Anisotropic Damage Model for Concrete Including Effect of Crack Closure in Compression

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ABSTRACT: *A new damage model for concrete is proposed, with the 5 following essential features: it accounts for (i) anisotropic damage, and (ii) asymmetry between tension and compression; (iii) stresses are continuous with respect to strain and damage; (iv) a complete loss of stress-bearing capacity is possible; (v) the model fits within the framework of "standard generalized materials" (as defined by Halphen and Nguyen), and the free energy is a convex function of strain and damage (considered separately). The essential novelty of the model is the convexity of the free energy with respect to damage, which entails nice mathematical properties in its numerical implementation. A von Mises-like dissipation potential is adopted for the evolution of damage. The very simple model thus defined involves only one material parameter in addition to elasticity constants. Numerical tests exhibit satisfactory predictions in simple tension, but somewhat less so in simple compression. Possible future improvements are envisaged in conclusion.*

INTRODUCTION

Constitutive models for concrete are too numerous for an exhaustive list to be possible here. Among these, many rely on the phenomenological theory of damage initiated by Kachanov [1]. This approach seems especially well suited for the description of the mechanical behaviour of concrete for 2 reasons. The first one is that the complexity of microscopic mechanisms, including the formation and propagation of multiple microcracks, their possible closure in compression, the presence of friction, etc., seems to preclude any other approach than a heuristic one. The second reason is that in many instances, the behaviour of concrete may safely be regarded as essentially elastic-brittle, which is the preferred domain of application of damage theory.

If one considers damage models accounting for anisotropy (arising from the preferred formation of cracks perpendicularly to the direction of major tension), candidates are much less numerous. Notable contributions are due, among others, to Murakami [2], Fichant et al. [3], Chaboche [4], Dragon et al. [5], Ladevèze [6], Yazdani and Karnawat [7], Frantziskonis and Desai [8]. Among these, the models of Fichant et al., Chaboche, Dragon et al. and Ladevèze attempt to account for the asymmetry between tension and compression, arising from crack closure in compression.

The aim of this work is to propose a first, simple version (leaving room for later improvements) of a model with the following features:

(i) Account for anisotropic damage, through consideration of a second-rank symmetric damage tensor.

(ii) Account for asymmetry between tension and compression, through consideration of the "positive and negative parts" of the strain tensor (combined with the damage tensor).

(iii) Satisfy the requirement of continuity of the stresses with respect to both strain and damage.

(iv) Allow for total damage (complete vanishing of stress).

(v) Fit within the framework of "standard generalized materials", as defined by Halphen and Nguyen [9], and involve a free energy convex with respect to both the strain and damage tensors.

Points (i) and (ii) do not require any further comment. Requirement (iii) seems obvious but, as noted by Chaboche [4], is not so easily fulfilled when combined with point (ii); several models in fact violate it. Requirement (iv) precludes for instance the use of the models of Dragon et al. [5] (who make the assumption of weak damage) and Ladevèze [6] (who allows for total damage only asymptotically). But the main novelty here is the fulfilment of point (v). Though not compulsory from a physical viewpoint, this feature is mathematically desirable because it leads to an easy numerical implementation of the model. Indeed it can be shown that the problem of finding, at a given point and for a fixed increment of strain, the final value of the damage parameter with a classical implicit algorithm, then reduces to finding the minimum of a convex function and is thus liable to efficient algorithms.

THE FREE ENERGY

Expression disregarding asymmetry between tension and compression

For clarity reasons, we first forget about the asymmetry between tension and compression. The (second-rank, symmetric) damage tensor is denoted $\mathbf{D}$. The specific free energy $\Phi(\boldsymbol{\epsilon}, \mathbf{D})$ is taken in the form

$$\Phi(\boldsymbol{\epsilon}, \mathbf{D}) = \frac{\lambda}{2} \left[\mathrm{tr}(\mathbf{B}.\boldsymbol{\epsilon})\right]^2 + \frac{\mu}{4} \,\mathrm{tr} \left[(\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})^2\right] \;, \quad \mathbf{B} \equiv \mathbf{1} - \mathbf{D} \tag{1}$$

$$\Rightarrow \boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\epsilon}} = \lambda \,\mathrm{tr}(\mathbf{B}.\boldsymbol{\epsilon}) \,\mathbf{B} + \frac{\mu}{2} \left[\mathbf{B}.(\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B}) + (\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B}).\mathbf{B}\right] . \tag{2}$$

