Theoretical analysis of a subsurface crack under cyclic surface loading

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ABSTRACT. *A general method for evaluating the fracture mechanics parameters of a subsurface crack parallel to the free surface of a semiplane is presented. A Weight Function (WF) with a matrix like structure is proposed to account for the coupling effects arising in non-symmetrical problems. An estimate of the WF accuracy is presented and a practical application is considered by evaluating the Stress Intensity Factors produced by a point like load travelling on the semiplane free surface. The complete analysis of this problem requires crack closure (either complete or partial) to be taken into account. Indeed different closure conditions are expected for different load positions or inclination. A theoretical method is proposed, that, starting from the matrix like structure of the WF, allows for the calculation of the Green Functions, by which the COD components of a subsurface crack s can be calculated under general loading conditions including those produced by crack closure.*

INTRODUCTION

Subsurface cracks have been experimentally observed in several mechanical components. The early stages of fatigue crack growth parallel to the external surface, under variable loading, are generally characterized by mixed fracture mode and dominates the onset damage mechanisms responsible for many failures such as spalling in rolling contact fatigue or pitting fatigue [1-3]. These phenomena have been studied by several authors in the framework of the fracture mechanics and many analyses have been carried out for determining the fracture mechanics parameters of the crack. The Finite Element (FE) method have been used extensively to evaluate the Stress Intensity Factors (SIFs) under complex loading conditions and to predict the preferred crack paths [4-6]. Unfortunately, FE analyses, since very powerful, are very time consuming, particularly when the crack propagation has to be predicted and a lot of SIF calculations have to be performed under several loading conditions of the cracked body. The Weight Function (WF) method turns out to be particularly efficient for solving this kind of problems. The authors [7-8] have recently presented a WF for a subsurface crack parallel to the external surface in a two-dimensional half space. The WF has been formulated with a matrix like structure to account for the coupling effects arising in non symmetrical problems and built up into a symmetrical and an anti-symmetrical components, as usual for embedded cracks, thus allowing for a straightforward evaluation of the FM parameters under a completely general loading condition.

In the present work an extended formulation of the WF is presented, to encompass a broader range of the ratio between the crack length and its distance from the free surface. The proposed WF is adopted for studying the cyclic SIF histories induced at the crack tips by a point like load travelling on the free surface of the semi-plane. The analysis is carried out for different ratios between crack length and crack distance from the free surface and for different inclinations of the travelling load, in order to simulate different friction conditions. Since the complete analysis of this problem requires to account also for the conditions of complete or partial crack closure, depending on the load position with respect to the crack and on the load inclination with respect to the free surface, a theoretical method is proposed, that, starting from the matrix like structure of the WF, allows for the calculation of the Green Functions (GF) by which the COD components can be calculated under general loading conditions, including the conditions of crack closure.

1 – DEFINITION OF THE PROBLEM

In an elastic semi-plane, an embedded crack having length 2*a* and distance *b* from the semi-plane surface is considered as represented in figure 1. As explained in [7-8] two local reference systems were introduced at the crack tips in order to provide a not ambiguous definition of the FM parameters (in particular regarding the sign of K_{II} for which a general definition is not adopted).

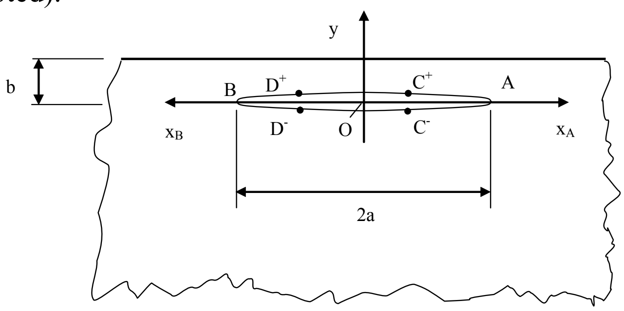

Fig.1 Subsurface crack parallel to the semiplane surface

The dimensionless ratio $r = a/b$ can be assumed as the only geometrical parameter necessary for defining the geometry. Since a matrix like formulation of the WF is necessary to account for the lack of symmetry of the problem, the following general expression was adopted for the WF, where for the tip A and the tip B, the anti-symmetrical contribution has to be summed to (+), or subtracted from (-) the symmetrical contribution, respectively:

$$\begin{pmatrix} K_I(r) \\ K_{II}(r) \end{pmatrix}^{A/B} = \int_0^a \left[\begin{pmatrix} h^{I\sigma}{}_S(x,r) & h^{I\tau}{}_S(x,r) \\ h^{II\sigma}{}_S(x,r) & h^{II\tau}{}_S(x,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma(x) \\ \tau(x) \end{pmatrix}_S \pm \begin{pmatrix} h^{I\sigma}{}_A(x,r) & h^{I\tau}{}_A(x,r) \\ h^{II\sigma}{}_A(x,r) & h^{II\tau}{}_A(x,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma(x) \\ \tau(x) \end{pmatrix}_A \right] \cdot dx \qquad (1)$$

