Microcracked Elastic Media

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ABSTRACT. *In the present paper we propose a new deterministic macroscopic theoretical approach to the strength analysis of cracked elastic bodies. By our numerical method we try to model evaluation of the strength by two different methods: (1) by gradually increasing of the applied exterior load, up to its critical value; (2) by ultrasonic through-transmission evaluation. The problem is studied under assumptions of a scalar theory that is related to the case of anti-plane problem. We give respective graphs on results of computations, which aim to establish some correlation relations between destructive and non-destructive testing.*

INTRODUCTION

Standard non-destructive techniques like ultrasonic evaluation may be applied to the problem of strength analysis of cracked elastic bodies, composed by materials such as concrete - Krautkramer [1] -. Application of such classical methods face significant obstacles when operating with strongly cracked media like concretes, granular materials, and so on. Thus it is not surprising that in the fundamental monograph on ultrasonic methods of evaluation only a few pages are devoted to such a non-standard medium as concrete.

A large amount of papers have been devoted to stochastic analysis of the microcracked elastic media - Ishimaru [2], Liu *et al* [3], and a more advanced approach is currently based on numerical Monte-Carlo simulation of microstructures as regular domain occupied by a body with a lot of deterministic defects. That similar approach was applied, but only experimentally, in recent papers of Tourin *et al.* [4]. Some interesting numerical results for media with microstructure were obtained in Fellinger *et al*. [5], and a very hard numerical investigation was undertaken in Eberhard *et al*.[6].

In the present paper, we propose a new approach to the deterministic strength analysis of the microcracked elastic solids. On the first hand, we further develop the method proposed in our previous work [7] on calculation of delay of the "time-of-flight" characteristics and respective decrease of the through-transmission ultrasonic velocity connected with this delay, an approach founded on the Ray Tracing method. On the other hand, we try to link the computed velocity with the value of the ultimate applied load which leads to material crushing with any particular cracked geometry, possessing known wave velocity, calculated in advance by the Ray Method.

It should be noted that here we deal only with a scalar model, when only one of two existing types of waves may propagate in the elastic medium. However, the proposed model may also be connected with the so-called “anti-plane” (or, shear, SH-) two-dimensional problem Achenbach [8].

SOME SIMPLE GEOMETRIES: STRENGTH ANALYSIS

The equations of equilibrium are described by the following system of partial differential equations and the components of the stress tensor are defined by the Hook’s relations, where μ and λ are classical elastic constants, and ū denotes the displacement vector, for the linear isotropic homogeneous elastic medium.

Let us first consider the two-dimensional plane *(y,z)*-strain, when the components of the displacement vector are $\bar{u}=\{0,\ u_y(y,z),\ u_z(y,z)\}$, and the non-trivial components of the stress tensor are $\sigma_{22} = \sigma_{yy}(y,z)$, $\sigma_{33} = \sigma_{zz}(y,z)$, $\sigma_{23} = \tau_{yz}(y,z)$. The general equations read in two-dimensional case as follows (1), with positions (2):

$$\begin{cases} \dfrac{\partial^2 u_y}{\partial y^2} + c^2 \dfrac{\partial^2 u_y}{\partial z^2} + \left(1-c^2\right) \dfrac{\partial^2 u_z}{\partial y \partial z} = 0 \\ \dfrac{\partial^2 u_z}{\partial z^2} + c^2 \dfrac{\partial^2 u_z}{\partial y^2} + \left(1-c^2\right) \dfrac{\partial^2 u_y}{\partial y \partial z} = 0, \end{cases} \tag{1}$$

$$\begin{gathered} \frac{\sigma_{yy}}{\lambda+2\mu} = \frac{\partial u_y}{\partial y} + \left(1-2c^2\right)\frac{\partial u_z}{\partial z}, \quad \frac{\sigma_{zz}}{\lambda+2\mu} = \left(1-2c^2\right)\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}, \\ \frac{\tau_{yz}}{\mu} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad c^2 = \frac{\mu}{\lambda+2\mu} = \frac{c_s^2}{c_l^2} < 1, \quad c_s^2 = \frac{\mu}{\rho}, \quad c_l^2 = \frac{\lambda+2\mu}{\rho}, \end{gathered} \tag{2}$$