Requirement (i) of the Introduction is satisfied and point (ii) is left out here. The satisfaction of requirement (iii) is a trivial consequence of eqn (2). This equation also immediately implies that for total damage ($\mathbf{D} = \mathbf{1}, \; \mathbf{B} = \mathbf{0}$), $\boldsymbol{\sigma} = \mathbf{0}$. Thus requirement (iv) is satisfied. This is obtained by considering the free *energy*, which is a function of *strain*, rather than the more customary (in the context of damage theory) free *enthalpy*, which is a function of *stress*. Consideration of the free enthalpy would lead to an expression containing the tensor $(\mathbf{1} - \mathbf{D})^{-1}$ so that the limiting value $\mathbf{D} = \mathbf{1}$ could not be reached. Also, it is easily shown that if one eigenvalue of $\mathbf{D}$, say D_1, is unity, the corresponding stress component σ_{11} is zero, as desired (vanishing stress in the direction of total damage). (However, the components σ_{12} and σ_{13} are zero only if the strain and damage tensors are diagonal in the same basis). Finally, the convexity of Φ with respect to both $\boldsymbol{\epsilon}$ and $\mathbf{D}$ (requirement (v)) results from that of the functions $x \mapsto x^2$ and $\mathbf{A} \mapsto \mathrm{tr}(\mathbf{A}^2)$, plus the linearity of $\mathrm{tr}(\mathbf{B}.\boldsymbol{\epsilon})$ and $\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B}$ with respect to $\boldsymbol{\epsilon}$ and $\mathbf{B}$.

Incorporation of asymmetry between tension and compression

The positive and negative parts of a real x are defined by $x_+ = \mathrm{Sup}(x, 0)$ and $x_- = \mathrm{Inf}(x, 0)$, and those of a symmetric matrix $\mathbf{A} = \sum A_i \mathbf{e}_i \otimes \mathbf{e}_i$ (the A_i and $\mathbf{e}_i$ being the eigenvalues and normalized eigenvectors of $\mathbf{A}$) by $\mathbf{A}_+ = \sum A_{i+} \mathbf{e}_i \otimes \mathbf{e}_i$ and $\mathbf{A}_- = \sum A_{i-} \mathbf{e}_i \otimes \mathbf{e}_i$.

In order to account for the asymmetry between tension and compression, we must distinguish between the positive and negative parts of $\boldsymbol{\epsilon}$ (possibly combined with $\mathbf{B}$); damage should appear only in the positive part, since cracks are ineffective in compression. More specifically, we consider a free energy of the modified form

$$\Phi(\boldsymbol{\epsilon}, \mathbf{D}) = \frac{\lambda}{2}\left[\operatorname{tr}(\mathbf{B}.\boldsymbol{\epsilon})\right]_+{}^2 + \frac{\lambda}{2}\left(\operatorname{tr}\boldsymbol{\epsilon}\right)_-{}^2 + \frac{\mu}{4}\operatorname{tr}\left[(\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})_+{}^2\right] + \mu\operatorname{tr}(\boldsymbol{\epsilon}_-{}^2) \quad (3)$$

$$\Rightarrow \boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\epsilon}} = \lambda\left[\operatorname{tr}(\mathbf{B}.\boldsymbol{\epsilon})\right]_+ \mathbf{B} + \lambda\left(\operatorname{tr}\boldsymbol{\epsilon}\right)_- \mathbf{1}$$

$$+ \frac{\mu}{2}\left[\mathbf{B}.(\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})_+ + (\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})_+ .\mathbf{B}\right] + 2\mu\,\boldsymbol{\epsilon}_- \,. \quad (4)$$

