LONG-TERM FRACTURE OF AN AGING TRANSVERSALLY-ISOTROPIC COMPOSITE WITH A CIRCULAR CRACK UNDER STATIONARY LOADING

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ABSTRACT

The problem of long-term fracturing of three-dimensional fiber reinforced aging viscoelastic composite with plane disk-shaped macrocrack under stationary loading is considered. The matrix fiber composite has hexagonal symmetry structure and consists of elastic isotropic fibers and aging viscoelastic isotropic matrix. The crack is oriented in the plane of isotropy. Only tensile loads normal to the plane of the crack are considered. The material is modeled as an anisotropic homogeneous linearly-viscoelastic medium with some averaged characteristics. The process zone in front of the crack tip is modeled as a Dugdale zone with a time-dependent stress. As a time operator of the matrix is assumed Maslov-Arutyunyan integral operator. The problem is solved on the basis of the Volterra principle and the long-term fracturing of viscoelastic bodies theory. The solution is derived to the nonlinear integral equation of crack growth. The irrational function of integral operators obtained in the solution is expended into a continued fraction of operators. As the continued fractions quickly converge towards its function only a few fractions is retained in the expansion. Numerical calculation of the nonlinear integral equations of crack growth is performed for specific material. Diagrams of kinetic curves and the service life of cracked body for different volume values of fibers and matrix are represented.

INTRODUCTION

In many cases the application of the Volterra principle to the solution of a quasistatic lineary-viscoelastic problems for homogeneous anisotropic materials comes down to the problem of representing irrational functions of the Volterra integral operators in the standard convolution-type form. Traditionally solution of the problem by expanding the function into a formal Tailor's row has a lot of complicated proceedures in practice. The solution of the problem by using different approximation schemes [1,2] has a lot of restrictions.

Alternative solution of the problem by expanding the function of integral operators into a continued fraction of operators on the bases of Thiele's formula was suggested in works [1,3]. By means of this method in works [1, 4, 5] were obtained solutions of long-term fracturing problems of lineary-viscoelastic materials with the different anisotropy properties.

But this method can not be applied to the non-different integral operators, which describe aging properties of materials, if the resolvent operator is not known.

In this paper, the operator with Maslov-Arutyunyan's kernel is assumed for the constitutive equation of an aging material, as the kernel of the resolvent operator was found by N.Kh. Arutyunyan [1].

PROBLEM FORMULATION

Let us consider internal plane disk-shaped mode I macrocrack of radius l_0 in a three-dimensional fiber reinforced aging viscoelastic composite under stationary tensile stress p applied in the infinity. The crack is lying in the plane of isotropy $x_1 x_2$ and between the layers of reinforced fibers.

The material of composite matrix is assumed as a homogeneous aging viscoelastic material and the composite reinforced fibers as a homogeneous elastic material.

Let us take Volterra integral operator $\lambda\mathbf{K}$ with Maslov-Arutyunyan's kernel [6] to define the constitutive equations of the aging viscoelastic material:

$$\lambda\mathbf{K}\ f(t) = \lambda\int_{\tau_1}^{t} \mathcal{K}(t,\tau)\, f(\tau)\, d\tau \tag{1}$$

$$\lambda\mathcal{K}(t,\tau) = -\mathcal{E}\frac{\partial}{\partial\tau}\left[\varphi(\tau)\cdot\left(1-e^{-\gamma\cdot(t-\tau)}\right)\right]. \tag{2}$$

Here, $\varphi(\tau)=C_0+\dfrac{A_1}{\tau}$ is the aging function, C_0, A_1, γ are the experimentally found parameters, $\mathcal{E}$ is Young's modulus, and $\lambda=\gamma\mathcal{E}$.

The kernel of the resolvent operator $\mathbf{R}(\lambda)$, which satisfy the equation

$$1-\lambda\mathbf{R}(\lambda)=[1+\lambda\mathbf{K}]^{-1}, \tag{3}$$

was found by N.Kh. Arutyunyan [6] and has the following form

$$\mathcal{R}(t,\tau,\lambda)=\varphi(\tau)-[\gamma\cdot\varphi(\tau)+\lambda\cdot\varphi^2(\tau)+\varphi'(\tau)]\int_{\tau}^{t}(\tau/s)^{\lambda A_1} e^{(\gamma+\lambda C_0)(\tau-s)}ds\,,\ \lambda\geq 0\,, \tag{4}$$

The composite material is assumed to be transversally-isotropic homogeneous linearly-viscoelastic aging medium with some averaged characteristics, and a crack to be on the symmetry planes of the anisotropy during all time of propagation.

