Simulation of Multiaxial Service Loading with a View to Fatigue Life Prediction

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***ABSTRACT*: *Service loading must be usually considered as a stochastic process. In order to evaluate a fatigue damage under such a loading, the cumulative damage representation over closed cycles is accepted. Therefore, the method of loading process simulation must guarantee all statistical process characteristics (probability density, power spectral density), discontinuous events occurrence (operating manoeuvres, shock and impact effects), as well as complete loading history with respect to the closed cycles series. Moreover, in the case of multiaxial loading , we must respect cross-correlated parameters and mutual dependence of discrete random events. In the paper, the algorithm of multiaxial loading simulation is presented. The method allows continuous monitoring of the damaging process, so that a fatigue life prediction of structure components under complex loading will be more accurate and effective.*

INTRODUCTION

For a fatigue damage development, stress condition in the structure surface is important, especially in the point of stress concentration. Introducing a co-ordination system *x*, *y*, *z* in this point, whereby axis *z* is identical with a perpendicular to a surface area, we can investigate normal stresses $\sigma_{x,}$ σ_y and a shear stress τ_z, which are generated in a material as a response to external excitation. There would be no problems if the stresses are not correlated, because all stresses would be then simulated separately as individual loading processes. Unfortunately, in real operation, we must consider a mutual correlation of the stresses, and that is a serious complication of the simulation.

Let us present a procedure how to “on-line” simulate such the processes. We will proceed from knowledge of their statistical characteristics, like a mean value, variance, auto-correlation function and cross-correlation function, which must be identified before a simulation period, on the basis of measured and evaluated data of a real load history. The principle of the procedure is based on the method of auto-regressive filtering, whereby formerly an ordinate of the first process is generated, subsequently an ordinate of the second process is generated and eventually an ordinate of the last process is generated, and all procedure is continuously repeated.

SIMULATION OF THE FIRST PROCESS

Let us consider, for example, that the normal stress σ_x is the first simulated process. Ordinates $\sigma_{x,i}$ of the process can be generated independently on other stresses according to relationship [1]

$$\sigma_{x,i} = m_x + a_{1,i} + \sum_{j=1}^{N_1} c_{1,j}\left(\sigma_{x,i-j} - m_x\right), \tag{1}$$

where m_x is a mean value of the loading σ_x, $\left\{c_{1,j}\right\}_{j=1}^{N_1}$ are filtering coefficients and $a_{1,i}$ are ordinates of generated white noise with zero mean value and with the variance

$$s_{a_1}^2 = s_x^2\left(1 - \sum_{j=1}^{N_1} c_{1,j}\gamma_{x,j}\right), \tag{2}$$

where s_x^2 is a variance of the loading σ_x and $\gamma_{x,j}$ are ordinates of the standard auto-correlation function of the loading σ_x, i.e.

$$\gamma_{x,j} = \frac{K_{x,j}}{s_x^2}, \tag{3}$$

where $K_{x,j} = \lim_{n\to\infty} \frac{1}{n-1}\sum_{i=1}^{n}\left(\sigma_{x,i} - m_x\right)\left(\sigma_{x,i-j} - m_x\right)$ is an auto-correlation function of the process σ_x, which is determined in the preliminary period.

Filtering coefficients can be identified solving of the system of N_1 algebraic equations

$$\boldsymbol{\gamma}_x = \boldsymbol{\Gamma}_x \boldsymbol{c}_1 \tag{4}$$

where

$$\boldsymbol{c}_1 = \begin{Bmatrix} c_{1,1} \\ c_{1,2} \\ . \\ . \\ . \\ c_{1,N_1} \end{Bmatrix}; \ \boldsymbol{\gamma}_x = \begin{Bmatrix} \gamma_{x,1} \\ \gamma_{x,2} \\ . \\ . \\ . \\ \gamma_{x,N_1} \end{Bmatrix}; \ \boldsymbol{\Gamma}_x = \begin{bmatrix} 1 & \gamma_{x.1} & . & . & . & \gamma_{x,N_1-1} \\ \gamma_{x,1} & 1 & . & . & . & \gamma_{x,N_1-2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \gamma_{x,N_1-1} & \gamma_{x,N_1-2} & . & . & . & 1 \end{bmatrix}. \tag{5}$$

In the most of practical cases, we can suppose that loading process has a gaussian nature, then a distribution of the white noise is gaussian, too. In the case of non-gaussian processes, some complications can arise owing to a central limit theorem. In the last resort, the procedure should be modified according to [2].

