the shear stress are calculated for the two vectors $\mathbf{C}_i$ and $\mathbf{C}_{i+k}$ with $k = 1,...,K$ according to the following expressions (derived from Eq. 6 for a two-value discrete sequence, *i* and *i* + *k*), Fig. 2 :

$$\mathbf{C}_m^{(i,i+k)} = \begin{bmatrix} C_{u,m}^{(i,i+k)} \\ C_{v,m}^{(i,i+k)} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(C_{u,i} + C_{u,i+k}\right) \\ \frac{1}{2}\left(C_{v,i} + C_{v,i+k}\right) \end{bmatrix}$$

$$C_a^{(i,i+k)} = \sqrt{\left(C_{v,i+k} - C_{v,m}^{(i,i+k)}\right)^2 + \left(C_{u,i+k} - C_{u,m}^{(i,i+k)}\right)^2} \tag{8}$$

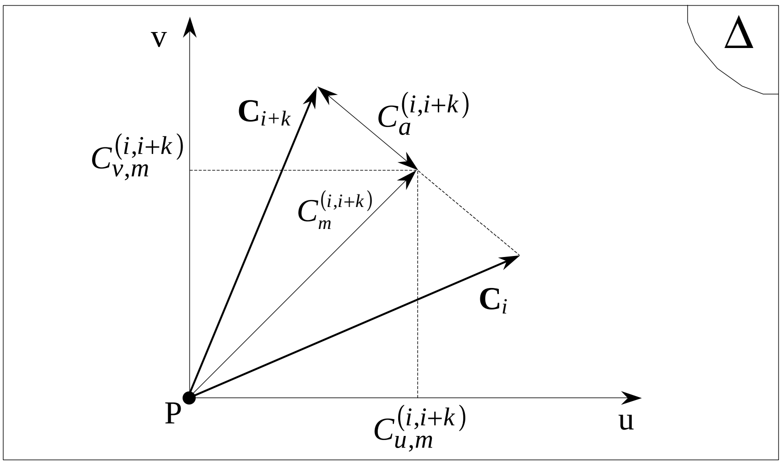

Figure 2: Definition of shear stress amplitude for a two-value discrete sequence, *i* and *i* + *k*

The vector $\mathbf{C}_{i+\bar{k}}$, where $\bar{k}$ is the time instant at which $C_a^{(i,i+k)}$ attains its maximum for $k = 1,...,K$, is retained in the new sequence $\mathbf{C}_j^*$.

Through the cycle counting of the variable N_j^* (by using the rainflow method), we can determine the maximum value $N_{\max,z}^*$ for the *z*-th resolved reversal. Moreover, the amplitude $C_{a,z}^*$ is obtained by applying Eq.

8 to the sequence $\mathbf{C}_j^*$, where now i and $i + k$ are related to the time instants defining the range of the z-th reversal. Then, according to Eq. 5, the z-th amplitude of the equivalent normal stress s_{eq} is given by:

$$s_{eq,\,a,\,z} = \sqrt{\left(N^*_{\max,z}\right)^2 + \left(\frac{s_{af}}{t_{af}}\right)^2 \left(C^*_{a,z}\right)^2} \qquad (9)$$

Using the Miner linear damage rule for $s_{eq,\,a,\,z}$, total damage at time T_0 is obtained as follows :

$$D(T_0) = \begin{cases} \displaystyle\sum_{z=1}^{Z} \frac{1}{2N_0\left(\dfrac{s_{af}}{s_{eq,a,z}}\right)^{m_s}} & \text{for } s_{eq,a,z} \geq c\, s_{af} \\ 0 \quad \text{for } s_{eq,a,z} \leq c\, s_{af} \end{cases} \qquad (10)$$

where Z is the total number of reversals (of N_j^*), determined through the rainflow method, at time T_0; N_0, s_{af} and m are parameters obtained from the S-N curve for uniaxial tension-compression (N_0 is the number of cycles at fatigue limit); c is a safety coefficient.

