Three-dimensional stress distributions ahead of sharply radiused V-notches in finite thick plates

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ABSTRACT. *By making use of the generalised plane strain hypothesis, an approximate stress field theory has been developed according to which the three-dimensional governing equations lead to a system where a bi-harmonic equation and a harmonic equation should be simultaneously satisfied. The former provides the solution of the corresponding plane notch problem, the latter provides the solution of the corresponding out-of-plane notch problem. The system can be applied not only to pointed three-dimensional V-notches but also to sharply radiused V-notches characterised by a notch tip radius small enough. Two examples are considered: an inclined elliptical hole in a thick plate under tension, and a uniaxially loaded shouldered thick plate. Limits and degree of accuracy of the analytical frame are discussed comparing theoretical results and numerical data from FE models. Practical consequences on early crack propogation angles are also documented.*

INTRODUCTION

Due to the inherent difficult of finding a complete solution of the elastic stress fields in notched or cracked three-dimensional solids, most of the analytical and numerical efforts in the literature have been devoted to the determination of the two-dimensional stress distributions. Pioneering studies on three-dimensional stress fields in cracked plates were carried out by Hartranft and Sih [1] and by Kassir and Sih [2].

In the ambit of three-dimensional elasticity, with the aim to simplify the governing equations, different plate theories (such as those due to Kirchhoff or Reissner) have been used to determine the approximate stress fields near the tip of a through crack in a thin elastic plate. For an extensive review on this topic, the reader should consult a paper by Zehnder and Viz [3].

The Kane and Mindlin theory, first proposed to analyse high frequency extensional vibrations [4], was used by Yang and Freund to study the state of stress in a thin elastic plate containing through-cracks [5] and by Kotousov and Lew [6] to discuss in detail the 'out-of-plane' mode ahead of cracks and sharp V-notches. The combined use of the Kane and Mindlin theory and the Bessel-function-eigen-expansion made it evident how an out-of-plane shear stress singularity always exists, in addition to Williams' in-plane singularities.

Dealing with 'blunt cracks' with a non-zero tip radius, the proof of the existence of the out-of-plane mode in three dimensional plates under remote in-plane shear loading has been discussed by Pook [7] for a parallel-sided notch with a semicircular small tip radius (ρ=0.01 and 0.1 mm). Pook demonstrated that Mode II and out-of-plane Mode cannot exist in isolation. If one of these modes is applied then the other is always induced. In order to describe the shape of cracks' displacements and to explain the link between the two modes, Volterra's "*distorsioni*" in a ring element were used [7].

Out-of-plane stress distributions have been documented in some recent papers by Berto et al. [8,9] for a variety of notch configurations. Recently Lazzarin and Zappalorto [10], by making use of the generalised plane strain hypothesis, have developed an approximate stress field theory according to which the three-dimensional governing equations lead to a system where a bi-harmonic equation and a harmonic equation should be simultaneously satisfied. The former provides the solution of the corresponding plane notch problem, the latter provides the solution of the corresponding out-of-plane notch problem.

Such a solution is reconsidered in the present work with the aim to:

- discuss the three-dimensional effects arising in a thick plate, infinitely extended in the x and y directions, weakened by an inclined elliptical hole. It will be prove the existence, besides the in-plane stress components linked to the well known Inglis solution, of two non-Inglis out-of-plane shear stress components;
- discuss the three-dimensional effects arising in a shouldered plate of finite thickness under tension. In particular it will be proved that the presence of a local out-of-plane singular mode the crack initiation angles vary through the thickness.

A NEW APPROACH TO THE THREE-DIMENSIONAL PROBLEM

Consider the Kane and Mindlin hypothesis for displacement components:

$$u_x = u(x,y) \qquad u_y = v(x,y) \qquad u_z = f(z)\,w(x,y) \qquad \text{(with } f(z)=bz\text{)} \tag{1}$$

It is easy to verify that as soon as displacement components are according to Eq. (1), the normal strains ε_{ii}, and γ_{xy}, are independent of z. As a consequence, by invoking the stress-strain relationships, also the stress components σ_{xx}, σ_{yy}, τ_{xy} and σ_{zz} are independent of z [10], while out-of-plane shear components result:

$$\tau_{yz} = G \times bz \frac{\partial w}{\partial y} \qquad \tau_{xz} = G \times bz \frac{\partial w}{\partial x} \tag{2}$$