SIMULATION OF THE SECOND PROCESS

Let us consider that a normal stress σ_y, which is stochastically dependent on σ_x, is the second simulated process. Ordinates $\sigma_{y,i}$ of this process can be generated as follows [3]

$$\sigma_{y,i} = m_y + a_{2,i} + \sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)} \left(\sigma_{x,i-j} - m_x\right) + \sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)} \left(\sigma_{y,i-j} - m_y\right), \tag{6}$$

where m_y is a mean value of the loading σ_y, $a_{2,i}$ are ordinates of generated white noise with zero mean value and $\left\{c_{2,j}^{(1)}\right\}_{j=0}^{N_{2,1}}, \left\{c_{2,j}^{(2)}\right\}_{j=1}^{N_{2,2}}$ are filtering coefficients.

After modification of Eq. 6, we get

$$\sigma_{y,i} - m_y = a_{2,i} + \sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)} \left(\sigma_{x,i-j} - m_x\right) + \sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)} \left(\sigma_{y,i-j} - m_y\right). \tag{7}$$

After multiplication of both sides of Eq. 7 by an expression $\left(\sigma_{y,i-l} - m_y\right)$, for $l = 0, 1, 2, \ldots, N_{2,2}$, we get

$$\left(\sigma_{y,i} - m_y\right)\left(\sigma_{y,i-l} - m_y\right) = a_{2,i}\left(\sigma_{y,i-l} - m_y\right) + \sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)} \left(\sigma_{x,i-j} - m_x\right)\left(\sigma_{y,i-l} - m_y\right) +$$

$$+\sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)}\left(\sigma_{y,i-j}-m_y\right)\left(\sigma_{y,i-l}-m_y\right);\ l=0,1,2,\dots,N_{2,2}. \tag{8}$$

If we consider that an auto-correlation function of the process σ_y is defined by the relationship

$$K_{y,l}=\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n}\left(\sigma_{y,i}-m_y\right)\left(\sigma_{y,i-l}-m_y\right) \tag{9}$$

and a cross-correlation function of processes σ_x and σ_y is defined as

$$K_{xy,l}=\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n}\left(\sigma_{x,i}-m_x\right)\left(\sigma_{y,i-l}-m_y\right), \tag{10}$$

we will get using a transformation of Eq. 8

$$K_{y,l}=\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n}a_{2,i}\left(\sigma_{y,i-l}-m_y\right)+\sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)}K_{xy,|l-j|}+$$
$$+\sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)}K_{y,|l-j|};\ l=0,1,2,\dots,N_{2,2}, \tag{11}$$

where we suppose a symmetry of the cross-correlation function.

The expression $\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n}a_{2,i}\left(\sigma_{y,i-l}-m_y\right)$ represents a cross-correlation function of a process σ_y and white noise a_2. Considering that a basic feature of a white noise is its non-correlation, we can rewrite Eq. 11 into form

$$K_{y,l}=s_{a_2}^2\delta_l+\sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)}K_{xy,|l-j|}+\sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)}K_{y,|l-j|};\ l=0,1,2,\dots,N_{2,2}\quad , \tag{12}$$

where $s_{a_2}^2$ is a variance of white noise a_2 and $\delta_l=\langle\begin{matrix}1,\ \text{for}\ l=0\\ 0,\ \text{for}\ l\neq 0\end{matrix}$.

After multiplication of both sides of Eq. 7 by the expression $\left(\sigma_{x,i-l}-m_x\right)$, for $l = 0, 1, 2, \dots, N_{2,1}$, we get

$$\left(\sigma_{y,i}-m_y\right)\left(\sigma_{x,i-l}-m_x\right)=a_{2,i}\left(\sigma_{x,i-l}-m_x\right)+\sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)}\left(\sigma_{x,i-j}-m_x\right)\left(\sigma_{x,i-l}-m_x\right)+$$

$$+\sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)}\left(\sigma_{y,i-j}-m_y\right)\left(\sigma_{x,i-l}-m_x\right);\ l=0,1,2,\ldots,N_{2,1}\ . \tag{13}$$

When we make, similarly like of Eq. 8, a transformation of Eq. 13, we will get

$$K_{xy,l}=\sum_{j=0}^{N_{2,1}} c_{2,j}^{(1)} K_{x,|l-j|}+\sum_{j=1}^{N_{2,2}} c_{2,j}^{(2)} K_{xy,|l-j|}\ ;\ l=0,1,2,..,N_{2,1}\ , \tag{14}$$

because white noise a_2 and a process σ_x are not cross-correlated.