As a crack model is applied here the modified Leonov-Panasyuk-Dugdale model [1,7] with the constant length of the process zone ($d = const$).

As a criterion of fracture is assumed criterion of critical opening displacement (COD)

$$\delta(r,t)\big|_{r=l(t)}=\delta_C. \tag{5}$$

The parameters of fracture model d and δ_C are assumed not to vary with age of material.

AVERAGED COMPOSITE CHARACTERISTICS

Averaged composite characteristics are determined by formula, which is obtained in work [8]:

$$\mathcal{E}_1=\frac{(\lambda_{11}^{**}-\lambda_{12}^{**})[\lambda_{33}^{**}(\lambda_{11}^{**}+\lambda_{12}^{**})-2\lambda_{13}^{**\,2}]}{\lambda_{11}^{**}\lambda_{33}^{**}-\lambda_{13}^{**\,2}},\ \nu_{12}=\frac{\lambda_{12}^{**}\lambda_{33}^{**}-\lambda_{13}^{**\,2}}{\lambda_{11}^{**}\lambda_{33}^{**}-\lambda_{13}^{**\,2}},\ \mathcal{E}_3=\lambda_{33}^{**}-\frac{2\lambda_{13}^{**\,2}}{\lambda_{11}^{**}+\lambda_{12}^{**}},$$

$$\nu_{13} = \lambda_{13}^{**} / (\lambda_{11}^{**} + \lambda_{12}^{**}) \,, \; G_{13} = \lambda_{44}^{**} / 2 \,, \tag{6}$$