Then the equilibrium equation in the z direction simply gives:

$$\nabla^2 w = 0 \tag{3}$$

where ∇^2 denotes the two-dimensional Laplacian operator. Invoking Eq. (3) the equilibrium equation in the x and y direction can be re-written as:

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial y \partial x} = 0 \tag{4}$$

Since the stress components σ_{xx}, σ_{yy} and τ_{xy} do not depend on z, we can introduce the classic Airy stress function $\phi(x,y)$ such that:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{5}$$

Doing so, Eq. (4) is automatically satisfied. At the same time, accounting for the generalised Hooke law for stresses and strains, the in-plane compatibility equation can be written as follows:

$$\nabla^4 \phi = \nu \nabla^2 \sigma_{zz} = 0 \tag{6}$$

the latter substitution being guaranteed by third of Beltrami-Mitchell's equations.

This mean that any three dimensional notch problem obeying to the displacement law given by Eq. (1) can be converted into a bi-harmonic problem (typical of plane stress or plane strain conditions) and a harmonic problem (typical of the out-of-plane shear case) according to the following system:

$$\nabla^4 \phi = 0 \qquad \nabla^2 w = 0 \tag{7}$$

Here w and ϕ are implicitly defined according to Eqs. (2) and (5), respectively. Equation (7a) is the common bi-harmonic equation providing the solution of the plane problem, whereas Eq. (7b) is, instead, the harmonic equation providing the solution of out-of-plane shear problem.

A NON-INGLIS SOLUTION FOR THE ELLIPTIC HOLE

Consider a slim inclined elliptic hole in a plate infinitely extended in the x and y directions and of finite thickness loaded in tension. Suppose β is the arbitrary orientation angle, as shown in figure 1.

The in-plane stress field solution for this problem is due to Inglis. However, as explained in the previous section, due to three-dimensional effects there exists, besides the common Inglis plane stresses, out-of-plane shear stress components, τ_{zx} and τ_{zy}.

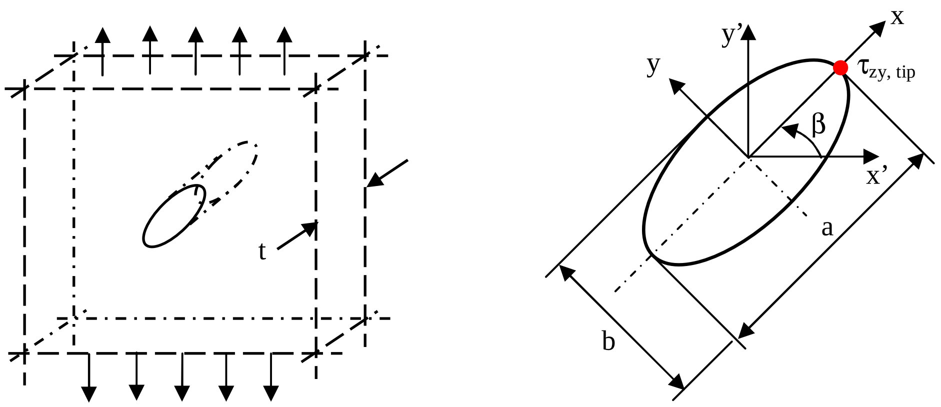

Figure 1. Inclined elliptic hole in a three-dimensional plate under tension

This non-Inglis shear stress field results from the solution of the harmonic equation,

