Development of damage in thin plates under cyclic loading

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ABSTRACT: *A simple thermoelastic-plastic problem is solved to show an effect of cyclic temperature fields of a constant amplitude on the development of damage in thin plates. The plate considered is a disc with a central circular hole whose material obeys the von Mises yield criterion with a constant yield stress. The temperature field is assumed to be independent of the space coordinates. It is shown that the equivalent plastic strain at the hole edge increases after each temperature cycle even if the temperature amplitude is relatively small. A strain-based approach to damage evolution is adopted to demonstrate an effect of such temperature cycles on fatigue life. Even though the very simple structure is considered, it is believed that the sensitivity of plastic strain to the variation of temperature is a general feature for thin plates of arbitrary in-plane shape and, therefore, this feature should be taken into account in numerical analyses.*

INTRODUCTION

Thin plates with holes have many structural applications. A hole is a stress concentrator where a plastic zone and fracture usually start to develop. A significant amount of research has been performed in the area of stress and strain analysis of such structures. In particular, thermomechanical problems have been considered in [1] – [5]. Even though closed form solutions involve more assumptions than numerical solutions, the former are very useful for studying qualitative effects. In particular, closed form elastic/plastic solutions may be obtained under the assumption of plane stress. Experimental observations indicate that for thin plates this assumption is much more appropriate than that of plain stain [6]. However, the application of computational models to plane stress problems leads to specific difficulties non-existent in other formulations [7]. In the case of thermoelastic-plastic problems, one of such difficulties may be related to the fact that the area of a plastic zone increases very rapidly with the temperature. It has been demonstrated in [5] where a closed form solution to a simple problem is given assuming that the temperature is a monotonic function of the time. Here this analysis is extended to include cyclic thermal loading. Also, an effect of such loading on the evolution of fatigue damage is discussed. The problem is as follows. A circular disc with a central circular hole is inserted into a container such that its outer radius is motionless during the process. The material of the disc is assumed to obey the Mises yield criterion. At the initial moment the disc has no stress. A cyclic temperature field and the constraints imposed on the structure may lead to the accumulation of plastic strain. Then, the evolution of fatigue damage can be found by applying strain-based approaches to fatigue life prediction. The main result of the present paper is that the temperature amplitude at which a significant plastic zone develops is very small. In other words, the development of plastic deformation and, as a result, of damage is very sensitive to the temperature field.

STATEMENT OF THE PROBLEM

Consider a thin disc of radius R_0 with a central circular hole of radius r_0, which is inserted into a rigid container of radius R_0. The state of stress is axisymmetric and two - dimensional $(\sigma_z = 0)$ in a cylindrical coordinate system $r\theta z$ with its z-axis coinciding with the axis of symmetry of the disc. The disc has no stress at the initial temperature. Thermal expansion caused by a cyclic change of temperature (Figure 1) relative to the reference state, T, and the constraints imposed on the disc affect the zero -stress state. The temperature field is assumed to be uniform. The material

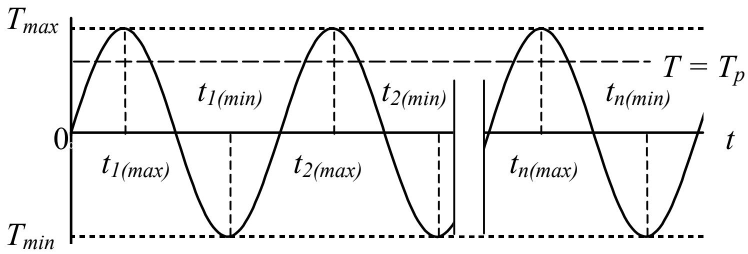

Figure 1: Cyclic temperature field – notation.

of the disc obeys the von Mises yield criterion. For the problem under consideration, this yield criterion may be written in the form

$$s_r^2 + \sigma^2 - \sigma s_r = k^2 \tag{1}$$

where k is the shear yield stress, a material constant independent of the temperature, σ is the hydrostatic stress and s_r is the deviatoric radial stress. Since the axial stress $\sigma_z = 0$, the radial and circumferential stresses are given by $\sigma_r = s_r + \sigma$ and $\sigma_\theta = 2\sigma - s_r$. The yield criterion (1) is satisfied by the following substitution

$$s_r = \omega 2k \sin\varphi / \sqrt{3} \quad \text{and} \quad \sigma = \omega k \left(\sqrt{3}\cos\varphi + \sin\varphi\right) / \sqrt{3} \tag{2}$$

where $\omega = \pm 1$ and φ is a function of r and T. The boundary conditions to the problem are

$$u = 0 \quad \text{at} \quad r = R_0 \tag{3}$$

and

$$\varphi = 5\pi/6 \quad \text{at} \quad r = r_0 \tag{4}$$

where u is the radial displacement. The condition (4) has been derived from the original condition $\sigma_r = 0$ at $r = r_0$ and Eq. (2).

