Growth of a Doubly Periodic Array of Fatigue Cracks in Anisotropic Elastic Medium

V. Bozhydarnyk, H. Sulym and Ia. Pasternak

Lutsk National Technical University, Lvivska Str. 75, 43018 Lutsk, Ukraine;
e-mail: pasternak@ukrpost.ua

ABSTRACT. *This study considers a doubly periodic array of cracks in the anisotropic elastic medium. The solution of the problem is reduced to a system of boundary integral equations, which are solved using the boundary element method. To determine the fracture propagation angle in the anisotropic medium Sih strain energy density criterion is applied. The utilized crack growth equation is based on the empirical Paris law.*

INTRODUCTION

The study of multiple cracks interaction is often reduced to the simulation of the regular arrays of congruent cracks. This approach is often used in rock mechanics, mechanics of composite materials etc.

There are three main approaches used in the boundary element (boundary integral equation) method for studying the doubly periodic sets of cracks and inclusions and effective properties of composite materials. The first one used by Liu [1] simulates media with multiple inclusions (fibers). The second approach considers only one representative volume element (RVE) of the composite material with a regular structure. Liu and Chen [2], Dong and Lee [3] adopted this approach for the use with the boundary element method (BEM). The third approach utilizes special boundary integral equations for periodic problems. Lin'kov and Koshelev [4] and Lin'kov [5] used the third approach and developed the complex variable BEM for studying of the doubly periodic arrays of cracks, holes and inclusions in the isotropic elastic medium. Clouteau et el. [6] derived the integral equations for a periodic 3D BEM. Due to its semi-analytical nature, this approach allows not only to determine the stress intensity factors for a doubly periodic cracks or a stress concentration on holes and inclusions, but also to study the effective properties of composite materials without additional consideration of the boundary of the RVE and the periodic conditions imposed on it. Thus, in numerical modeling only the boundary of a crack is considered, which significantly decreases the size of the resulting system of equations. The shape of the RVE is defined by two fundamental periods, which form the lattice.

The third approach is widely used for accurate analysis of doubly periodic sets of cracks and thin inclusions. Wang [7] presented extremely accurate and efficient method for computing the interaction of a set or multiple sets of general doubly periodic cracks in isotropic elastic medium. Xiao and Jiang [8] studied the orthotropic medium with doubly periodic cracks of unequal size under antiplane shear. Chen et al. [9] have studied various multiple crack problems in elasticity.

However, the study of anisotropic solids containing doubly periodic arrays of cracks is a challenging problem. Therefore, this paper is focused on the development of the efficient BEM approach for the analysis of regular sets of cracks and their growth.

BOUNDARY INTEGRAL EQUATIONS FOR DOUBLY PERIODIC PROBLEMS

The static equilibrium equations in the reference coordinate system $Ox_1x_2x_3$ can be given in the form [10]

$$\sigma_{ij,j} + f_i = 0 \quad \left(i, j = 1,2,3\right), \tag{1}$$

where σ_{ij} is a stress tensor; f_i is a body force vector. Here and further, the Einstein summation convention is assumed. The comma at subscript denotes the differentiation with the respect to the coordinate indexed after the comma, i.e. $u_{i,j} \equiv \partial u_i / \partial x_j$.

Under the assumption of small strains the constitutive relations of linear anisotropic elasticity are as follows [10]

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km}, \tag{2}$$

where $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$ is a strain tensor; u_i is a displacement vector; C_{ijkm} are the elastic stiffnesses (elastic moduli). With respect to the symmetry properties of the elasticity tensor

$$C_{ijkm} = C_{jikm} = C_{kmji}, \tag{3}$$

Eq. (2) can be rewritten in the following form:

$$\sigma_{ij} = C_{ijkm}u_{k,m}. \tag{4}$$

Consider the 2D stress/strain field, in which displacements do not change with the x_3 coordinate of a solid, i.e. $u_{i,3} \equiv 0$. Thus, the mechanical fields at the arbitrary cross-section of a solid normal to x_3 axis are the same. In this case, the equilibrium equation (1) takes the form

$$\sigma_{ij,j} + f_i \equiv C_{ijkm}u_{k,jm} + f_i = 0 \quad \left(i,k = 1,..,3;\ j,m = 1,2\right). \tag{5}$$