The analytical expressions of the WF were defined by fulfilling their asymptotical properties (either for $r \to 0$ or $x \to a$). In particular, as the *r* ratio approaches zero, the crack can be assimilated to the Griffith crack in an infinite body for which the following uncoupled relationships hold:

$$r \to 0 \qquad h^{I\sigma}{}_S(x,r) = h^{II\tau}{}_A(x,r) = \frac{2}{\sqrt{\pi \cdot a}} \cdot \left(1 - \left(\frac{x}{a}\right)^2\right)^{-1/2}$$

$$h^{I\sigma}{}_A(x,r) = h^{II\tau}{}_S(x,r) = \frac{2}{\sqrt{\pi \cdot a}} \frac{x}{a} \cdot \left(1 - \left(\frac{x}{a}\right)^2\right)^{-1/2} \tag{2}$$

$$h^{II\sigma}{}_S(x,r) = h^{I\tau}{}_S(x,r) = h^{II\sigma}{}_A(x,r) = h^{I\tau}{}_A(x,r) = 0$$

Therefore, for solving the integral equations (1) the following expressions were assumed, by which, adopting the notations $M = I$ or II (mode of facture), μ=σ or τ (nominal stress component) and C=S or A (symmetrical or antisymmetrical), the WF components are expressed either:

$$h_C^{M\mu}(x,r,a) = \frac{2}{\sqrt{\pi \cdot a}} \cdot \sum_{i=0}^{n} c_{C\ i}^{M\mu}(r) \cdot \left[1 - \left(\frac{x}{a}\right)^2\right]^{i-\frac{1}{2}} \tag{3}$$

or

$$h_C^{M\mu}(x,r,a) = \frac{2}{\sqrt{\pi \cdot a}} \cdot \frac{x}{a} \cdot \sum_{i=0}^{n} c_{C\ i}^{M\mu}(r) \cdot \left[1 - \left(\frac{x}{a}\right)^2\right]^{i-\frac{1}{2}} \tag{4}$$

Equation (3) holds for $h_S^{I\sigma}(x,r,a)$, $h_S^{II\sigma}(x,r,a)$, $h_A^{I\tau}(x,r,a)$ and $h_A^{II\tau}(x,r,a)$, whereas eqn. (4) holds for $h_A^{I\sigma}(x,r,a)$, $h_A^{II\sigma}(x,r,a)$, $h_S^{I\tau}(x,r,a)$ and $h_S^{II\tau}(x,r,a)$. In the present analysis the number of terms of the power expansion was limited to 3, so that n=2 in the summations (3) and (4). In order to fulfil the general asymptotic conditions, expressed by (2), the first coefficients corresponding to i=0 for the diagonal and off-diagonal terms of the WF are respectively:

$$c_{S\ i}^{M\mu}(r) = c_{A\ i}^{M\mu}(r) = 1 \text{ for } i = 0 \qquad \text{and } M\mu = I\sigma \text{ or } II\tau \tag{5a}$$

$$c_{S\ i}^{M\mu}(r) = c_{A\ i}^{M\mu}(r) = 0 \text{ for } i = 0 \qquad \text{and } M\mu = II\sigma \text{ or } I\tau \tag{5b}$$

The other coefficients corresponding to i>0 depend on the r ratio. In order represent the dependence over a wide range of r the following function was considered:

$$c_{(S/A)_i}^{M\mu}(r) = \frac{r^{Ai}}{B_i^{\ Ai} + r^{Ai}} \cdot C_i \cdot r^{D_i} + E_i \cdot r^{F_i} \tag{6}$$

The constant parameters A_i, B_i, C_i, D_i, E_i, F_i were determined by least square fitting the SIF values calculated for reference loading conditions at different r ratios. In particular, having assumed n=2, eight linearly independent nominal stress distributions applied on the crack edges were considered, as described in [8]. A data base of SIFs values was built up by performing several FE analyses with r ratio varying over a wide range [0.005, 40], in order to encompass the two limit conditions for the crack: an embedded Griffith crack and a shallow delamination immediately beneath the free surface. The SIFs calculated by using the WF for the reference cases were compared with the original values obtained by the FE analysis. The relative differences between the WF and the FE SIFs for both Modes showed a satisfactory agreement being generally within 1.0% in the whole considered range of r.