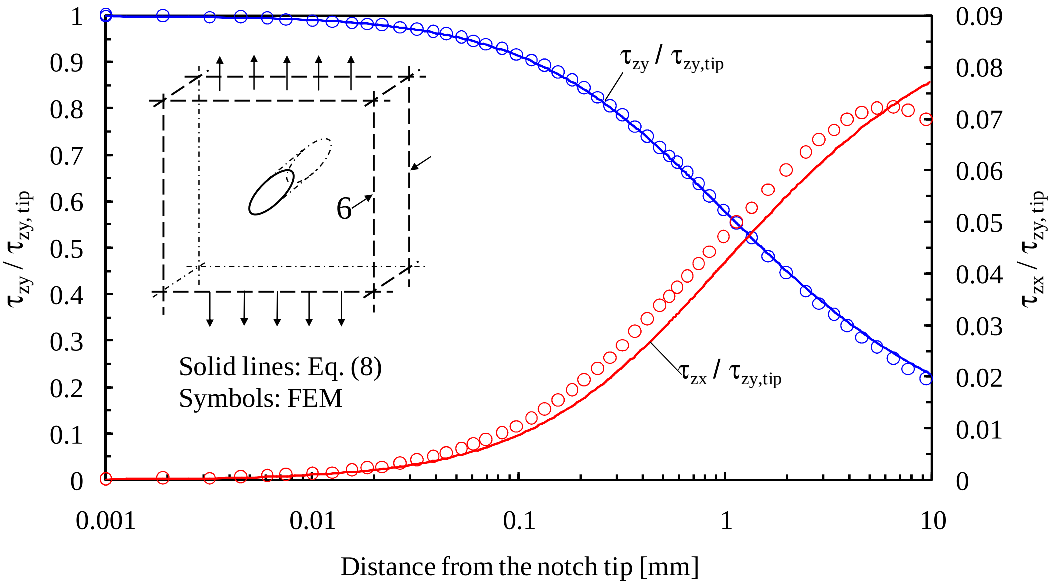

Figure 2. Plot of the stress components τ_{zx} and τ_{zy} along the *x*-direction for an inclined elliptical hole (a=1 mm, b=0.1 mm and β=45°) and comparison with Eq. (8). Distance from the mid plane z=2.5 mm.

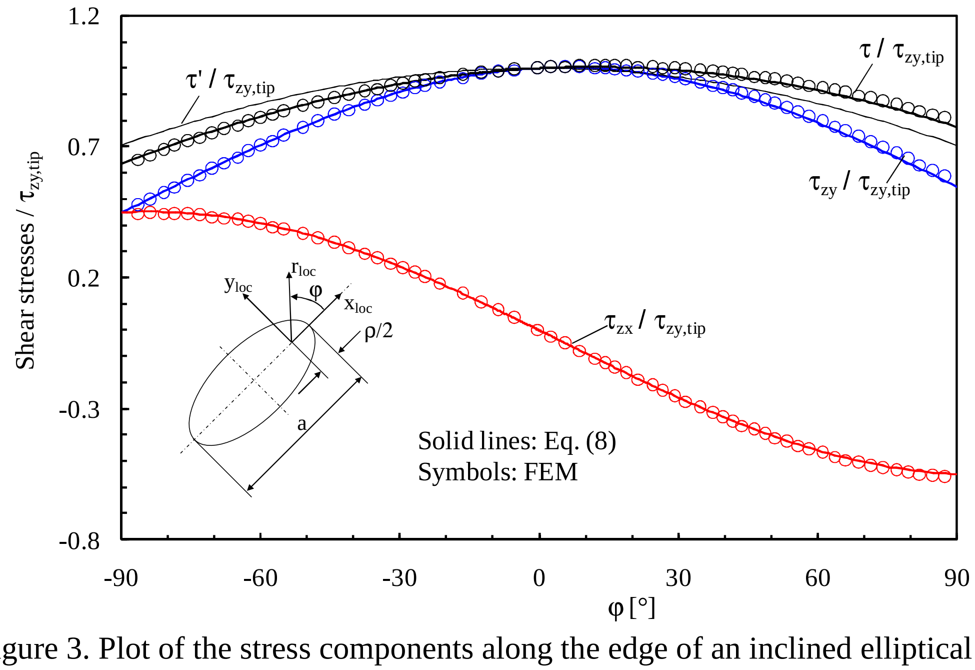

Figure 3. Plot of the stress components along the edge of an inclined elliptical hole (a=1 mm, b=0.1 mm and β=45°) and comparison with Eq. (8). Plate thickness t= 6mm. Distance from the mid plane z=2.5 mm.