Consider a doubly periodic set of cracks, which are modeled by the lines Γ_s ($s \in \mathbb{Z}$) of displacement discontinuities (Fig. 1). Due to the translational symmetry of the considered doubly periodic problem, the discontinuities of tractions Σt_i^s and displacements Δu_i^s are identical for each of the contours Γ_s ($s \in \mathbb{Z}$). Therefore, the system of boundary integral equations of a doubly periodic problem, which give the solution to Eq. (5), can be written in the following form [11]

$$\frac{1}{2}\Sigma u_i^0(\mathbf{y}) = \int_{\Gamma_0^+} U_{ij}^{\mathrm{dp}}(\mathbf{x},\mathbf{y})\Sigma t_j^0(\mathbf{x})d\Gamma(\mathbf{x}) - \int_{\Gamma_0^+} T_{ij}^{\mathrm{dp}}(\mathbf{x},\mathbf{y})\Delta u_j^0(\mathbf{x})d\Gamma(\mathbf{x}) + I_i^\infty(\mathbf{y}),$$
$$\frac{1}{2}\Delta t_i^0(\mathbf{y}) = n_j^+(\mathbf{y})\left[\Xi_{ij}^\infty + \int_{\Gamma_0^+} D_{ijk}^{\mathrm{dp}}(\mathbf{x},\mathbf{y})\Sigma t_k^0(\mathbf{x})d\Gamma(\mathbf{x}) - \int_{\Gamma_0^+} S_{ijk}^{\mathrm{dp}}(\mathbf{x},\mathbf{y})\Delta u_k^0(\mathbf{x})d\Gamma(\mathbf{x})\right], \tag{6}$$

where the doubly periodic kernels $\mathbf{K}^{\mathrm{dp}} = \left[U_{ij}^{\mathrm{dp}}, T_{ij}^{\mathrm{dp}}, D_{ijk}^{\mathrm{dp}}, S_{ijk}^{\mathrm{dp}}\right]$ are explicitly defined in Ref. [11], functions I_i^∞ and Ξ_{ik}^∞ define the external load [11].

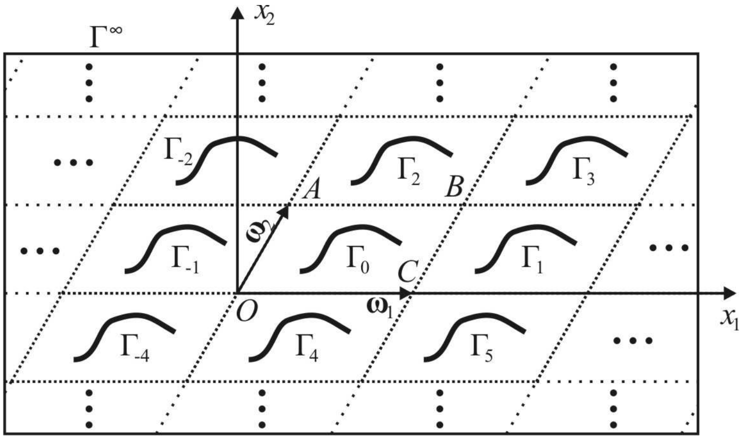

Figure 1. A doubly periodic set of curved cracks

Average strains are as follows [11]

$$\begin{aligned}\langle u_{i,1}\rangle &= -\Delta\hat{u}_i^{(2)}\omega_{x_2}^{(1)} \Big/ \left(\omega_{x_1}^{(1)}\omega_{x_2}^{(2)} - \omega_{x_1}^{(2)}\omega_{x_2}^{(1)}\right) + u_{i,j}^{\mathrm{hom}}(\Xi_{km}),\\ \langle u_{i,2}\rangle &= \quad \Delta\hat{u}_i^{(2)}\omega_{x_1}^{(1)} \Big/ \left(\omega_{x_1}^{(1)}\omega_{x_2}^{(2)} - \omega_{x_1}^{(2)}\omega_{x_2}^{(1)}\right) + u_{i,j}^{\mathrm{hom}}(\Xi_{km}),\end{aligned} \tag{7}$$

where $u_{i,j}^{\mathrm{hom}}(\sigma_{km})$ are gradients of displacements in the homogeneous medium caused by the far-field load σ_{km}; $\boldsymbol{\omega}^{(1)} = \left[\omega_{x_1}^{(1)}, \omega_{x_2}^{(1)}\right]^{\mathrm{T}}$ and $\boldsymbol{\omega}^{(2)} = \left[\omega_{x_1}^{(2)}, \omega_{x_2}^{(2)}\right]^{\mathrm{T}}$ are the period vectors (see Fig. 1). Cyclic constants are equal to [11]

$$\begin{aligned}\Delta u_i^{(k)} &= u_{i,j}^{\mathrm{hom}}(\Xi_{km})\omega_{x_j}^{(k)} + \Delta\hat{u}_i^{(k)}, \quad \Delta\hat{u}_i^{(1)} = 0,\\ \Delta\hat{u}_i^{(2)} &= \int_{\Gamma_0^+}\left[U_{ij}^*(\mathbf{x})\Sigma t_j^0(\mathbf{x}) - T_{ij}^*(\mathbf{x})\Delta u_j^0(\mathbf{x})\right]d\Gamma(\mathbf{x}).\end{aligned} \tag{8}$$