It has been shown in [5] that the radius of the elastic-plastic boundary increases very rapidly with the temperature. In particular, the rise in temperature at which the entire disc becomes plastic, T_p, is small. This temperature rise is within the range of conventional engineering applications and, therefore, it is of importance to study the development of fatigue damage in the case of temperature fields with $T_{max} \geq T_p$ (Figure 1) where T_{max} is the constant temperature amplitude. Such a study is performed in the present paper assuming that $|T_{max}| = |T_{min}|$ (Figure 1).

STRESS ANALYSIS

Monotonic thermal loading of the stress-free disc leads to the initiation of a plastic zone at the hole and, then, to its growth until the entire disc becomes plastic [5]. The basic assumption in the present paper, which will be verified *a posteriori*, is that the plastic zone starts to develop at the hole in each temperature cycle and that other plastic zones do not appear in the disc. It will be seen later that if this assumption is satisfied, the increase in plastic strain after each cycle can be found from the stress solutions at $T = T_{max}$ and $T = T_{min}$. Since the entire disc is plastic at these temperatures, substituting Eq. (2) into the only non-trivial equilibrium equation $\partial\sigma_r/\partial r + (\sigma_r - \sigma_\theta)/r = 0$ and solving this equation along with the boundary condition (4) gives

$$\frac{r}{r_0} = \frac{\sqrt{\sqrt{3}}}{\sqrt{\sqrt{3}\sin\varphi - \cos\varphi}}\exp\left[\frac{\sqrt{3}}{2}\left(\varphi - \frac{5\pi}{6}\right)\right] \tag{5}$$

It is clear that φ is an increasing function of *r* in the interval $r_0 \le r \le R_0$. Therefore, it attains its maximum value φ_m at $r = R_0$. It follows from Eq. (5) that the value of φ_m is determined by

$$\frac{R_0}{r_0} = \frac{\sqrt{\sqrt{3}}}{\sqrt{\sqrt{3}\sin\varphi_m - \cos\varphi_m}}\exp\left[\frac{\sqrt{3}}{2}\left(\varphi_m - \frac{5\pi}{6}\right)\right] \tag{6}$$

It is convenient to use φ as a new independent variable in place of *r*. To this end it is necessary to express the derivative of any function with respect to *r* in terms of its derivative with respect to φ. It follows from Eq. (5) that

$$\frac{\partial}{\partial r} = \frac{\left(\cos\varphi - \sqrt{3}\sin\varphi\right)}{r\left(\sqrt{3}\cos\varphi - \sin\varphi\right)}\frac{\partial}{\partial\varphi} \tag{7}$$

It is important to note that Eqs. (2) and (5) show that the stress distribution is independent of the time and, therefore, of the temperature. Of course, this is a consequence of the fact that the entire disc is plastic. In the process of loading (and unloading) there is a temperature range such that the entire disc is elastic and another range such that there are two zones, plastic and elastic. In these stages of the process the stress field depends on the temperature. However, these stages do not influence the plastic strain distribution at $T = T_{max}$ and $T = T_{min}$, as will be seen later.

It follows from Eq. (2) that the stress increments at the end of each time interval within which the temperature field varies monotonically are

$$\Delta s_r = \varsigma 4k \sin\varphi / \sqrt{3} \quad \text{and} \quad \Delta\sigma = \varsigma 2k\left(\sqrt{3}\cos\varphi + \sin\varphi\right) / \sqrt{3} \tag{8}$$

where $\varsigma = 1/2$ corresponds to the interval $0 \le t \le t_{1(max)}$, $\varsigma = 1$ to the intervals $t_{n(min)} \le t \le t_{n+1(max)}$, and $\varsigma = -1$ to the intervals $t_{n(max)} \le t \le t_{n(min)}$ (Figure 1).

The stress solution for any instant of the time at which there is an elastic zone can be obtained without finding the plastic strain distribution in the plastic zone. This has been demonstrated in [5] for the case of monotonic loading. Such a solution has been obtained for the problem under consideration and used to verify the aforementioned assumption. In all cases considered, it has been found that the assumption is correct. In other words, within each time interval where the temperature is a monotonic function of the time the plastic zone starts to develop at the hole and then its radius increases until the entire disc becomes plastic. Other plastic zones do not appear within these time intervals.