These differences are within the estimated range of accuracy for the FE SIF evaluation thus indicating the adequacy of the chosen functions (eqn. 6) for interpolating the FE results in the whole r range.

THE SIFs PRODUCED BY A POINT LIKE TRAVELLING LOAD

With reference to figure 2, a plane body carrying a subsurface crack loaded by a force uniformly distributed through thickness having intensity *P* (force per unit thickness) was considered.

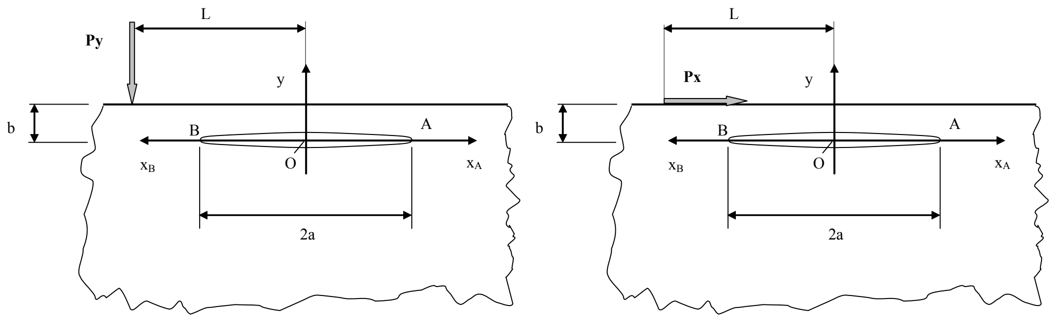

Fig. 2: point like load moving on the surface

The normal (P_y) and tangential (P_x) forces were applied at a parametrical distance L from the crack centre thus reproducing the conditions of a travelling force. Inertia forces were neglected. Material is considered linear elastic and no contact between crack edges is taken into account. Under these assumptions material overlapping is permitted even though without physical meaning.

The nominal stress produced either by P_y or P_x in the uncracked body to be used in eqn. (1) can be deduced by the analytical Boussinesq solution [9] and subdivided in a symmetrical and an anti-symmetrical component. By solving the integral equation in eqn. (1) the K_I and K_{II} values for different relative load positions (L/a) and r were calculated, and two examples of the obtained trends are reported in figure 3, together with results of a FE analysis of the problem. In this case a characteristic SIF values: $K_o = P \cdot \sqrt{\frac{\pi}{a}}$, was adopted to normalize the numerical results. A very good agreement between FE and WF results was found, being the relative difference in the order of 1%. As regards the load normal to the body surface and pointing inwards the semiplane, SIFs histories with K_I always negative are predicted (fig. 3). For this condition it is therefore reasonable to predict an always completely closed crack, subjected only to reversed cycles of K_{II}. On the contrary, for loads tangential to the free surface very complex K_I and K_{II} histories are predicted (fig.4) and conditions of partial or complete crack closure are expected. The problem of a travelling force inclined with respect to the semiplane surface can be evaluated, by neglecting the contact between crack surfaces, as a superimposition of the effects of normal and tangential forces. This analysis is however consistent, from a physical point of view, only if the crack is completely open during the load movement. In the case of partial crack closure the evaluation of the SIFs is a non linear problem, as the boundary conditions are unknown a priori and they depend on the applied load. The knowledge of the closed crack region is necessary for the evaluation of the contact stresses on the crack surfaces, by which the effective SIFs can be determined via WF, as a superimposition of their effects to those of external stresses. The problem can be efficiently faced if the Green Function (GF) giving the COD components for a general loading condition is determined.