FATIGUE CRACK GROWTH SIMULATION

Boundary integral equations (6) can be solved numerically using the direct boundary element method. Since the singularity of corresponding kernels of the periodic and doubly periodic BIEs is the same as that of the nonperiodic BIEs, for the numerical evaluation of the weakly, strongly and hypersingular integrals one can utilize quadratures and polynomial transformations of Ref. [12], which smooth the integrand. While studying cracks and thin inclusions it is convenient to use the special shape functions [12], which accounts the square root singularity at the crack tip. This allows determination of the stress intensity factors using the following relation [12]

$$\mathbf{k}^{(1)} = \lim_{s \to 0} \sqrt{\frac{\pi}{8s}} \mathbf{L} \cdot \Delta \mathbf{u}(s), \tag{9}$$

where $\mathbf{k}^{(1)} = [K_{II}, K_{I}, K_{III}]^{T}$ is a vector of stress intensity factors (SIF); $\mathbf{L}$ is a real Barnett – Lothe tensor [10]. Mechanical fields in a local system of coordinates $Ox_1'x_2'$ with the origin O placed at the crack tip and Ox_1' axis directed along a tangent to the crack are as follows [11]

$$\mathbf{u} = \sqrt{\frac{2}{\pi}} \operatorname{Re}\left\{ \mathbf{A} \left\langle \sqrt{Z_*} \right\rangle \mathbf{B}^{-1} \mathbf{k}^{(1)} \right\}, \quad \boldsymbol{\varphi} = \sqrt{\frac{2}{\pi}} \operatorname{Re}\left\{ \mathbf{B} \left\langle \sqrt{Z_*} \right\rangle \mathbf{B}^{-1} \mathbf{k}^{(1)} \right\}, \tag{10}$$

where $\left\langle \sqrt{Z_*} \right\rangle = \operatorname{diag}\left[\sqrt{Z_1}, \sqrt{Z_2}, \sqrt{Z_3} \right]$; $Z_\alpha = x_1' + p_\alpha x_2' = r(\cos\theta + p_\alpha \sin\theta)$; p_α are Stroh eigenvalues [10]; (r, θ) are local polar coordinates; $\mathbf{A}$ and $\mathbf{B}$ are Stroh matrices, which are completely defined by the elasticity tensor C_{ijkm} [10]; $\boldsymbol{\varphi}$ is a stress function. The stress field near the crack tip equals [10, 11]

$$\begin{aligned} \boldsymbol{\sigma}_1 &= [\sigma_{i1}] = -\boldsymbol{\varphi}_{,2} = -\frac{1}{\sqrt{2\pi}} \operatorname{Re}\left\{ \mathbf{B} \left\langle p_* Z_*^{-1/2} \right\rangle \mathbf{B}^{-1} \mathbf{k}^{(1)} \right\}, \\ \boldsymbol{\sigma}_2 &= [\sigma_{i2}] = \boldsymbol{\varphi}_{,1} = \frac{1}{\sqrt{2\pi}} \operatorname{Re}\left\{ \mathbf{B} \left\langle Z_*^{-1/2} \right\rangle \mathbf{B}^{-1} \mathbf{k}^{(1)} \right\}. \end{aligned} \tag{11}$$

Based on Eqs. (10) and (11) one can calculate the strain energy density (SED) near the crack tip

$$W(r, \theta) = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \left(u_{i,2} \varphi_{i,1} - u_{i,1} \varphi_{i,2} \right) = rS(\theta), \tag{12}$$

where $S(\theta)$ is a SED factor [13].

As stated by Sih [13], crack initiation will start in a radial direction along which the strain energy density $S(\theta)$ is a minimum. Additionally one should consider the hoop stress $\sigma_{\theta\theta}$, which is to be positive in the crack growth direction.

The crack growth rate is defined by the empirical Paris law

$$\frac{da}{dN} = A(\Delta K)^n, \tag{13}$$

where da/dN is a rate of crack growth per cycle; A and n are empirical constants; and $\Delta K = K_{I\max} - K_{I\min}$.

The boundary element method used allows efficient simulation of fatigue crack growth. Since only boundary mesh is required, there is no need to remesh the entire RVE, but only to add two new boundary elements at both tips of the crack inclined at an angle defined by the Sih SED criterion [13].