Then Eq. 12 and Eq. 14 represent $(N_{2,1}+N_{2,2}+2)$ algebraic equations for the same number of unknowns: $\left\{c_{2,j}^{(1)}\right\}_{j=0}^{N_{2,1}}$, $\left\{c_{2,j}^{(2)}\right\}_{j=1}^{N_{2,2}}$ and $s_{a_2}^2$.

SIMULATION OF THE THIRD PROCESS

It remains the last simulated process – shear stress τ_z, which is stochastically dependent on normal stresses σ_x and σ_y. Ordinates of the process τ_z can be generated according to relationship [3]

$$\begin{aligned}\tau_{z,i}&=m_z+a_{3,i}+\sum_{j=0}^{N_{3,1}} c_{3,j}^{(1)}\left(\sigma_{x,i-j}-m_x\right)+\sum_{j=0}^{N_{3,2}} c_{3,j}^{(2)}\left(\sigma_{y,i-j}-m_y\right)+\\&+\sum_{j=1}^{N_{3,3}} c_{3,j}^{(3)}\left(\tau_{z,i-j}-m_z\right),\end{aligned} \tag{15}$$

where m_z is a mean value of loading τ_z, $a_{3,i}$ are ordinates of generated white noise with zero mean value and $\left\{c_{3,j}^{(1)}\right\}_{j=0}^{N_{3,1}}$, $\left\{c_{3,j}^{(2)}\right\}_{j=0}^{N_{3,2}}$ and $\left\{c_{3,j}^{(3)}\right\}_{j=1}^{N_{3,3}}$ are filtering coefficients.

Shifting a term m_z into the left side of Eq. 15, after multiplication of both sides of the modified equation by the expression $\left(\tau_{z,i-l}-m_z\right)$, for $l = 0, 1, 2, \ldots, N_{3,3}$, and after subsequent transformation, similarly like in the previous paragraph, we get a system of $(N_{3,3}+1)$ algebraic equations

$$K_{z,l}=s_{a_3}^2\delta_l+\sum_{j=0}^{N_{3,1}} c_{3,j}^{(1)} K_{xz,|l-j|}+\sum_{j=0}^{N_{3,2}} c_{3,j}^{(2)} K_{yz,|l-j|}+$$

$$+\sum_{j=1}^{N_{3,3}} c_{3,j}^{(3)} K_{z,|l-j|}\ ;\ l=0,1,2,\dots,N_{3,3}\ , \tag{16}$$

where $s_{a_3}^2$ is a variance of white noise a_3, K_z is an auto-correlation function of the process τ_z and K_{xz}, K_{yz} are cross-correlation functions of processes σ_x and τ_z, respectively σ_y and τ_z.

Similarly, after multiplication of a modified Eq. 15 by the term $\left(\sigma_{y,i-l}-m_y\right)$, for $l = 0, 1, 2, \dots, N_{3,2}$, and after subsequent transformation, we get a system of $(N_{3,2}+1)$ algebraic equations

$$K_{yz,l}=\sum_{j=0}^{N_{3,1}} c_{3,j}^{(1)} K_{xy,|l-j|}+\sum_{j=0}^{N_{3,2}} c_{3,j}^{(2)} K_{y,|l-j|}+\sum_{j=1}^{N_{3,3}} c_{3,j}^{(3)} K_{yz,|l-j|}\ ;\ l=0,1,2,..,N_{3,2}\ . \tag{17}$$

In the same way, after multiplication by $\left(\sigma_{x,i-l}-m_x\right)$, for $l = 0, 1, 2, \dots, N_{3,1}$, we get $(N_{3,1}+1)$ algebraic equations

$$K_{xz,l}=\sum_{j=0}^{N_{3,1}} c_{3,j}^{(1)} K_{x,|l-j|}+\sum_{j=0}^{N_{3,2}} c_{3,j}^{(2)} K_{xy,|l-j|}+\sum_{j=1}^{N_{3,3}} c_{3,j}^{(3)} K_{xz,|l-j|}\ ;\ l=0,1,2,..,N_{3,1}\ . \tag{18}$$

Eq. 16, 17 and 18 represent altogether $(N_{3,1} + N_{3,2} + N_{3,3} + 3)$ algebraic equations for the same number of unknowns: $\left\{c_{3,j}^{(1)}\right\}_{j=0}^{N_{3,1}}$, $\left\{c_{3,j}^{(2)}\right\}_{j=0}^{N_{3,2}}$, $\left\{c_{3,j}^{(3)}\right\}_{j=1}^{N_{3,3}}$ and $s_{a_3}^2$.