Here ρ is the mass density, and c expresses the ratio of the transverse and longitudinal wave speeds [8]. The general equations should be completed by the following boundary conditions (3), if we start from the very simple problem about a shear of an infinite layer by a constant shear load τ_0.

$$y=0:\ u_z=0;\quad y=h:\ \tau_{yz}=\tau_0;\quad |z|<\infty. \tag{3}$$

The exact solution to equations (1)--(3) is (4); the solution depends only on the coordinate *y*, so the problem is in fact one-dimensional.

$$u_y \equiv 0, \quad u_z = \frac{\tau_0}{\mu} y, \quad \sigma_{yy} = \sigma_{zz} \equiv 0, \quad \tau_{yz} \equiv \tau_0 . \tag{4}$$

It should be noted that the well known classical result of the theory of elasticity, which establishes that tangential stress vanishes on the principal elementary areas of normal stress and vice versa [9]. The basic conclusion, which can be extracted from consideration of this very simple problem is that typically, when a specimen is loaded by a shear exterior force, one of the principal values of the tangential stress is achieved on the area elements parallel to the line of the applied load.

Let us pass to a slightly more complex two-dimensional problem about a rectangular specimen with the sizes a×h, loaded again by a constant uniform shear tangential stress τ_0.

This problem is equivalent to the previous one if the length of the rectangle would be a = ∞. Really, in the considered anti-plane problem the components of the displacement vector are $\bar{u} = \{0,0,w(x,y)\}$, with the only non-trivial components of the stress tensor being (5). For all that equations of equilibrium reduce here to the single scalar Laplace equation (6) with boundary condirions (7),

$$\sigma_{13} = \tau_{xz}(x,y) = \mu \frac{\partial w(x,y)}{\partial x}, \qquad \sigma_{23} = \tau_{yz}(x,y) = \mu \frac{\partial w(x,y)}{\partial y}. \tag{5}$$

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = 0, \tag{6}$$

$$\begin{array}{ll} y=0: \quad w=0, \ (0<x<a); & y=h: \quad \tau_{yz} = \tau_0, \ (0<x<a); \\ x=0: \quad \tau_{xz}=0, \ (0<y<h); & x=a: \quad \tau_{xz} = 0, \ (0<y<h); \end{array} \tag{7}$$

that corresponds to the case when the lower face is perfectly coupled with an absolutely rigid foundation, the upper face is loaded by the tangential stress τ_0 applied in direction of the *z-axis*, the left and the right faces are free of tangential stress. The so posed problem seems to be two-dimensional (but really one-dimensional), however one can easily check that its solution is given again by (6): $w(x,y)=(\tau_0/\mu)\, y$, which automatically satisfy the partial differential equation (6) and the required boundary conditions (7).

CRACKED PROBLEMS

We now consider a single horizontal crack located somewhere in the rectangle (Fig.1).

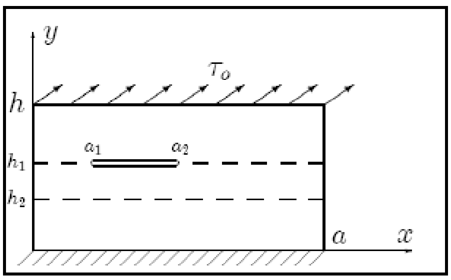

Figure1

Let us represent the full solution of this problem as a sum of the one related to a problem about tangential stress τ_O applied to the boundary surface of the uncracked body and a "perturbed" solution, which is caused just by presence of the crack. If in the full problem we assume that the faces of the cracks are free of load, then for the new unknown function w(x,y), we have the same Laplace equation (6) with the following boundary conditions since the applied load $\tau_{yz}=\tau_O$ does not generate any normal stress τ_{xz} on the lateral boundaries (x=0, and x=a) of the rectangular domain.