Eq. (3), and can be written as [11]:

$$\tau_{zx} = \frac{\tau_{zy,\,tip}\, b}{a^2 - b^2}\left[b \tan\beta \frac{\sinh 2\xi}{\cosh 2\xi - \cos 2\eta} - a\left(\tan\beta + \frac{\sin 2\eta}{\cosh 2\xi - \cos 2\eta}\right)\right]$$
$$\tau_{zy} = \frac{\tau_{zy,\,tip}\, b}{a^2 - b^2}\left[a \frac{\sinh 2\xi}{\cosh 2\xi - \cos 2\eta} + b\left(\tan\beta \frac{\sin 2\eta}{\cosh 2\xi - \cos 2\eta} - 1\right)\right] \qquad (\beta \neq \frac{\pi}{2}) \qquad (8)$$

where curvilinear coordinates η and ξ can be given as a function of x and y [11].

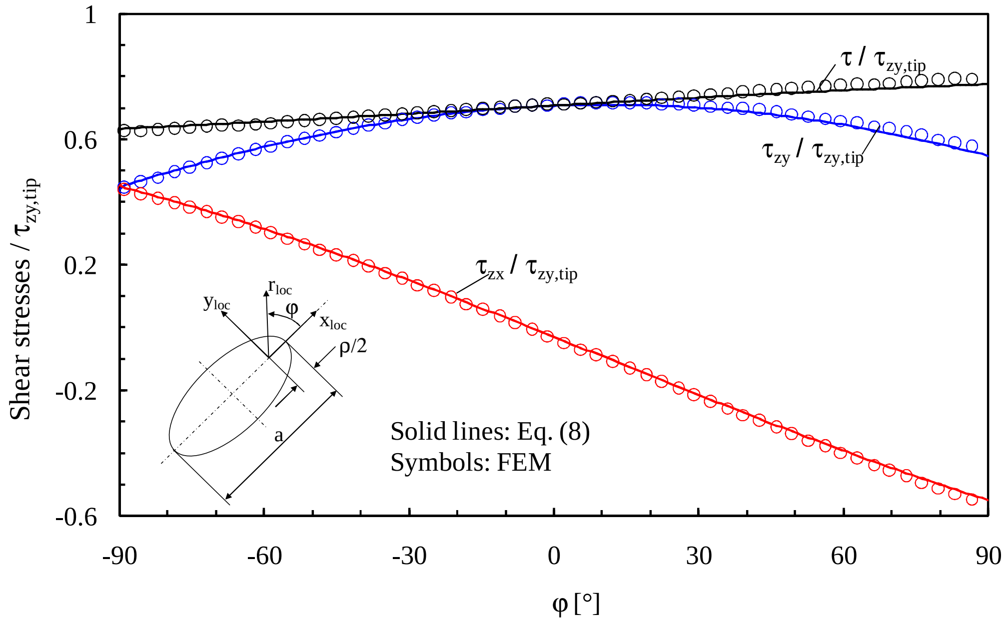

Figure 4. Plot of the stress components of an inclined elliptical hole (a=1 mm, b=0.1 mm and β=45°) along a circular path of radius r_{loc}= 0.01 mm and comparison with Eq. (8). Plate thickness t= 6mm. Distance from the mid plane z=2.5 mm.

It is evident that, due the presence of an inclination angle, β, the out-of-plane shear stress components have both a symmetrical and an antimetrical part. Eq. (8) allows also to determine the shear stress resultant, $\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$.

In order to validate this theoretical result some finite element analyses have been carried out on a uiniaxially loaded wide plate, of finite thickness, containing an inclined elliptical hole with a=1 mm, b=0.1 mm and β=45°. 20 node brick elements have been used to carry out the numerical investigations, with a very fine mesh pattern, in order to get the desired degree of accuracy of the results. Figures 2-4 show a comparison between the theorical prediction, Eq. (8), and the numerical results. It is evident that in all the cases the agreement is very satisfactory.

A THREE-DIMENSIONAL SHOULDERED PLATE

Another interesting example of application is represented by a thick shouldered plate under tension with a zero notch root radius. In this case the out-of-plane shear stresses have a symmetrical part which result in a singular stress distribution to be considered besides the common Williams' mode I and mode II singularities [6, 10].