ESTIMATION OF A FATIGUE LIFE

Simulated stresses σ_x, σ_y and τ_z can be immediately used as control parameters for loading systems. Service life then can be predicted experimentally.

For computational estimation of fatigue life, we can make a conversion into uniaxial loading and use knowledge of a cumulative damage theory for such a case. Then, we can calculate ordinates of equivalent uniaxial stress according to mostly used Mises hypothesis, "on-line" to the simulation procedure, using a relationship

$$\sigma_{eq,i}=\sqrt{\sigma_{x,i}^2+\sigma_{y,i}^2-\sigma_{x,i}\sigma_{y,i}+3\tau_{z,i}^2}\ . \tag{19}$$

For multiaxial loading, although, there is a problem that a direction of a principal stress is continuously changed. It is well-known that a direction of principal stress is deviated from x-axis in the xy-plane by the angle

$$\varphi = \frac{1}{2} arctan \frac{2\tau_z}{\sigma_x - \sigma_y} \ . \tag{20}$$

Therefore, it is necessary for an effective conversion that this angle should be constant in any moment, i.e. it must hold a condition

$$\tau_{z,i} = const\left(\sigma_{x,i} - \sigma_{y,i}\right); \ \text{for all}\ i \ . \tag{21}$$

According to this presumption, we can then convert a biaxial loading into uniaxial one, and from the generated ordinates $\sigma_{eq,i}$, we can identify, using a rainflow algorithm, both loading amplitudes $\sigma_{eq,a,i}$ and corresponding mean values $\sigma_{eq,m,i}$, see [4].

If we use a procedure shown in [5], we could also calculate in each moment a correction to residual deformations according to the general relationship

$$\Delta\varepsilon_m = \sqrt[n]{\frac{1}{K}}\left(\pm \sqrt[n]{|\sigma_1|} \mp \sqrt[n]{|\sigma_N|} \pm \frac{2}{\sqrt[n]{2}} \sum_{i=2}^{N} \sqrt[n]{|\sigma_i - \sigma_{i-1}|} \right), \tag{22}$$

where n and K are parameters of a cyclic deformation curve of a material

$$\sigma = \pm 2K \left(\frac{|\varepsilon - \varepsilon_e|}{2} \right)^n \ ; \ \text{for}\ \varepsilon \gtrless 0, \text{respectively}, \tag{23}$$

where ε_e is an elastic strain component and $\{\sigma_i\}_{i=1}^{N}$ are stresses corresponding to peaks of unclosed hysteresis loops during a time history in a stress-strain diagram.

Then we can determine ordinates of an equivalent stress amplitude for a reversed cycle in the form

$$\sigma^*_{eq,a,i} = \sigma_{eq,a,i} + \psi_\sigma \sigma_{eq,m,i} + \psi_\varepsilon \Delta\varepsilon_{eq,m,i} \ , \tag{24}$$

where parameters ψ_σ and ψ_ε can be determined from the triaxial Haigh diagram [5].

Now, we must determine a number n_0 of stress amplitude levels, and stress amplitudes calculated according to Eq. 24 must by assorted into these levels as $\left\{\sigma^*_{eq,a,j}\right\}_{j=1}^{n_0}$ values. For estimation of number of cycles to failure N_f, we can then use the relationship [6]

$$N_f = \frac{N_0}{k\sum_{j=1}^{n_0}\left(\frac{\sigma^*_{eq,a,j}}{\sigma_C}\right)^q p\left(\sigma^*_{eq,a,j}\right)}\ , \qquad (25)$$

where k is a material constant (usually $k \approx 1$), σ_C is an endurance limit, N_0 is a corresponding number of cycles, q is the slope of a Wöhler curve for reversed loading and p is an occurrence probability.

CONCLUSIONS

According to the method, structure design would be more effective than according to standard methods, and material can be economised, too.

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