where

$$\left.\begin{aligned}
&\lambda_{11}^{**} = \lambda_{22}^{**} = [3\lambda_{11}^{*} + 2\lambda_{13}^{*} + 3\lambda_{33}^{*} + 4\lambda_{44}^{*} + c(5\lambda_{11}^{*} - 2\lambda_{13}^{*} - 3\lambda_{33}^{*} - 4\lambda_{44}^{*})]/8 \,,\\
&\lambda_{12}^{**} = [\lambda_{11}^{*} + 6\lambda_{13}^{*} + \lambda_{33}^{*} - 4\lambda_{44}^{*} - c(\lambda_{11}^{*} - 8\lambda_{12}^{*} + 6\lambda_{13}^{*} + \lambda_{33}^{*} - 4\lambda_{44}^{*})]/8 \,,\\
&\lambda_{13}^{**} = [\lambda_{12}^{*} + \lambda_{13}^{*} - c(\lambda_{12}^{*} - \lambda_{13}^{*})]/2 \,, \; \lambda_{33}^{**} = \lambda_{11}^{*} - c(\lambda_{11}^{*} - \lambda_{33}^{*}) \,,\\
&\lambda_{44}^{**} = [\lambda_{11}^{*} - \lambda_{12}^{*} + 2\lambda_{44}^{*} - c(\lambda_{11}^{*} - \lambda_{12}^{*} - 2\lambda_{44}^{*})]/4 \,,
\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
&\lambda_{11}^{*} + \lambda_{12}^{*} = c_1(\lambda_{11}^{(1)} + \lambda_{12}^{(1)}) + c_2(\lambda_{11}^{(2)} + \lambda_{12}^{(2)}) - \frac{c_1 c_2 (\lambda_{11}^{(1)} + \lambda_{12}^{(1)} - \lambda_{11}^{(2)} - \lambda_{12}^{(2)})^2}{c_1(\lambda_{11}^{(2)} + \lambda_{12}^{(2)}) + c_2(\lambda_{11}^{(1)} + \lambda_{12}^{(1)}) + 2m} \,,\\
&\lambda_{11}^{*} - \lambda_{12}^{*} = c_1(\lambda_{11}^{(1)} - \lambda_{12}^{(1)}) + c_2(\lambda_{11}^{(2)} - \lambda_{12}^{(2)}) - \frac{c_1 c_2 (\lambda_{11}^{(1)} - \lambda_{12}^{(1)} - \lambda_{11}^{(2)} + \lambda_{12}^{(2)})^2}{c_1(\lambda_{11}^{(2)} - \lambda_{12}^{(2)}) + c_2(\lambda_{11}^{(1)} - \lambda_{12}^{(1)}) + \dfrac{2mn}{n + 2m}} \,,\\
&\lambda_{13}^{*} = c_1\lambda_{12}^{(1)} + c_2\lambda_{12}^{(2)} - \frac{c_1 c_2 (\lambda_{11}^{(1)} + \lambda_{12}^{(1)} - \lambda_{11}^{(2)} - \lambda_{12}^{(2)})(\lambda_{12}^{(1)} - \lambda_{12}^{(2)})}{c_1(\lambda_{11}^{(2)} + \lambda_{12}^{(2)}) + c_2(\lambda_{11}^{(1)} + \lambda_{12}^{(1)}) + 2m} \,,\\
&\lambda_{33}^{*} = c_1\lambda_{11}^{(1)} + c_2\lambda_{11}^{(2)} - \frac{2 c_1 c_2 (\lambda_{12}^{(1)} - \lambda_{12}^{(2)})^2}{c_1(\lambda_{11}^{(2)} + \lambda_{12}^{(2)}) + c_2(\lambda_{11}^{(1)} + \lambda_{12}^{(1)}) + 2m} \,,\\
&\lambda_{44}^{*} = c_1\lambda_{44}^{(1)} + c_2\lambda_{44}^{(2)} - \frac{c_1 c_2 (\lambda_{44}^{(1)} - \lambda_{44}^{(2)})^2}{c_1\lambda_{44}^{(2)} + c_2\lambda_{44}^{(1)} + s} \,,\\
&2m = \left(\frac{c_1}{\lambda_{11}^{(1)} - \lambda_{12}^{(1)}} + \frac{c_2}{\lambda_{11}^{(2)} - \lambda_{12}^{(2)}} \right)^{-1}, \; 2n = \left(\frac{c_1}{\lambda_{11}^{(1)} + \lambda_{12}^{(1)}} + \frac{c_2}{\lambda_{11}^{(2)} + \lambda_{12}^{(2)}} \right)^{-1}, \; s = \left(\frac{c_1}{\lambda_{44}^{(1)}} + \frac{c_2}{\lambda_{44}^{(2)}} \right)^{-1},\\
&\lambda_{11}^{(j)} = \frac{E^{(j)}\nu^{(j)}}{(1 - 2\nu^{(j)})(1 + \nu^{(j)})} + \frac{E^{(j)}}{1 + \nu^{(j)}} \,, \; \lambda_{12}^{(j)} = \frac{E^{(j)}\nu^{(j)}}{(1 - 2\nu^{(j)})(1 + \nu^{(j)})} \,, \; \lambda_{44}^{(j)} = \frac{E^{(j)}}{2(1 + \nu^{(j)})} \,,\\
&j = 1, 2 \,, \; c = c_{14} / c_1 \,.
\end{aligned}\right\} \tag{8}$$

Here, c_1 is specific volume of all reinforcing fibers, $c_2 = 1 - c_1$ is specific volume of the matrix, c_{14} is specific volume of fibers that is perpendicular to the plane of isotropy $x_1 x_2$, superscript (1) denots the material of reinforcing fibers and superscript (2) denotes the matrix material.

Composite viscoelastic characteristics can be obtained from Eqn. 6,7, and 8 after replacing elastic constants by corresponding time operators.

As the fiber material is elastic, the time operators $\overline{\mathbf{E}}^{(1)}$ and $\overline{\nu}^{(1)}$ may be considered as constants. Let's assume the matrix time operator $\overline{\nu}^{(2)}$ is constant, and write the time operator $\overline{\mathbf{E}}^{(2)}$ in form

$$\overline{\mathbf{E}}^{(2)} = E^{(2)}(1 - \lambda \mathbf{R}(\lambda)) \,. \tag{9}$$

Here, $\lambda = \gamma \mathcal{E}^{(2)}$, and $\lambda \mathbf{R}(\lambda)$ is the Volterra integral operator with the kernel from Eqn. 4.