However, the numerical simulation is stable if one appends crack tip elements of approximately the same length Δa. This implies in different number ΔN of cycles per simulation step. The latter is determined from Eq. (13) as

$$\Delta N \approx \frac{\Delta a}{A(\Delta K)^n} . \tag{14}$$

If Δa is small enough, the numerical results are convergent.

EFFECTIVE PROPERTIES OF A CRACKED ANISOTROPIC MATERIAL

To determine the effective properties of a cracked anisotropic material one has to solve the problem for 5 linearly independent variants of an average stress $\langle \sigma_{ij} \rangle$, the best of which are

1) $\langle \sigma_{11} \rangle = 1$ Pa, $\langle \sigma_{ij} \rangle = 0$ $\left((i \neq 1) \wedge (j \neq 1) \right)$;
2) $\langle \sigma_{22} \rangle = 1$ Pa, $\langle \sigma_{ij} \rangle = 0$ $\left((i \neq 2) \wedge (j \neq 2) \right)$;
3) $\langle \sigma_{32} \rangle = 1$ Pa, $\langle \sigma_{ij} \rangle = 0$ $\left((i \neq 3) \wedge (j \neq 2) \right)$;
4) $\langle \sigma_{31} \rangle = 1$ Pa, $\langle \sigma_{ij} \rangle = 0$ $\left((i \neq 3) \wedge (j \neq 1) \right)$;
5) $\langle \sigma_{12} \rangle = 1$ Pa, $\langle \sigma_{ij} \rangle = 0$ $\left((i \neq 1) \wedge (j \neq 2) \right)$.

Then a matrix

$$\langle \mathbf{c} \rangle = \begin{bmatrix} \langle u_{1,1}^{(1)} \rangle & \cdots & \langle u_{1,1}^{(5)} \rangle \\ \langle u_{2,2}^{(1)} \rangle & \cdots & \langle u_{2,2}^{(5)} \rangle \\ \langle u_{3,2}^{(1)} \rangle & \cdots & \langle u_{3,2}^{(5)} \rangle \\ \langle u_{3,1}^{(1)} \rangle & \cdots & \langle u_{3,1}^{(5)} \rangle \\ \langle u_{1,2}^{(1)} \rangle + \langle u_{2,1}^{(1)} \rangle & \cdots & \langle u_{1,2}^{(5)} + u_{2,1}^{(5)} \rangle \end{bmatrix}^{-1} \tag{15}$$

is the matrix of effective properties of MEE composite material, which relates an average stress vector $\langle \boldsymbol{\sigma} \rangle = \left[\langle \sigma_{11} \rangle, \langle \sigma_{22} \rangle, \langle \sigma_{32} \rangle, \langle \sigma_{31} \rangle, \langle \sigma_{12} \rangle \right]^{\mathrm{T}}$ with an average strain vector $\langle \boldsymbol{\varepsilon} \rangle = \left[\langle \varepsilon_{11} \rangle, \langle \varepsilon_{22} \rangle, \langle 2\varepsilon_{32} \rangle, \langle 2\varepsilon_{31} \rangle, \langle 2\varepsilon_{12} \rangle \right]^{\mathrm{T}}$: $\langle \boldsymbol{\sigma} \rangle = \langle \mathbf{c} \rangle \langle \boldsymbol{\varepsilon} \rangle$. The superscript ahead of $\langle u_{i,j} \rangle$ in Eq. (15) denotes the variant of the applied average load. The values of $\langle u_{i,j}^{(k)} \rangle$ used in Eq. (15) are determined using Eq. (7).

Proposed approach utilizes special boundary integral equations (6), which use the doubly periodic kernels [11]. Thus, for determination of the effective properties of a medium with a doubly periodic array of cracks one has to consider only the surface of one crack, because Eqs. (6), (7) already include the periodic boundary conditions at the unit cell. The shape of the RVE is defined by the two periods $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$, which form the lattice.

NUMERICAL EXAMPLE

Consider a glass/epoxy plate containing a doubly periodic array of cracks of an initial length $2a$. Cracks form a square lattice, and the RVE side is two times greater than the length of the crack ($d = 4a$). Mechanical properties of the glass/epoxy are as follows: $E_1 = 48.26$ GPa; $E_2 = 17.24$ GPa; $\nu_{12} = 0.29$; $G_{12} = 6.89$ GPa. Material symmetry axes are inclined to the Cartesian axes as shown in Fig. 2. Crack paths are obtained for a single crack (dashed lines) and for doubly periodic cracks (solid lines) for different values of the angle α, which defines the direction of the material symmetry axes.