Requirements (i) and (ii) are now satisfied. The model however suffers from the following defects. The term $\lambda\ [\operatorname{tr}(\mathbf{B}.\boldsymbol{\epsilon})]_+\ \mathbf{B}$ of the stresses (pertaining to tension) should vanish precisely when the term $\lambda\ (\operatorname{tr}\ \boldsymbol{\epsilon})_-\ \mathbf{1}$ (pertaining to compression) becomes non-zero, and vice versa. This is unfortunately not true because $\operatorname{tr}(\mathbf{B}.\boldsymbol{\epsilon})$ and $\operatorname{tr}\ \boldsymbol{\epsilon}$ do not vanish simultaneously, except if $\mathbf{B}$ is a multiple of the unit tensor. Also, "changes of regime" in the terms $\frac{\mu}{2}\left[\mathbf{B}.(\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})_+ + (\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})_+ .\mathbf{B}\right]$ and $2\mu\ \boldsymbol{\epsilon}_-$ are not simultaneous either because the eigenvalues of $\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B}$ and $\boldsymbol{\epsilon}$ do not vanish simultaneously, except if $\mathbf{B}$ and $\boldsymbol{\epsilon}$ are diagonal in the same basis. Numerical study of the model predictions in simple cases however shows that these deficiencies are of little practical importance. Requirement (iii) is satisfied because the functions $x \mapsto x_\pm$ and $\mathbf{A} \mapsto \mathbf{A}_\pm$ are continuous. Concerning requirement (iv), if $\mathbf{D} = \mathbf{1}$ ($\mathbf{B} = \mathbf{0}$), the terms of the stresses pertaining to tension vanish, as desired. Also, if one eigenvalue of $\mathbf{D}$, say D_1, is unity, the terms pertaining to tension in the component σ_{11} vanish. The convexity of Φ with respect to $\boldsymbol{\epsilon}$ and $\mathbf{D}$ (requirement (v)) is a consequence of the convexity of the functions $x \mapsto x_\pm{}^2$, $\mathbf{A} \mapsto \operatorname{tr}(\mathbf{A}_\pm{}^2)$ and the linearity of $\operatorname{tr}(\mathbf{B}.\boldsymbol{\epsilon})$, $\operatorname{tr}\ \boldsymbol{\epsilon}$, $\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B}$ with respect to $\boldsymbol{\epsilon}$ and $\mathbf{B}$. The only non-trivial property here is the convexity of the functions $\mathbf{A} \mapsto \operatorname{tr}(\mathbf{A}_\pm{}^2)$, established in Badel's thesis [10].

It may incidentally be remarked that simultaneously satisfying requirements (ii) and (v) is a difficult task. The form (3) was arrived at only after several unsuccessful trials. For instance, such appealing expressions as $\operatorname{tr}\left[(\sqrt{\mathbf{B}}.\boldsymbol{\epsilon}.\sqrt{\mathbf{B}})_+{}^2\right]$, $\operatorname{tr}\left[(\mathbf{B}.\boldsymbol{\epsilon}.\mathbf{B})_+{}^2\right]$, $\operatorname{tr}(\mathbf{B}.\boldsymbol{\epsilon}_+{}^2.\mathbf{B})$, all turn out to be non-convex with respect to either $\mathbf{D}$ or $\boldsymbol{\epsilon}$.

Incorporation of bounds on the damage tensor eigenvalues

The eigenvalues D_i of $\mathbf{D}$ should vary between 0 (no damage) and 1 (total damage in the direction of the eigenvalue). The evolution equation for $\mathbf{D}$ (see below) will ensure the conditions $\dot{D}_i \geq 0$ (increasing damage), and therefore

$D_i \geq 0$ since $\mathbf{D} = \mathbf{0}$ initially. The conditions $D_i \leq 1$ can be enforced, in an admittedly somewhat formal and artificial, but mathematically natural way, by modifying expression (3) into

$$\Phi(\boldsymbol{\epsilon}, \mathbf{D}) = \frac{\lambda}{2}\,[\mathrm{tr}(\mathbf{B}.\boldsymbol{\epsilon})]_+{}^2 + \frac{\lambda}{2}\,(\mathrm{tr}\;\boldsymbol{\epsilon})_-{}^2$$

$$+\frac{\mu}{4}\,\mathrm{tr}\left[(\mathbf{B}.\boldsymbol{\epsilon} + \boldsymbol{\epsilon}.\mathbf{B})_+{}^2\right] + \mu\,\mathrm{tr}(\boldsymbol{\epsilon}_-{}^2) + \mathcal{I}_{(-\infty,1]}(\max\, D_i)\;, \qquad (5)$$

$\mathcal{I}_\mathcal{C}$ denoting the indicatrix function of the convex set $\mathcal{C}$ ($\mathcal{I}_\mathcal{C}(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathcal{C}$, $+\infty$ if $\mathbf{x} \notin \mathcal{C}$). This function in (5) acts as a "barrier" preventing the D_i from exceeding unity. Its introduction preserves convexity with respect to $\mathbf{D}$ because the domain ($\max\, D_i \leq 1$) is convex in the space of symmetric matrices (consequence of Rayleigh-Ritz's theorem).