We should recall relations between the stresses and the function w(x,y) given by (8). Here "Δ" designates the difference between the values of displacement Δw= w(x, h_1+0) - w(x, h_1-0):

$$\begin{aligned} &y=0:\ \ w=0,\ \ (0<x<a); \qquad y=h:\ \ \tau_{yz}=\tau_0,\ \ (0<x<a);\\ &x=0:\ \ \tau_{xz}=0,\ \ (0<y<h); \qquad x=a:\ \ \tau_{xz}=0,\ \ (0<y<h);\\ &y=h_1:\begin{cases}\Delta w=0,\ (0<x<a_1)\cup(a_2<x<a);\\ \tau_{yz}=-\tau_0,\ (a_1<x<a_2);\end{cases}\end{aligned} \tag{8}$$

In order to construct numerical solution, we apply the Boundary Element Method (BEM), in its form which is usually called "the displacement discontinuity method" [10]; the technique since it is described in detail in the works [11] [12]. The last two integrals results (in this work only one: $\tau_{xy}\ (x,y)\ /\mu)$

$$\frac{\tau_{yz}(x,y)}{\mu}=-\frac{\varepsilon g_i}{4\pi}\frac{y^2+(\varepsilon/2)^2-(x_i-x)^2}{[\,y^2+(x_i-x+\varepsilon/2)^2\,][\,y^2+(x_i-x-\varepsilon/2)^2\,]}, \tag{9}$$

We would like to construct solution of the problem (boundary value Eq.8) by representing the total length of the crack as a union of small sub-intervals of the length ε, with the central points (x_i, 0), I =1,2,...,I.

We prefer to follow an approximate approach [11] founded on the physical hypothesis that, when studying stress concentration in a vicinity of the crack, the influence of the regular boundary conditions on the outer boundaries of the domain to this stress concentration is very weak. Such an approach will appear very efficient in the next sections, when the number of cracks may reach as high value as several hundred.

Under conditions of the discussed hypothesis the full solution to the single-crack problem can be obtained taking into account linearity of the problem. Really, due to the linearity, the full tangential stress over the crack surface at the point (x_i, 0) is a superposition of contributions from all elementary solutions:

$$\frac{\tau_{yz}(x_i,0)}{\mu}==\frac{\varepsilon}{4\pi}\sum_{j-i}^{I}\frac{g_j}{[\,(x_j-x_i)^2-(\varepsilon/2)^2\,]}, \tag{10}$$

that is obtained if we note that y = 0 over the crack faces.

Now complete formulation of the problem requires to satisfy the only remaining boundary condition given by the last line of (8), that reduces the problem to the linear algebraic system regarding the unknown quantities g_i:

$$\frac{\varepsilon}{4\pi}\sum_{j=1}^{I}\frac{g_j}{\left[\left(x_j - x_i\right)^2 - \left(\varepsilon/2\right)^2\right]} = -\frac{\tau_0}{\mu}, \quad i = 1, 2, \dots, I. \tag{11}$$

As soon as the system (11) is solved, the both stresses can be calculated at arbitrary point of the medium as a superposition of the contributions given by (9):

$$\frac{\tau_{yz}(x,y)}{\mu} = -\frac{\varepsilon}{4\pi}\sum_{i=1}^{I}\frac{\left[y^2 + (\varepsilon/2)^2 - \left(x_j - x\right)^2\right]g_j}{\left[y^2 + \left(x_j - x + \varepsilon/2\right)^2\right]\left[y^2 + \left(x_j - x - \varepsilon/2\right)^2\right]}, \tag{12}$$

Fig. 2 demonstrates the stress concentration along two different cross-sections, in the case when the crack is located at the center of the rectangular domain with a = 4, and the length of the crack is $a_2 - a_1 = 1$. An interesting conclusion can be extracted from consideration of the curve 1. Since the part of the domain between the line $y = h_1$ of the crack and the applied load τ_0 distributed over the line y = h is under condition of equilibrium and since the total normal force acting to this sub-domain over the interval $y = h_1$, $a_1<x<a_2$ is equal to zero, this immediately implies that over some other sub-intervals of the crack's line the stress must be higher than the applied load τ_0. In fact, our graph shows that this is so indeed, and the value of the stress gradually decreases with distance always remaining higher than the unit value: $\tau_{yz} > 1$. An absolutely different situation takes place over the line $y = h_2 < h_1$. The principal conclusion from this consideration is that this is not so easy to *a priori* predict, in which sub-domains of the considered sample the stress increases and in which sub-domains it decreases.