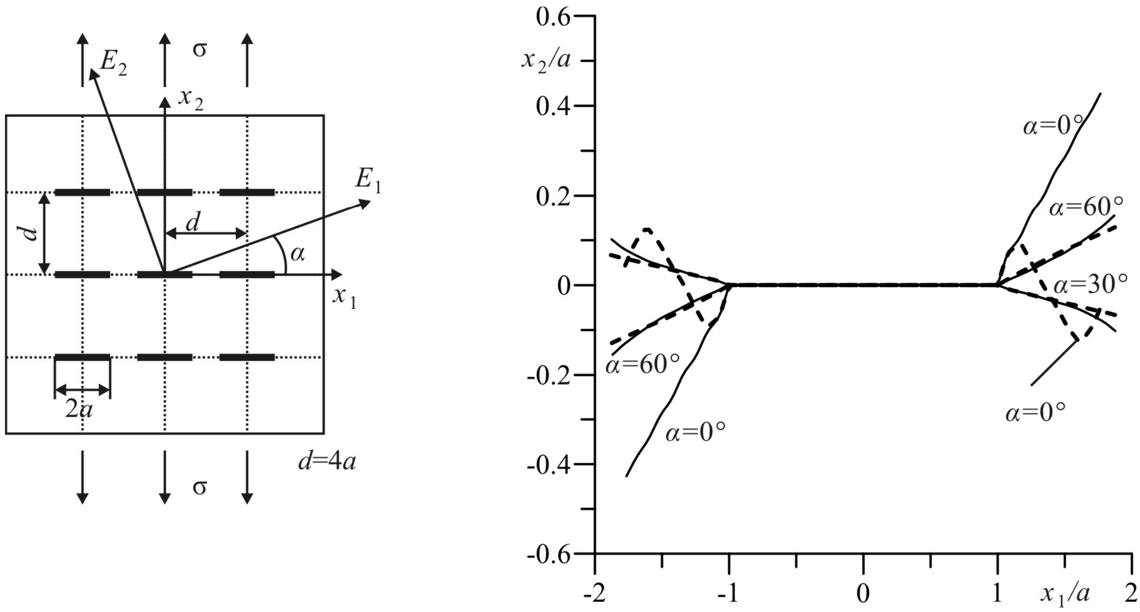

Figure 2. Doubly periodic array of fatigue cracks in glass/epoxy

One can see in Fig. 2 that the biggest inclination of a crack path is observed for the case, when material symmetry axes coincide with the Cartesian ones. In this case due to the anisotropy of the material, the path of a single crack is winding. For different angles α crack paths have different direction, and the inclination of the crack path is bigger for a doubly periodic cracks. Thus, crack interaction and anisotropy of a plate are essential for study of crack propagation.

CONCLUSION

The account of material anisotropy and crack interaction is significant for determination of a crack path. The developed boundary element approach allows accounting both.

REFERENCES

1. Liu, Y. (2006) *Int. J. Numer. Meth. Engng.* **65**, 863–881
2. Liu, Y.J., Chen, X.L. (2003) *Electronic Journal of Boundary Elements* **1**(2), 316–335.
3. Dong, C.Y., Lee, K.Y. (2005) *Int. J. Fract.* **133**, 389–405.
4. Lin'kov, A.M., Koshelev, V.F. (1999) *J. Chinese Institute of Engineers* **22**(6), 709–720.
5. Lin'kov, A.M. (2002) *Boundary integral equations in elasticity theory*, Springer.
6. Clouteau, D., Elhabre, M.L., Aubry, D. (2000) *Comp. Mech.* **25**(6), 567–577.
7. Wang, G.S. (2004) *Theor. Appl. Fract. Mech.* **42**, 249–294.
8. Xiao, J., Jiang, C. (2009) *Acta Mechanica Solida Sinica* **22**(1), 53–63.
9. Chen, Y.Z., Hasebe, N., Lee, K.Y. (2003) *Multiple crack problems in elasticity*, Vol. 1. WIT.
10. Ting, T.C.T. (1996) *Anisotropic elasticity: theory and applications*, Oxford University Press.
11. Pasternak, Ia. (2012) *Eng. Anal. Bound. Elem.* **36**(5), 799–811.
12. Pasternak, Ia. (2011) *Eng. Anal. Bound. Elem.* **35**(4), 678–690.
13. Sih, G.C. (1991) *Mechanics of fracture initiation and propagation: surface and volume energy density applied as failure criterion*, Kluwer Academic Publishers.