THE DISSIPATION POTENTIAL AND THE REVERSIBILITY DOMAIN

Within the framework of generalized standard materials, the evolution of $\mathbf{D}$ is governed by some convex dissipation potential $\Delta(\dot{\mathbf{D}})$ as follows:

$$\mathbf{F}^D \in \partial\Delta(\dot{\mathbf{D}}) \;\Leftrightarrow\; \dot{\mathbf{D}} \in \partial\tilde{\Delta}(\mathbf{F}^D)\;,\;\mathbf{F}^D = -\frac{\partial\Phi}{\partial\mathbf{D}} \qquad (6)$$

where the symbols ∂ and $\tilde{}$ denote the sub-gradient and Legendre-Fenchel transform of a convex function. This evolution equation automatically ensures the non-negativeness of the dissipation $\mathbf{F}^D : \dot{\mathbf{D}}$.

The dissipation potential

The simplest possible expression of $\Delta(\dot{\mathbf{D}})$ is $k\left\|\dot{\mathbf{D}}\right\|$ where k is a positive material constant and $\left\|\dot{\mathbf{D}}\right\| = (\dot{\mathbf{D}} : \dot{\mathbf{D}})^{1/2}$ the norm of $\dot{\mathbf{D}}$. However, to ensure the conditions $\dot{D}_i \geq 0$, we modify this expression into

$$\Delta(\dot{\mathbf{D}}) = k\left\|\dot{\mathbf{D}}\right\| + \mathcal{I}_{[0,+\infty)}(\min\,(\dot{\mathbf{D}})_i) \qquad (7)$$

where the $(\dot{\mathbf{D}})_i$ denote the eigenvalues of $\dot{\mathbf{D}}$. The indicatrix function prevents the $(\dot{\mathbf{D}})_i$ from becoming negative, which suffices to warrant that the D_i are non-decreasing (see Badel's thesis [10]). The constant k in (7) is the only adjustable material parameter in the model.

The reversibility domain
The function $\Delta(\dot{\mathbf{D}})$ being positively homogeneous of degree 1, its Legendre-Fenchel transform $\tilde{\Delta}(\mathbf{F}^D)$ is the indicatrix function of some convex set in the space of thermodynamic forces $\mathbf{F}^D$, which is the reversibility domain. It is shown in Badel's thesis [10] that this set is defined by

$$f(\mathbf{F}^D) = \left\|\mathbf{F}^D{}_+\right\| - k \leq 0 \ , \ \left\|\mathbf{F}^D{}_+\right\| = (\mathbf{F}^D{}_+ : \mathbf{F}^D{}_+)^{1/2} \ . \tag{8}$$

Condition $(6)_2$ then takes the familiar, plasticity-theory-like form

$$f(\mathbf{F}^D) \leq 0 \ , \ \dot{\mathbf{D}} = \eta \frac{\partial f}{\partial \mathbf{F}^D} \ , \ \eta \geq 0 \ , \ f(\mathbf{F}^D)\eta = 0 \ . \tag{9}$$

NUMERICAL EXAMPLES

Model predictions are presented in Figures 1 for simple tension and 2 for simple compression; the material constants are $E = 30$ GPa, $\nu = 0.2$ and $k = 3\,10^{-4}$ MPa. The axial stress σ_{11} is displayed as a function of axial strain ϵ_{11} and lateral strain ϵ_{22}, and also volumetric strain tr $\boldsymbol{\epsilon}$ for simple compression. In simple tension, the results are qualitatively satisfactory. Note that the single parameter k governs (together with E) both the peak stress and the subsequent stress drop; also, although this is not shown here, damage being anisotropic, it remains nil in direction 2 (since $\epsilon_{22} < 0$) so that one could apply some non-zero stress σ_{22} after σ_{11}. Predictions in simple compression are less satisfactory: 1) damage (arising from the positive ϵ_{22}) occurs for too small values of $|\sigma_{11}|$; 2) the stress drop quickly ceases and the behaviour again becomes elastic with an only slightly degraded stiffness (because although damage becomes total in direction 2, it remains nil in direction 1).

DIRECTIONS FOR POSSIBLE IMPROVEMENTS

Although the model proposed does possess new, nice features, it obviously suffers from its excessive simplicity, as could be anticipated from its single adjustable parameter. Drawback 1) just evidenced can be remedied by considering a non-constant k of the form $k_0 - k_1$ (tr $\boldsymbol{\epsilon})_-$; then it becomes larger in simple compression than in simple tension. Another possibility is to replace $\boldsymbol{\epsilon}$ by $\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^i$ in (5) where $\boldsymbol{\epsilon}^i$ is some irreversible strain, for instance of the form $\alpha\ \mathbf{D}.(\mathbf{1} - \mathbf{D})^{-1}$; this modifies the expression of $\mathbf{F}^D$ so that damage occurs