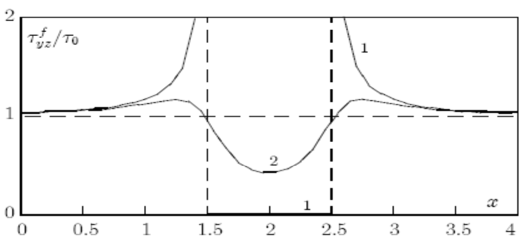

Figure 2.

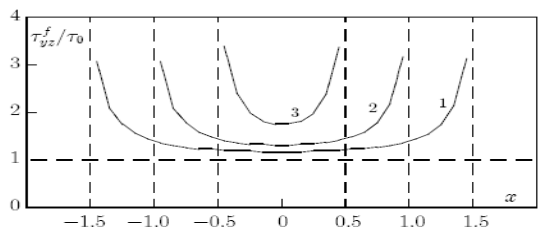

Figure 3

Now let us pass to the study of a pair of equal collinear cracks situated close to each other. In this case the problem can be reduced again to the same linear algebraic system (11), where the boundary elements should be distributed over the both cracks. If we wish to keep the same length ε of each small "displacement discontinuity" element, then the total number of nodes increases in two times compared with the previous problem. Fig. 3 shows distribution of the stress on the interval between the cracks in the case when the length of both of them is 2.

To investigate the strength properties of the microcracked elastic medium we simulate the problem by a deterministic many-crack geometry (Fig 4-5). We assume that the specimen is under the same conditions of loading by a uniformly distributed tangential stress τ_0 as in the previous simple problems. Let us give the governing equations to apply the developed BEM method in this case. Let us consider the pair of elementary displacement discontinuities which are described by the data $\{x_i,y_i,n_i,t_i\}$ and $\{x_j,y_j,n_j,t_j\}$ in the Cartesian system (x,y). Here the first two quantities designate coordinates of the central point, and the quantities n, t are directed along the elementary crack and in direction to its normal. If we have the total number of elements to be equal I, then complete contribution of all elements to τ_i is a superposition of elementary expressions [11] that can be written symbolically as (13a), for any elements *i*.

Let us come back to the strength analysis of the microcracked body where we assume that the same uniformly distributed tangential stress τ_0 is applied to the upper boundary of the considered cracked rectangular specimen. Representing again the full solution, i.e. as a sum of solutions related to the uncracked body and a perturbed one, we can conclude that the latter is described by free outer boundaries of the domain and cracks' faces loaded by some tangential stress that is evidently equal to $\tau_i = -\tau_0 n_{iy}$ on the surface of the i-th crack. In this case the full problem, with the use of the basic relation, is reduced to the linear algebraic system (13b).

$$\text{a)}\quad \tau_i = \sum_{j=1}^{I} K_{ij}\, g_j, \qquad \text{b)}\quad \sum_{j=1}^{I} K_{ij}\, g_j = -\tau_0 n_{iy}, \quad i = 1, 2, \ldots, I. \qquad (13)$$

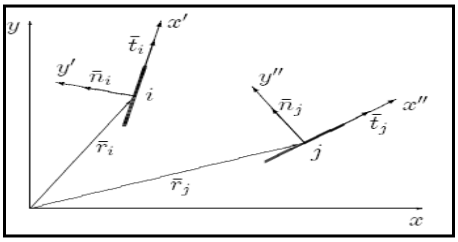

Figure 4.

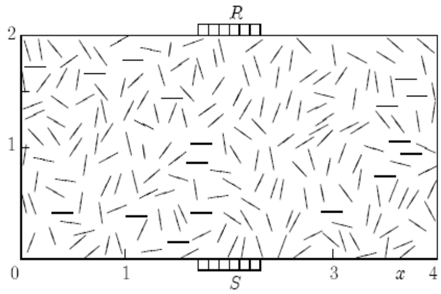

Figure 5

CONCLUSIONS BY NUMERICAL SIMULATION AND COMPARISON WITH ULTRASONIC EVALUATION

In our computations we considered a rectangular specimen of the size 4 cm×2 cm possessing a large number N of cracks of equal length 2 mm randomly distributed in the specimen (see Fig. 7 where N=200 cracks are presented). We varied the number of cracks N from N=0 up to N = 450. For each crack we chose M = 10 boundary elements, then the total number of elements could achieve the value I = 4500, and the maximum dimension of linear algebraic system we solved was 4500 × 4500. In order to connect the developed BEM technique with the strength analysis of this elastic body, we covered the total domain by a mesh with horizontal and vertical lines, with the step 0.5 mm, so that no crack tip coincides with any mesh node, to avoid infinite stresses (analysis of computations, see [12]). Obviously, in order to achieve destruction of the material under such conditions, the exterior load should be amplified in τ_*/R_c times.