If the composite reinforcing fibers are tough enough in comparison with matrix, the following time operators can be considered as constants

$$\overline{\mathbf{E}}_1 \approx \mathcal{E}_1,\ \overline{\mathbf{E}}_3 \approx \mathcal{E}_3,\ \overline{\nu}_{12} \approx \nu_{12},\ \overline{\nu}_{13} \approx \nu_{13}. \tag{10}$$

After substitution of Eqn. 9 in Eqn. 8,7 the time operator $\overline{\mathbf{G}}_{13}$ can be expressed in the following form

$$\overline{\mathbf{G}}_{13} = \mathcal{G}_{13}\left[1 + \sum_{i=1}^{4} \alpha_i \mathbf{R}(\beta_i)\right]. \tag{11}$$

Here, α_i and β_i are coefficients that are determined during the algebraic transformations of resolvent operators $\mathbf{R}(\lambda_i)$.

So, only one time operator $\overline{\mathbf{G}}_{13}$ describes composite viscoelastic properties.

CRACK OPENING EQUATION

According to the Volterra principle, the crack opening (the double normal displacement of one bank) in the lineary-viscoelastic material is determined by equation of crack opening in elastic material, in which the elastic constants must be replaced by the corresponding time operators of the viscoelastic material. From here and [4], it follows that the crack opening can be expressed by

$$\delta(x, l(t)) = \mathbf{T}\left(\overline{\mathbf{E}}_{ij}\right) \cdot g(x, l(t)), \tag{12}$$

where

$$g(x, l) = \frac{4}{\pi}\sigma \int_{\arcsin\frac{l}{l+d}}^{\psi} \sqrt{l^2 - x^2 \sin^2(\theta)}\, d\theta, \quad \psi = \begin{cases} \pi/2, & 0 \le x \le l, \\ \arcsin\dfrac{l}{x}, & l \le x \le l + d, \end{cases} \tag{13}$$

$$\mathbf{T}\left(\overline{\mathbf{E}}_{ij}\right) = \sqrt{(1 - \nu_{13}\nu_{31})\frac{1}{\mathcal{E}_1 \mathcal{E}_3}} \cdot \sqrt{2\left(\sqrt{(1 - \nu_{12}^2)(1 - \nu_{13}\nu_{31})\frac{\mathcal{E}_1}{\mathcal{E}_3}} - (1 + \nu_{12})\nu_{13}\right) + \frac{\mathcal{E}_1}{\overline{\mathbf{G}}_{13}}}. \tag{14}$$

Using the COD criterion (see Eqn. 5), one can derive from Eqn. 12:

$$\mathbf{T}\left(\overline{\mathbf{E}}_{ij}\right) \cdot g(l(t), l(t)) = \delta_c, \tag{15}$$

where $l(t)$ is the sought function. This is the nonlinear integral equation of crack growth in the viscoelastic aging material.

To efficiently solve Eqn. 15 we need to represent operator $\mathbf{T}\left(\overline{\mathbf{E}}_{ij}\right)$ in the standard convolution-type form

$$\mathbf{T} f(t) = \mathcal{T}^0 \left[f(t) + \int_{\tau_1}^{t} \Pi(t, \tau) f(\tau)\, d\tau \right] \tag{16}$$

Here, $\mathcal{J}^0$ and $\Pi(t,\tau)$ should be expressed in terms of $\mathbf{T}(\overline{\mathbf{E}}_{ij})$.

On the bases of square root function expansion into a continued fraction

$$\sqrt{1+x} = 1 + 2\mathop{\mathrm{K}}_{k=1}^{\infty}\frac{0.25x}{1} = 1 + 2\cfrac{0.25x}{1+\cfrac{0.25x}{1+\cfrac{0.25x}{1+\dots}}}, \tag{17}$$

one can expand the function of integral operators into a continued fraction of operators as follows

$$\sqrt{1+\sum_{i=1}^{4}\mu_i\mathbf{R}(\eta_i)} = 1 + \cfrac{0.5\sum_{i=1}^{4}\mu_i\mathbf{R}(\eta_i)}{1+\cfrac{0.25\sum_{i=1}^{4}\mu_i\mathbf{R}(\eta_i)}{1+\cfrac{0.25\sum_{i=1}^{4}\mu_i\mathbf{R}(\eta_i)}{1+\dots}}} \overset{\mathcal{M}}{\approx} 1 + 2\mathop{\mathrm{K}}_{k=1}^{\mathcal{M}}\frac{0.25\sum_{i=1}^{4}\mu_i\mathbf{R}(\eta_i)}{1}. \tag{18}$$