STRAIN ANALYSIS

The elastic strain increments are obtained from the Duhamel-Neumann law and the distribution of the stress increments (8). In particular, the increments of the radial and circumferential strains are given by

$$\begin{aligned} \Delta\varepsilon_r^e &= \varsigma\left(2k/E\right)\left[\left(1-2\nu\right)\cos\varphi + \sqrt{3}\sin\varphi\right] + \varsigma 2\alpha T_{max} \\ \Delta\varepsilon_\theta^e &= \varsigma\left(2k/E\right)\left[\left(2-\nu\right)\cos\varphi - \sqrt{3}\nu\sin\varphi\right] + \varsigma 2\alpha T_{max} \end{aligned} \tag{9}$$

where E is the Young's modulus, ν is the Poisson's ratio, α is the thermal coefficient of linear expansion. These material properties are assumed to be independent of the temperature. It follows from the associated flow rule that

$$\xi_r^p s_\theta = \xi_\theta^p s_r \tag{10}$$

where ξ_r^p and ξ_θ^p are the plastic portions of the strain rate tensor and s_θ is the deviatoric circumferential stress, $s_\theta = \sigma_\theta - \sigma$. At small strains, $\xi_r^p = \partial\Delta\varepsilon_r^p / \partial t$ and $\xi_\theta^p = \partial\Delta\varepsilon_\theta^p / \partial t$ where $\Delta\varepsilon_r^p$ and $\Delta\varepsilon_\theta^p$ are the plastic strain increments. It has already been mentioned that the stress field is temperature independent if the entire disc is plastic. However, Eq. (5) is valid in the plastic zone even if there is an elastic zone. Therefore, the stress field in the plastic zone is independent of the temperature in any stage of the process. Since the ratio s_r/s_θ is independent of the time, Eq. (10) may be immediately integrated, with the use of Eq. (2), to give

$$\Delta\varepsilon_r^p\left(\sqrt{3}\cos\varphi - \sin\varphi\right) = 2\Delta\varepsilon_\theta^p \sin\varphi \tag{11}$$

The total strain increments must satisfy the compatibility equation

$$\Delta\varepsilon_r - \Delta\varepsilon_\theta - r\,\partial\Delta\varepsilon_\theta/\partial r = 0 \tag{12}$$

Assuming that $\Delta\varepsilon_r = \Delta\varepsilon_r^e + \Delta\varepsilon_r^p$ and $\Delta\varepsilon_\theta = \Delta\varepsilon_\theta^e + \Delta\varepsilon_\theta^p$ and using Eqs. (7), (9) and (11), Eq. (12) can be transformed to the following linear inhomogeneous differential equation for $\Delta\varepsilon_\theta^p$

$$\partial\Delta\varepsilon_\theta^p/\partial\varphi + \sqrt{3}\Delta\varepsilon_\theta^p + \varsigma\left(2\sqrt{3}k/E\right)\left(\cos\varphi - \sqrt{3}\sin\varphi\right) = 0 \tag{13}$$

Let Δu be the increment of the radial displacement. Then, the total increment of the circumferential strain is $\Delta\varepsilon_\theta = \Delta u/r$ and it vanishes at $r = R_0$ due to Eq. (3). Therefore, the boundary condition to Eq. (13) should be derived from the equation $\Delta\varepsilon_\theta^e + \Delta\varepsilon_\theta^p = 0$ at $r = R_0$ ($\varphi = \varphi_m$). Substituting Eq. (9) into this equation gives

$$\Delta\varepsilon_\theta^p = -\varsigma(2k/E)\left[(2-\nu)\cos\varphi_m - \sqrt{3}\nu\sin\varphi_m\right] - \varsigma 2\alpha T_{max} \tag{14}$$

at $\varphi = \varphi_m$. The minimum value of T_{max} at which the entire disc becomes plastic, T_p, corresponds to the condition $\Delta\varepsilon_\theta^p = 0$ at $r = R_0$, since, as has been already mentioned, the plastic zone starts at the hole and expands until the entire disc becomes plastic. Therefore, Eq. (14) leads to

$$\alpha T_p = (k/E)\left[\sqrt{3}\nu\sin\varphi_m - (2-\nu)\cos\varphi_m\right] \tag{15}$$

Integrating Eq. (13) with the use of Eq. (14) gives

$$\Delta\varepsilon_\theta^p = \varsigma\frac{k}{E}\left[\begin{array}{l}\sqrt{3}\left(\sin\varphi - \sqrt{3}\cos\varphi\right) - \\ \quad -\left(1-2\nu\right)\left(\cos\varphi_m + \sqrt{3}\sin\varphi_m\right) - \left(2E\alpha/k\right)T_{max}\end{array}\right]e^{-\sqrt{3}(\varphi-\varphi_m)} \qquad (16)$$