The distributions of the three stress intensity factors, K_1, K_2 and K_3 along the plate thickness are shown, for two examples, in figures 4 and 5. It is evident that there is a wide zone within the plate thickness where K_1 and K_2 are almost constant. Conversely K_3 has a linear trend (in agreement with the basic hypotheses formulated in section 2) up to a maximum besides which it decrease going towards zero on the free-of-stress surface of the plate.

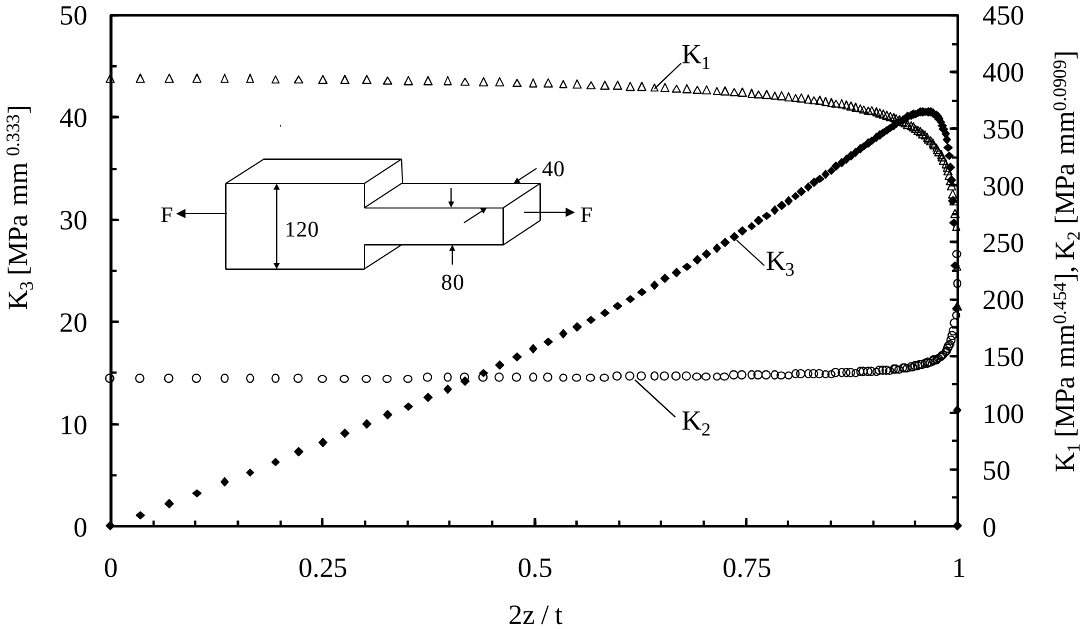

Figure 5. Plot of K_i along the thickness for a shouldered plate with t=40 mm.

The distribution of the notch stress intensity factors allows us to draw some comments on the crcak initiation angle. In agreement with [12] it can be assumed that a crack will emanate from the sharp V-notch tip into the direction which is perpendicular to the stress σ_1 which represents the maximum principals stress acting on a virtual cylindrical surface around the notch tip. Under the hypothesis of plane strain, such a principal stress can be determined as:

$$\begin{aligned}\sigma_1 &= \frac{\sigma_{\theta\theta}+\sigma_{zz}}{2}+\frac{1}{2}\sqrt{(\sigma_{\theta\theta}-\sigma_{zz})^2+4\tau_{z\theta}^2} \\ &= K_1 r^{\lambda_1-1} f_1^{(+)} - K_2 r^{\lambda_2-1} f_2^{(+)} + \sqrt{(K_1 r^{\lambda_1-1} f_1^{(-)} - K_2 r^{\lambda_2-1} f_2^{(-)})^{\,2} + \frac{2}{\pi} K_3^2 r^{2(\lambda_3-1)} \cos^2(\lambda_3)\theta}\end{aligned} \tag{10}$$

where:

$$f_1^{(\pm)}(\theta) = \frac{1}{2\sqrt{2\pi}} \frac{(1+\lambda_1 \pm 4\nu)\cos(1-\lambda_1)\theta + \chi_1(1-\lambda_1)\cos(1+\lambda_1)\theta}{1+\lambda_1+\chi_1(1-\lambda_1)}$$
$$f_2^{(\pm)}(\theta) = \frac{1}{2\sqrt{2\pi}} \frac{(1+\lambda_2 \pm 4\nu)\sin(1-\lambda_2)\theta + \chi_2(1+\lambda_2)\sin(1+\lambda_2)\theta}{1-\lambda_2+\chi_2(1+\lambda_2)} \tag{12}$$

and λ_i [13, 14] are the eigenvalues of the V-notch problem, while χ_i are coefficients dependent on the notch angle [15].

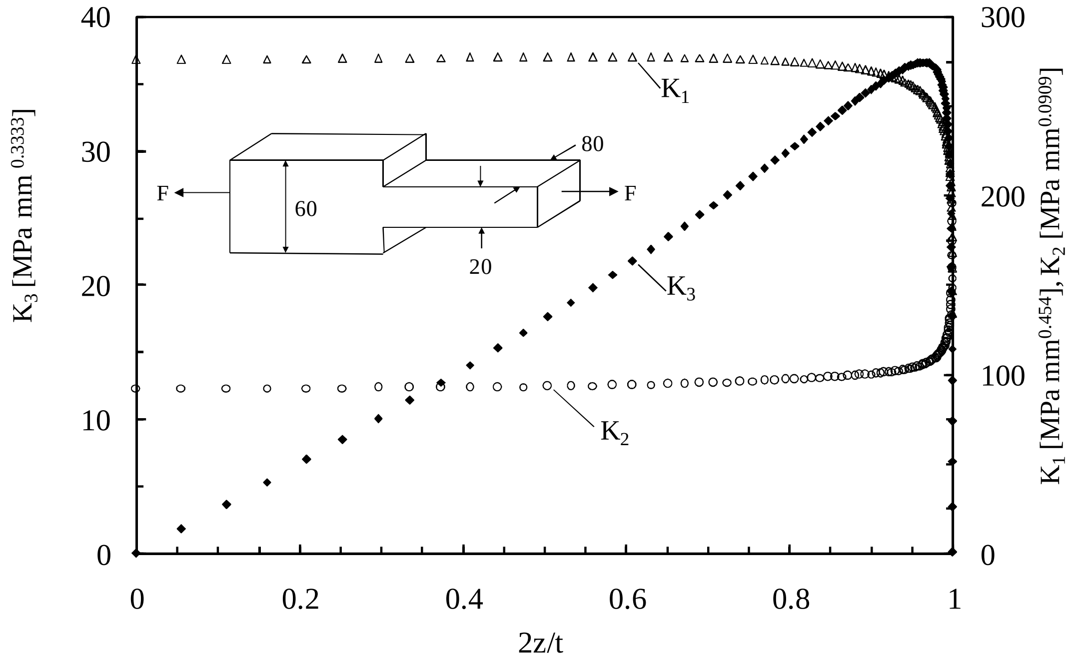

Figure 6. Plot of K_i along the thickness for a shouldered plate with t=80 mm.

Due to the assumption that the crack initiation growth direction is perpendicular to σ_1, the angles of crack initiation, θ_0 and ψ_0, can be determined from the following conditions:

$$\left.\frac{\partial \sigma_1}{\partial \theta}\right|_{\theta=\theta_0} = 0 \qquad \left.\frac{\partial^2 \sigma_1}{\partial \theta^2}\right|_{\theta=\theta_0} < 0 \tag{13}$$

and:

$$\psi_0 = \frac{1}{2}\mathrm{Arg}\left[K_1 r^{\lambda_1 - 1} f_1^{(-)}(\theta_0) - K_2 r^{\lambda_2 - 1} f_2^{(-)}(\theta_0) + i\frac{2K_3}{\sqrt{2\pi}} r^{\lambda_3 - 1}\cos(\lambda_3)\theta_0\right] \tag{14}$$

It is evident from Eq. (13) and (14) that the angles θ^* and ψ^* are independent of the distance from the notch tip only for the crack case (λ_1=λ_2=λ_3=0.5).