So, for operator $\mathbf{T}(\overline{\mathbf{E}}_{ij})$ one can derive from Eqn. (14):

$$\mathbf{T}(\overline{\mathbf{E}}_{ij}) \overset{\mathcal{M}}{\approx} \mathcal{J}^0\left[1+\sum_{i=1}^{\mathcal{N}}\gamma_i\mathbf{R}(\lambda_i)\right]. \tag{19}$$

Usually, continued fraction of operator $\mathbf{T}(\overline{\mathbf{E}}_{ij})$ quickly converge towards $\mathbf{T}(\overline{\mathbf{E}}_{ij})$ and, therefore, the number $\mathcal{M}$ in Eqn. (18) can be taken not more than $\mathcal{M} \le 3$.
From Eqn. 19 it follows that the kernel of operator $\mathbf{T}(\overline{\mathbf{E}}_{ij})$ in Eqn. 16 can be written as

$$\Pi(t,\tau) = \sum_{i=1}^{\mathcal{N}}\gamma_i\mathcal{R}(t,\tau,\lambda_i) \tag{20}$$

in the convolution-type form.

SUBCRITICAL CRACK GROWTH

The subcritical crack growth can be divided into three periods [1,7], namely:
incubation period during which the crack only opens but does not grow;
transition period where the growth of the crack starts; and
the main period of slow growth of the crack up to unstable growth.

Incubation Period

Based on Eqns. 15, 16, and 20, for macrocracks when $d \ll l$, the equation for determination of incubation period duration t_* in terms of stress intensity factors has the form

$$\frac{\mathcal{K}_{Ic}}{\mathcal{K}_I} = 1 + \int_{\tau_1}^{t_*}\Pi(t_*,\tau)\, d\tau, \tag{21}$$

where $\mathcal{K}_I = 2p\sqrt{l_0/\pi}$, $\Pi(t,\tau)$ is the kernel of integral operator $\mathbf{T}(\overline{\mathbf{E}}_{ij})$.
The safe value of the stress intensity factor [1,7] $\mathcal{K}_I = \mathcal{K}_{Is}$ is determined from Eqn. 21 by tending $t_* \to \infty$.

Transition Period

Based on Eqns. 15,16, and 20 and approximation of crack opening in the process zone [9], the equation of crack growth during this period can be written as

$$\frac{K_{Ic}}{K_I}=1+F\big((l(t)-l_0)/d\big)\int_{\tau_1}^{t_*}\Pi(t,\tau)\ d\tau+\int_{t_*}^{t}\Pi(t,\tau)F\big((l(t)-l(\tau))/d\big)\ d\tau\,, \tag{22}$$

where $F(s)=\sqrt{1-s}+(s/2)\ln\left((1-\sqrt{1-s})/(1+\sqrt{1-s})\right)$.

The end of transition period t_1 is evaluated from Eqn 22 by the condition when the length attains the value, $l(t_1)=l_0+d$.

Main Period

Due to the Eqns. 15,16, and 20, the equation of the crack growth during the main period has the form:

$$\frac{K_{Ic}}{K_I}=1+\int_{t'}^{t}\Pi(t,\tau)F\big((l(t)-l(\tau))/d\big)d\tau\,, \tag{23}$$

where t' is determined by equation $l(t)-l(t')=d$.

At the time t_2, when the crack achieves the critical length $l(t_2)=l_*$, the dynamic growth of the crack starts.

As the time of dynamic growth is very small, the service life ΔT of the cracked aging viscoelastic material is determines by expression

$$\Delta T=t_2-\tau_1. \tag{24}$$

AN EXAMPLE

Let’s consider a specific example with the following characteristics of materials:

Reinforcing fibers (fiberglass) – $E^{(1)}=0.7\cdot 10^{11}\,\mathrm{Pa}$, $\nu^{(1)}=0.2$;

Matrix (concrete) – $E^{(2)}=2\cdot 10^{10}\,\mathrm{Pa}$, $\nu^{(2)}=0.167$, $C_0=0.918\cdot 10^{-10}\,\mathrm{Pa}^{-1}$,

$A_1=4.918\cdot 10^{-10}\,\mathrm{day}\cdot\mathrm{Pa}^{-1}$, $\gamma=0.026\,\mathrm{day}^{-1}$.