Using the incompressibility equation for plastic strains, $\Delta\varepsilon_r^p + \Delta\varepsilon_\theta^p + \Delta\varepsilon_z^p = 0$, and Eq. (11), it is possible to find that $\Delta\varepsilon_r^p = \Delta\varepsilon_z^p = -\Delta\varepsilon_\theta^p/2$ at $\varphi = 5\pi/6$ (or $r = r_0$). In the case under consideration, the equivalent plastic strain is defined by $\Delta\varepsilon_{eq}^p = \sqrt{(2/3)\left[\left(\Delta\varepsilon_r^p\right)^2 + \left(\Delta\varepsilon_\theta^p\right)^2 + \left(\Delta\varepsilon_z^p\right)^2\right]}$. Therefore, $\Delta\varepsilon_{eq}^p = \left|\Delta\varepsilon_\theta^p\right|$ at $r = r_0$. It follows from Eq. (16) that

$$\Delta\varepsilon_{eq}^p = \frac{k}{E}\left|\varsigma\left[\begin{array}{l}2\sqrt{3} - \left(1-2\nu\right)\left(\cos\varphi_m + \sqrt{3}\sin\varphi_m\right)e^{-\sqrt{3}(5\pi/6-\varphi_m)} - \\ \qquad\qquad -\left(2E\alpha/k\right)T_{max}e^{-\sqrt{3}(5\pi/6-\varphi_m)}\end{array}\right]\right| \qquad (17)$$

at $r = r_0$. This equation and Eq. (6) show that the increment $\Delta\varepsilon_{eq}^p$ is the same after each cycle $T_{max} \to T_{min} \to T_{max}$. Therefore, the accumulated strain after any number of cycles can be found from Eq. (17) with no difficulty.

DISCUSSION AND CONCLUSIONS

The closed-form solution found in the paper determines the plastic strain increment in a disc with a hole subject to cyclic thermal loading. The solution is valid if $T_{max} \geq T_p$ (Figure 1). The value of T_p can be found from Eqs. (6) and (15). For a mild steel ($\alpha = 18\times10^{-6}$ per ^{0}C, $k/E = 0.635\times10^{-3}$, ν = 0.33), these equations give $T_p \approx 41^0\text{C}$ if $r_0/R_0 \to 0$. For other configurations the value of T_p is even smaller. Thus, the range of applicability of the solution begins at a relatively small amplitude of the temperature field. Once the distribution of plastic strain is found, different approaches may be applied for fatigue damage prediction [8] and in many cases the problem reduces to simple integration. In particular, it has been shown [8] that in some cases this integration results in the Coffin-Manson relationship. Since this relationship is a well established engineering

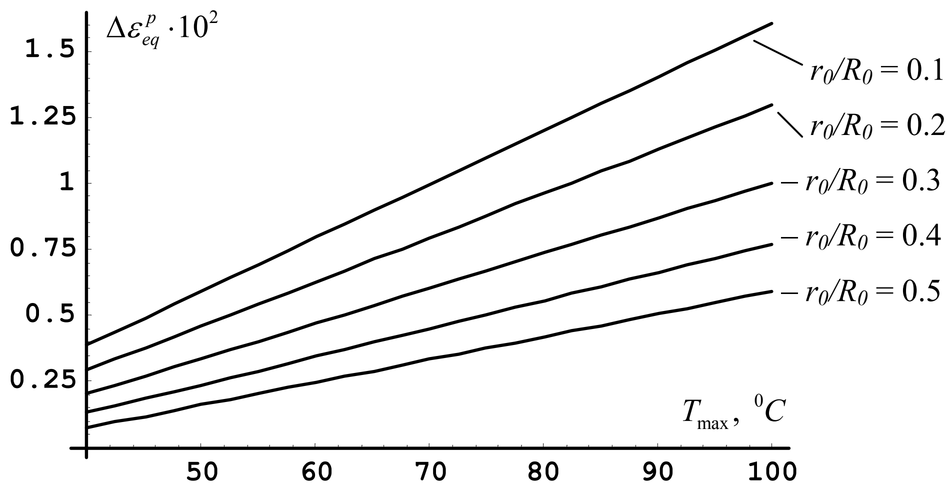

Figure 2: Variation of the plastic strain increment at the hole edge with the temperature amplitude for several disc geometries.

expression for characterizing the total fatigue life [9], it will be used here for illustrative purpose. To apply this approach, it is only necessary to calculate the plastic strain amplitude. This amplitude may be found from Eq. (17). For the mild steel, the plastic strain increment resulting from the temperature change from T_{min} to T_{max} (or from T_{max} to T_{min}) is illustrated in Figure 2 for several disc geometries.

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