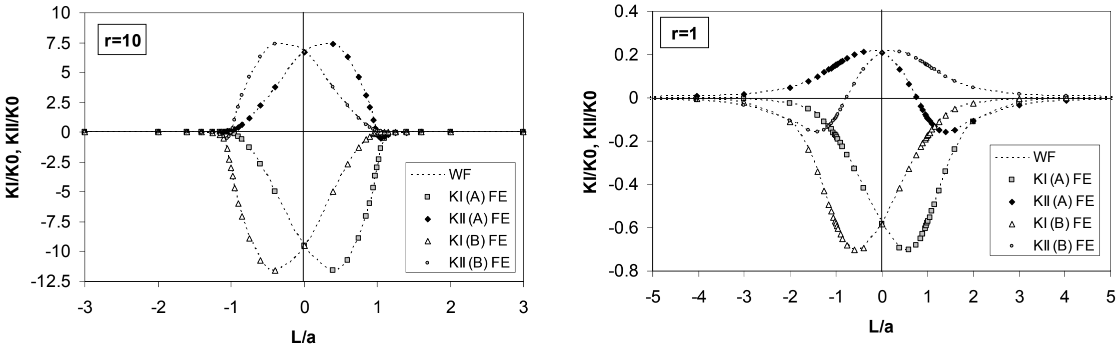

Fig. 3 SIF produced by the normal (compressive) force P_y travelling on the surface for two r ratios

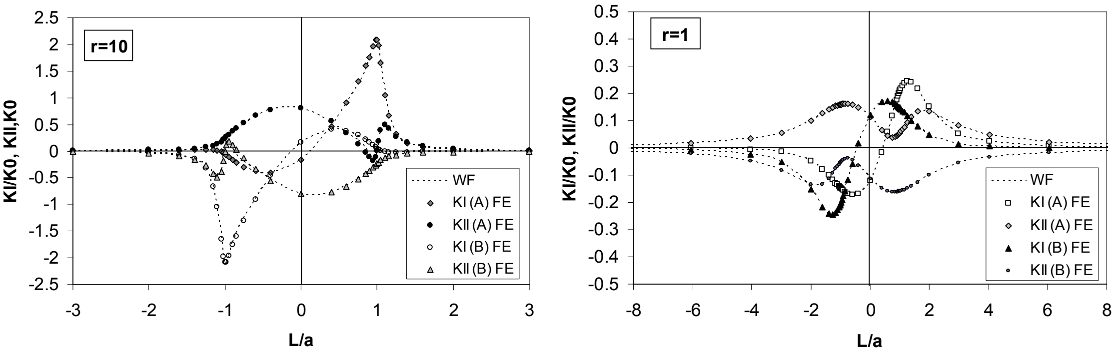

Fig. 4 SIF produced by the tangential force P_x travelling on the surface for two r ratios

EVALUATION OF THE GREEN'S FUNCTION FOR THE COD

On the basis of the WF definition, the COD components u and v, indicating the relative displacements of corresponding points C^+,C^- and D^+,D^- on crack edges (figure 1), in the x and y direction, can be calculated for symmetrical and anti-symmetrical load cases by using the symmetrical and anti-symmetrical components of the WF respectively. Any loading condition can be subdivided into a symmetric and an anti-symmetric components, so that the corresponding COD components can be calculated as the summation of the CODs originated by the two loading conditions. By generalizing the formulation reported in [10] the following equations hold:

$$v^{A/B}(x,a,r) = \frac{2}{H}\int_x^a \left[h_{I\sigma S}(x,b,r)\cdot K_{IS}(b,r) + h_{II\sigma S}(x,b,r)\cdot K_{IIS}(b,r)\right]db$$
$$\pm\frac{2}{H}\int_x^a \left[h_{I\sigma A}(x,b,r)\cdot K_{IA}(b,r) + h_{II\sigma A}(x,b,r)\cdot K_{IIA}(b,r)\right]db \qquad (7a)$$

$$u^{A/B}(x,a,r) = \frac{2}{H}\int_x^a \left[h_{I\tau S}(x,b,r)\cdot K_{IS}(b,r) + h_{II\tau S}(x,b,r)\cdot K_{IIS}(b,r)\right]db$$
$$\pm\frac{2}{H}\int_x^a \left[h_{I\tau A}(x,b,r)\cdot K_{IA}(b,r) + h_{II\tau A}(x,b,r)\cdot K_{IIA}(b,r)\right]db \qquad (7a)$$

where, as for eqn. (1), A/B refers to the right and left crack tips, H is equal to E (Young modulus) for plane stress and $E/(1-\nu^2)$ for plane strain (ν is the Poisson's ratio). The values (K_{IS},K_{IIS}) and (K_{IA},K_{IIA}) are produced respectively by the symmetric and anti-symmetric load cases and, by recalling eqn. (1), can be evaluated as follows:

$$\begin{pmatrix} K_{IS}(b,r) \\ K_{IIS}(b,r) \end{pmatrix}^{A/B} = \int_0^b \left[\begin{pmatrix} h^{I\sigma}{}_S(x',r) & h^{I\tau}{}_S(x',r) \\ h^{II\sigma}{}_S(x',r) & h^{II\tau}{}_S(x',r) \end{pmatrix}\cdot\begin{pmatrix} \sigma(x') \\ \tau(x') \end{pmatrix}_S\right]\cdot dx' \qquad (8a)$$

$$\begin{pmatrix} K_{IA}(b,r) \\ K_{IIA}(b,r) \end{pmatrix}^{A/B} = \int_0^b \left[\begin{pmatrix} h^{I\sigma}{}_A(x',r) & h^{I\tau}{}_A(x',r) \\ h^{II\sigma}{}_A(x',r) & h^{II\tau}{}_A(x',r) \end{pmatrix}\cdot\begin{pmatrix} \sigma(x') \\ \tau(x') \end{pmatrix}_A\right]\cdot dx' \qquad (8b)$$

By indicating the WF with a matrix notation as follows:

$$[W(x,b,r)]_{S/A} = \begin{bmatrix} h_{I\sigma}(x,b,r) & h_{I\tau}(x,b,r) \\ h_{II\sigma}(x,b,r) & h_{II\tau}(x,b,r) \end{bmatrix}_{S/A} \qquad (9)$$

After introducing the expressions of eqns. (8) into eqns. 7, and changing the order of integration [10], the following expression is obtained:

$$\begin{bmatrix} v(x,a) \\ u(x,a) \end{bmatrix}^{A/B} = \frac{2}{H}\cdot\int_0^a\left[\int_{\max(x,x')}^a [W(x,b,r)]_S^T\cdot[W(x',b,r)]_S\,db\right]\cdot\begin{pmatrix} \sigma(x') \\ \tau(x') \end{pmatrix}_S\cdot dx'$$
$$\pm\frac{2}{H}\cdot\int_0^a\left[\int_{\max(x,x')}^a [W(x,b,r)]_A^T\cdot[W(x',b,r)]_A\,db\right]\cdot\begin{pmatrix} \sigma(x') \\ \tau(x') \end{pmatrix}_A\cdot dx' \qquad (10)$$

Where $[\ ]^T$ is the transpose matrix. By introducing the following 2x2 matrices:

$$G(x,x',r)_S = \begin{bmatrix} G_{v\sigma}(x,x',r) & G_{v\tau}(x,x',r) \\ G_{u\sigma}(x,x',r) & G_{u\tau}(x,x',r) \end{bmatrix}_S = \int_{\max(x,x')}^a [W(x,b,r)]_S^T\cdot[W(x',b,r)_S]db \qquad (11a)$$

$$G(x,x',r)_A = \begin{bmatrix} G_{v\sigma}(x,x',r) & G_{v\tau}(x,x',r) \\ G_{u\sigma}(x,x',r) & G_{u\tau}(x,x',r) \end{bmatrix}_A = \int_{\max(x,x')}^a [W(x,b,r)]_A^T\cdot[W(x',b,r)_A]db \qquad (11b)$$

Equation (10) can be rewritten as:

$$\begin{bmatrix} v(x,a,r) \\ u(x,a,r) \end{bmatrix}^{A/B} = \frac{2}{H}\cdot\left\{\int_0^a [G(x,x',r)_S]\cdot\begin{pmatrix} \sigma(x') \\ \tau(x') \end{pmatrix}_S dx' \pm \int_0^a [G(x,x',r)_A]\cdot\begin{pmatrix} \sigma(x') \\ \tau(x') \end{pmatrix}_A\cdot dx'\right\} \qquad (12)$$