Table 1 shows how converges continued fraction of operator $\mathbf{T}(\bar{\mathbf{E}}_{ij})$(see Eqn.19) in the space of positive functions E ($c_1=0.4$, $c_{14}=0.1$).

TABLE 1

CONVERGENCE OF CONTINUED FRACTION OF OPERATOR $\mathbf{T}(\bar{\mathbf{E}}_{ij})$

	$\left\Vert\mathbf{T}(\bar{\mathbf{E}}_{ij})\right\Vert_E/\mathbf{T}(\bar{\mathbf{E}}_{ij})$				
τ_1, day	M=1	M=2	M=3	M=4	M=10
7	1.6303	1.515	1.5319	1.5295	1.5298
21	1.4498	1.3738	1.3845	1.3830	1.3831
120	1.3755	1.3165	1.3245	1.3234	1.3235

In the numerical calculations of the specific example it accepts $\mathcal{M} = 4$.

Figure 1 shows numerical calculation of the service life of cracked aging composite vs. specific volume of fibers that lay in the plane of isotropy $\xi = c_1 - c_{14}$ for different ages of material (1-$\tau_1 = 1\ days$, 2-$\tau_1 = 21\ days$, 3-$\tau_1 = 120\ days$). The calculation is performed for the following data: $c_{14} = 0.1$, $d/l_0 = 0.05$, $p/p_* = 0.8$, were $2p_*\sqrt{l_0/\pi} = K_{Ic}$.

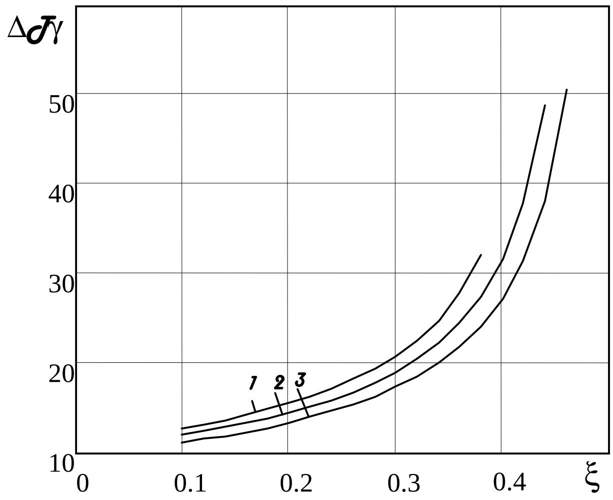

Figure 1: The diagram of service life vs. the specific volume of reinforcing fibers.

Figure 2 shows numerical calculation of kinetic curves of crack growth for different ages of material (1-$\tau_1 = 1\ days$, 2-$\tau_1 = 21\ days$, 3-$\tau_1 = 120\ days$). The calculation is performed for the same data as in Figure 1 and $c_1 = 0.4$.

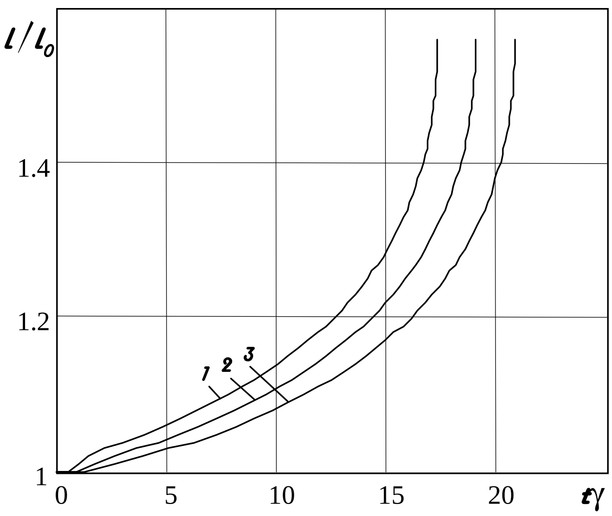

Figure 2: The diagram of kinetic curves.

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