(Stochastic) results of such calculations, which were carried out by the proposed deterministic numerical method, versus number of cracks, are shown by black disks in Fig. 6. Curve 1 represents the same pattern as a certain smoothed curve.

Since our principal goal is to establish correlation between the strength and the change of ultrasonic velocity with the growth of the number of cracks, we also performed computations on simulation of the through-transmission ultrasonic technique. The transmitter is designated by the letter "S", and the receiver -- by the letter "R" (diameter 6 mm, with frequency *f* = 5MHz). We applied the Ray Tracing method which was proposed and discussed in detail in our previous work [7]. In that work we introduced the concept of the so-called mean “time delay” and of the "seeming" wave velocity. If in the uncracked medium the wave velocity is v_0 then time delay gives for the velocity of the cracked medium the following value (14a) where L=20 mm is the distance between the transmitter and the receiver. The values of the quantity v_1 versus number of cracks calculated deterministically are marked in Fig. 6 by a lower set of open circles, and the line 2 shows a respective smoothed curve. Another possible way to treat change of the velocity is the natural time delay $(T_s)_2$ of the leading front of the impulse, arising due to the natural fact that with increase of the number of cracks for a strongly cracked medium there is no "free path" (without re-reflections) for ultrasonic rays. The velocity in this case also can be calculated analogously to (14a), with (14b).

$$\text{a)} \quad v_1 = \frac{v_0}{1 + v_0 \left(T_s \right)_1 / L}, \qquad \text{b)} \quad v_2 = \frac{v_0}{1 + v_0 \left(T_s \right)_2 / L}. \qquad (14)$$

Results of calculations are reflected by the upper open circles in Fig. 7, and in a smoothed form -- by the line 3. Since the principal goal of the present work is to establish correlation between the strength properties and the wave speed, we re-draw Fig. 6 in the form where wave velocity is shown versus the applied force, in Fig. 7.

We thus can conclude:

- When comparing the two different treatments of the time delay and related decrease of the through-transmitted velocity, we should agree that the classical delay of the leading front of signal gives more reliable results, when analyzing the strength-velocity graphs (i.e. operation with the the quantity v_2/v_0 is more reliable than with v_1/v_0). In experiments the wave velocity abates not more than twice for extremely cracked media that is in a perfect agreement with the line 3. At the same time, line 2 demonstrates decrease of the wave velocity almost to zero value, that is physically not realistic and does not confirm by experiments.
- Physically, the initial stage of the loading the wave velocity weakly depends upon the number of cracks, being approximately constant. In fact this is because for small number of cracks there are always some ultrasonic rays which, emitting from the transmitter, reach the receiver in the same manner if there would be no crack, and so the delay is equal to zero that involves no change in the through-transmitted velocity.
- In contrast with the feature marked in 2, the strength of the material significantly decrease with increasing of number of cracks even when there are a few cracks. This evident observation leads to the phenomenon describing weak dependence of v versus

N for small N. Perhaps, just this fact yielded so heated debates in literature and even led to a justified point of view that the value of the wave velocity cannot predict the strength of the material. In fact, this is so indeed in the initial stage of the loading.

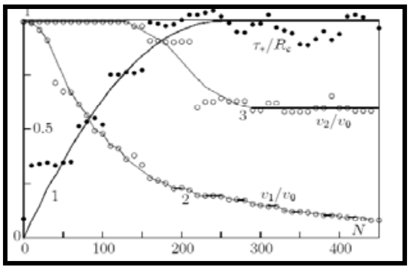

FIG. 6

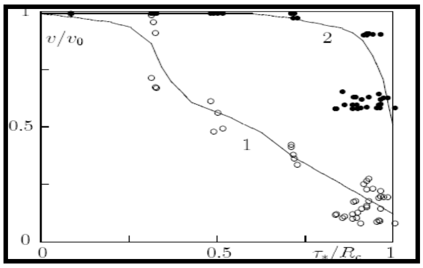

FIG. 7

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