Modelling of cracks by the Strong Discontinuities Approach

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ABSTRACT. The paper concerns a general variational derivation of all the state equations of a continuum medium with dissipative interfaces in the context of the Strong Discontinuities Approach. The kinematics is based on an enhanced enrichment displacement field, that satisfies a priori the boundary conditions and the continuity requirements. The variational formulation is established for a damaging material under general constitutive hypotheses. The numerical implementation of the strong discontinuity is developed within the Elements with Embedded Discontinuities method. Some considerations about the numerical implementation and the equilibrium condition at the interface are presented.

KEYWORDS. Interfaces; Strong Discontinuities; Variational Formulation; Elements with Embedded Discontinuities

INTRODUCTION

One of the most important causes that can produce structural failure is material cracking evolving into collapse mechanisms. The simulation of the behaviour of structures and components with discontinuities has become an important research topic. The number of experimental and analytical studies has led to the conclusion that the cracking process in continuum media is preceded by a strain-localization phenomenon, characterized by the formation of strain localization zones in which damage and other inelastic effects accumulate, gradually turning into macroscopically observable discontinuities or cracks.

These phenomena can be effectively described by means of models that incorporate the kinematics of strong discontinuities obtained by an enrichment of the displacement field with a discontinuous term. Such models are based on the Strong Discontinuities Approach originally introduced in [1,2]. Different numerical implementation of the method in the Finite Element Method have been proposed in the literature, that can be collected into the two classes of Elements with Embedded Discontinuities [3] and eXtended Finite Element Method [4]. The main difference between the methods is in the approximation of the displacement field, requiring additional nodal degrees of freedom in the XFEM, while in EED the enrichment is element-wise and the additional degree of freedom can be condensed, so that the dimension of the discretized problem does not increase. It can be shown that under special choices of the enrichment variables the two approaches fully coincide.

In the paper the equations ruling the problem of the enriched continuum are obtained from a generalized four-field Hu-Washizu variational principle. The equilibrium, compatibility and constitutive equations constitute the Euler-Lagrange stationarity conditions of the functional. Because of the topology of the problem, an additional equilibrium condition at the interface is obtained. It guarantees for the continuity of the stress across the interface. This equation is well known as the orthogonality condition between the stress and the enhanced deformation field. The possible choices of the approximations introduced in the discretized principle give raise to the different implementations of the method. Usually, in order to satisfy the interface equilibrium condition, a Petrov-Galerkin approximation is introduced. However the resulting stiffness matrix is non symmetric.

Because of the fact that traditional discretized enrichment function usually does not satisfy at all points on the boundary the imposed constraint, in the paper an enhancement of strong discontinuity kinematics is also proposed.

THE MODEL

The paper presents a variational formulation of the equilibrium problem for a continuum Ω characterized by an elastic-plastic damaging behavior, in which the growth of interfaces *S* takes places. The presence of pre-assigned physical interfaces is also considered. The growth or the activation of an interface is ruled by a specific activation function, based on a cohesive fracture like criterion. In the general formulation the medium and the interface are ruled by different constitutive equations, defined by distinct free energy and dissipation functionals. The strong form of the equilibrium and compatibility conditions is presented, with special attention to the equilibrium conditions at the interfaces and to the satisfaction of the Dirichelet boundary conditions. Similarities and differences with respect to other formulations in the literature are highlighted. The obtained weak formulation allows an effective numerical implementation of the interface model, able to predict both the occurrence of the discontinuity and its direction; no tracking algorithm is introduced.

Kinematics

In [6] the classical kinematics of the Strong Discontinuities Approach is used to develop a structural model for the simulation of growth and propagation of interface inside a continuum medium.
Let *S* be an interface embedded within a continuous body occupying the domain Ω. Let Ω_φ be a subset of Ω containing the discontinuity and such that S divides Ω_φ into two subdomains, Ω_φ^+ and Ω_φ^- respectively. The normal $\mathbf{n}$ is oriented toward the interior of Ω_φ^+. The boundary of Ω_φ is divided by the surface *S* in two parts. According to the position of the interface, a portion of the boundary of Ω_φ can belong to the boundary of Ω. The geometry of the problem is depicted in Fig. 1.