Thus demonstrating that $[G(x,x',r)]$ represents the Green's functions (GF) as it relates the load applied on the crack faces to the local displacement. Considering the power law expansion proposed for the WF (eqns. 3-4) and taking into account of the combination between the different WF terms in the matrix product of eqn. (11), the GF can be obtained by solving 3·(n+1)^2 integrals of one variable. Having assumed n=2 for the WF, the number of integrals to be evaluated is 27. However, by considering that due to the asymptotic properties of the WF, some of the WF coefficients are zero, the evaluation of only 22 of the 27 integrals is necessary. In particular, the following three classes of integrals have to be determined:

$$I_{1\,kj}(x,x') = \int_{\max(x,x')}^{a} \frac{1}{b}\cdot\left(1-\frac{x}{b}\right)^{\left(k-\frac{1}{2}\right)}\cdot\left(1-\frac{x'}{b}\right)^{\left(j-\frac{1}{2}\right)} db \qquad k,j,=0,n \tag{13a}$$

$$I_{2\,kj}(x,x') = \int_{\max(x,x')}^{a} \frac{1}{b^2}\cdot\left(1-\frac{x}{b}\right)^{\left(k-\frac{1}{2}\right)}\cdot\left(1-\frac{x'}{b}\right)^{\left(j-\frac{1}{2}\right)} db \qquad k,j,=1,n \tag{13b}$$

$$I_{3\,kj}(x,x') = \int_{\max(x,x')}^{a} \frac{1}{b^3}\cdot\left(1-\frac{x}{b}\right)^{\left(k-\frac{1}{2}\right)}\cdot\left(1-\frac{x'}{b}\right)^{\left(j-\frac{1}{2}\right)} db \qquad k,j,=0,n \tag{13c}$$

Integrals of type I_1 and I_3 were analytically solved by using a recursive strategy, whereas I_2 integrals were reduced to the solution of elliptic integrals. By knowing the analytical GFs, the COD components can be determined at any location of the crack for any loading condition by eqn. (12) when the nominal stress components $\sigma(x)$ and $\tau(x)$ are known on the crack edge. The problem of crack closure can therefore be faced in an efficient way by using the procedure explained in [11] for non symmetrical problems, that accounts for couplings effects, active between normal stresses and tangential displacement and between normal displacement and tangential stress. An example of the COD v component calculated by the proposed GF approach and by the FE modelling is shown in figure 5, where the conditions of partial crack closure produced by a load P inclined by an angle of 45° with respect to the normal at the free surface and pointing inward the semiplane are plotted.

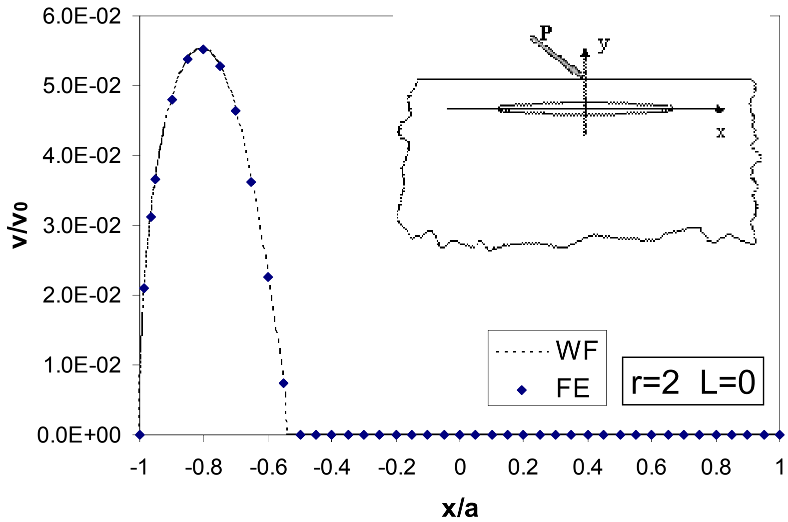

Fig. 5: v components of COD calculated by the GFs method and by the FE analysis. The v values are normalised by the following characteristic parameter $v_0=a\cdot\sigma_o/E$ ($\sigma_0=2P/\pi a$)

CONCLUSION

An analytical matrix like formulation of the WF for a subsurface crack parallel to the free surface of a semi-infinite body was proposed. The results of a Finite Element analysis carried out for several independent loading cases were used for evaluating the numerical coefficients of the WF. The obtained WF reproduced the FE results with a good accuracy.

The problem of a load travelling on the free surface of a semi-infinite body carrying a subsurface crack was then studied. The conditions of partial crack closure were initially disregarded and under these hypothesis the SIFs calculated by the WF were in very good agreement with those determined by FE analysis, thus showing the usefulness of the WF. A theoretical analysis is then proposed to account for the effects on SIFs exerted by the contact phenomena between crack faces in the case of partial crack closure. Starting from the matrix like formulation of the WF and considering the theoretical definition of the COD, the Green's Function for the COD was proposed in the form of symbolic integrals.

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