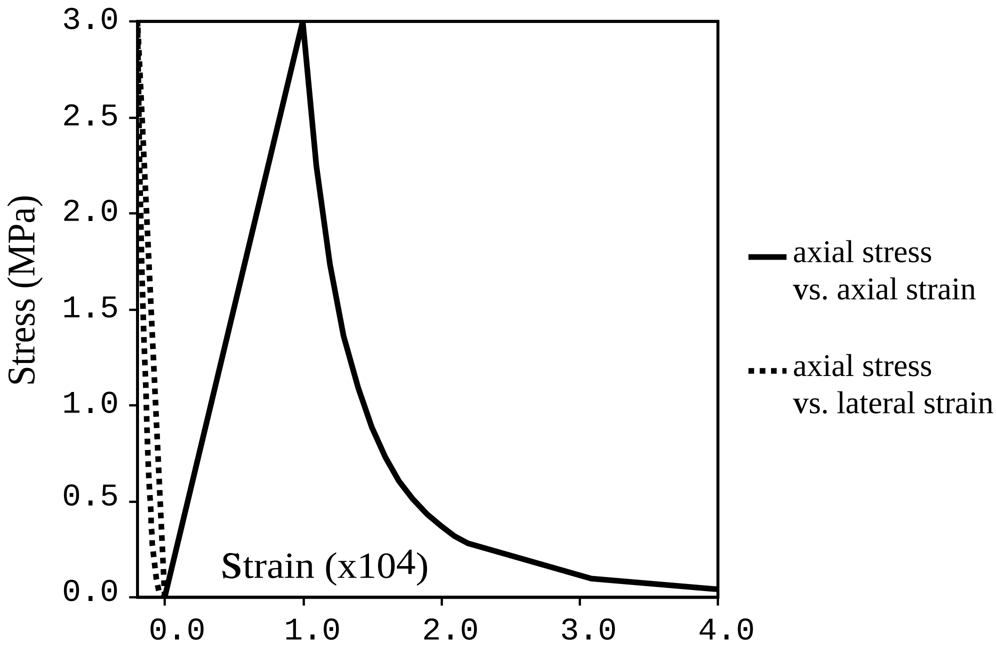

Figure 1: Model predictions in simple tension.

later in simple compression. Drawback 2) is connected to the fact that damage is considered to arise from the sole positive components of $\boldsymbol{\epsilon}$. It can be remedied by introducing damage also in the "compression terms" of Φ; but this requires introducing distinct damage tensors in the "tension" and "compression" terms, otherwise the desired asymmetry would be lost. All these developments are currently under progress.

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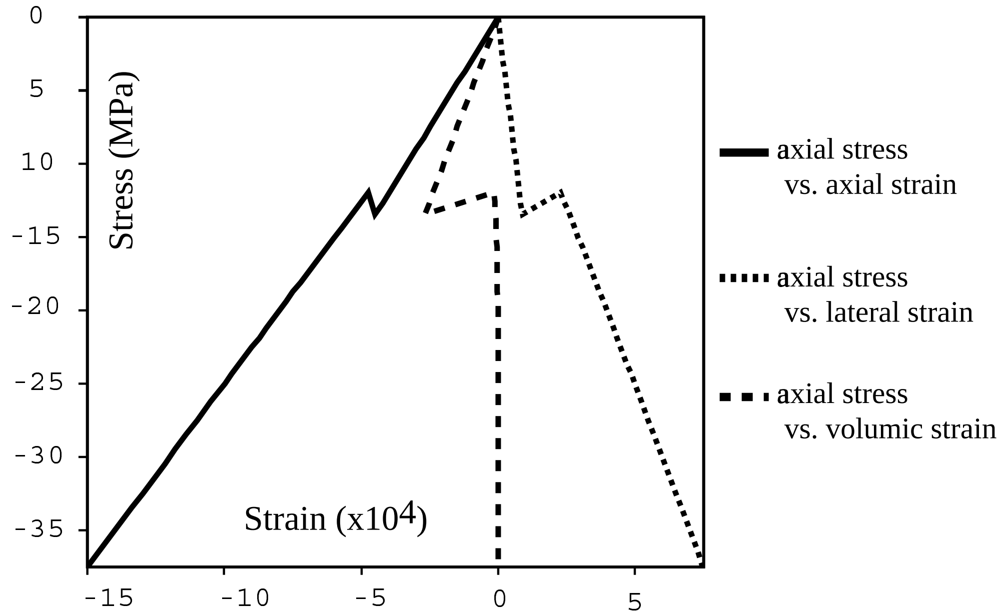

Figure 2: Model predictions in simple compression.

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