If the total damage $D(T_0)$ is higher than or equal to the unity, the above criterion predicts the structural component failure for $T < T_0$, viceversa in the case of $D(T_0) < 1$. Hence, the calculated fatigue life of the component is given by:

$$T_{cal} = \frac{T_0}{D(T_0)} \qquad (11)$$

EXPERIMENTAL APPLICATION

Now the present criterion is assessed by analysing the results obtained from fatigue tests on round specimens made of 10HNAP steel, subjected to a combination of random proportional bending and torsion [6]. Such a steel presents a fine-grained ferritic-pearlitic structure, and its mechanical properties are : tensile strength R_m = 566 MPa, yield stress R_e = 418 MPa, Young modulus E = 215 GPa, Poisson ratio ν = 0.29. The characteristic values of the S-N curve for cyclic uniaxial tension-compression with loading ratio equal to -1 are : s_{af} = 252.3 MPa (for $N_0 = 1.282 \times 10^6$ cycles) and

m_s = 9.82. The shear stress fatigue limit t_{af} is equal to 182.0 MPa. The coefficient c is assumed to be equal to 0.5.

Stationary and ergodic random loading with zero expected value, normal probability distribution and wide-band frequency spectrum (0-60 Hz) has been applied to the above specimens. High-cycle fatigue tests have been carried out for four combinations of proportional torsional, $M_T(t)$, and bending, $M_B(t)$, moments. The 14 specimens tested under $M_T(t)/M_B(t)=1$ are analysed in the following.

The biaxial proportional random stress state $s_{xx}(t)$ and $s_{xy}(t)$ is calculated from the total moment $M(t)=\sqrt{M_T^2(t)+M_B^2(t)}$. For each root-mean-square value of $M(t)$, the experimental fatigue life T_{exp} has been determined. The theoretical procedure presented in the previous sections is applied to such experimental data, and fatigue life T_{cal} is calculated. The comparison between experimental and theoretical predictions is illustrated in Fig. 3, showing a fairly good agreement.

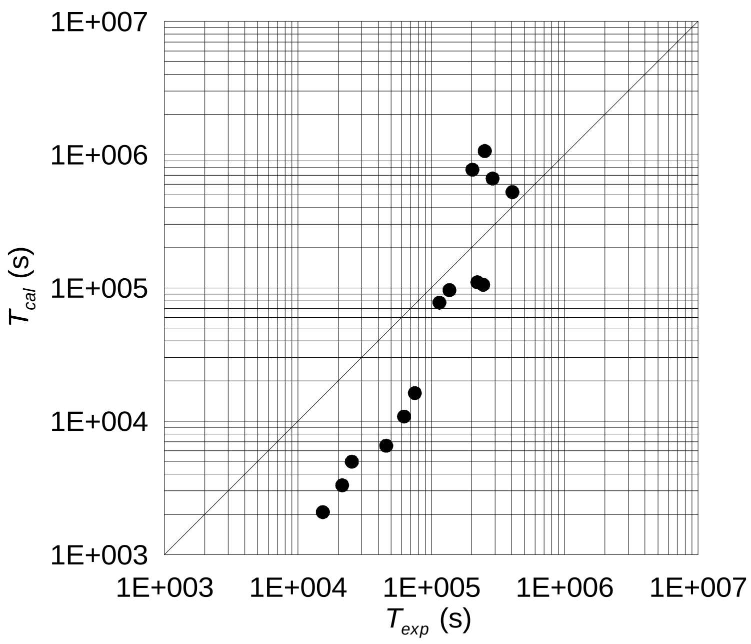

Figure 3: Comparison between experimental and theoretical fatigue lives

CONCLUSIONS

A multiaxial high-cycle fatigue criterion, based on the critical plane approach, for random loading is herein presented. An equivalent stress is defined as a nonlinear combination of the maximum normal stress and the amplitude of shear stress, acting on the critical plane. A cycle counting through the rainflow method is performed using the normal stress as the counting variable. A damage accumulation model is applied to the amplitude spectrum of the equivalent stress, in order to estimate fatigue endurance. The comparison between theoretical and experimental results appears fairly satisfactory for the cases analysed.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the research support for this work provided by the Italian Ministry for University and Technological and Scientific Research (MIUR).

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