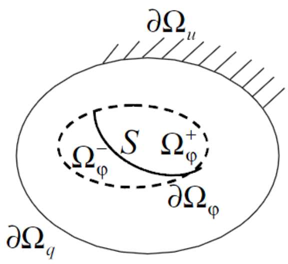

Figure 1: Domain Ω and discontinuity surface *S*.

Across the interface *S* the displacement field is discontinuous and the jump is denoted by $[[\mathbf{u}]]_S$. The displacement field is usually given by the sum of a continuous differentiable function $\bar{\mathbf{u}}$ defined in Ω plus the function $\tilde{\mathbf{u}}$, continuous and differentiable everywhere except on the interface S, so that the kinematics of the Strong Discontinuities is ruled by the following equations:

$$\mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t) + \tilde{\mathbf{u}}(\mathbf{x},t)$$

$$\begin{aligned} \tilde{\mathbf{u}}^+(\mathbf{x},t) &- \tilde{\mathbf{u}}^-(\mathbf{x},t) = [\![\mathbf{u}]\!]_S(\mathbf{x},t) \quad \forall \mathbf{x} \in S \\ \tilde{\mathbf{u}}(\mathbf{x},t) &= 0 \quad \text{on} \quad \partial\Omega_\varphi \end{aligned}$$

$$\tilde{\mathbf{u}}(\mathbf{x},t) = \bar{M}_S(\mathbf{x})[\![\mathbf{u}]\!](\mathbf{x},t)$$

The enhanced enrichment function $\overline{M}_S = M_S(\mathbf{x})N_S(\mathbf{x}) = (H_S(\mathbf{x}) - \varphi(\mathbf{x}))N_S(\mathbf{x})$ vanishes on the boundary of region Ω_φ, also when it falls on the restrained boundary of Ω, and presents an unit jump across *S*. H_S is the Heaviside

function related to the surface *S* and defined on the domain Ω_φ and function φ is continuous, differentiable, defined on Ω_φ and such that it assumes the unit value on $\partial\Omega^+_\varphi$. Function $[\![\mathbf{u}]\!]$ is a regular function on Ω, such that $[\![\mathbf{u}]\!] = [\![\mathbf{u}]\!]_S$ on *S*.
The F.E. form of the discretized displacement field is given by [6]

$$\mathbf{u}(\mathbf{x}) = \sum_{i \in N_m} N_i(\mathbf{x})\hat{\mathbf{u}}_i + [\![\mathbf{u}]\!] \left[H_S(\mathbf{x}) - \sum_{j \in S_m{}^+} N_j(\mathbf{x}) \right] \bar{N}_S(\mathbf{x})$$

where N_i are the shape functions defining the approximation of the displacement field, $\hat{\mathbf{u}}_i$ are nodal degrees of freedom, and the first sum is extended over the set of all the nodes of the finite element mesh, while S^+_m is the set of the enriched nodes belonging to Ω^+_φ. The domain Ω_φ coincides with the band of elements that are cut by the discontinuity and the interpolation of function $[\![\mathbf{u}]\!]$ is made element-wise. In this way, the nodal degrees of freedom coincide with the nodal displacements and the jump function can be treated as an internal variable. Function $[\![\mathbf{u}]\!]$ is supposed to be constant inside the element. Function $\overline{N}_S$ plays the role of annihilating the enriched component of the displacement field on the restrained portion of the boundary. The presence of function $\overline{M}_S$ in discretizing the enrichment field guarantees the satisfaction of boundary conditions in all the possible configuration of the interface inside the bulk, as schematically illustrated in Fig. 2.

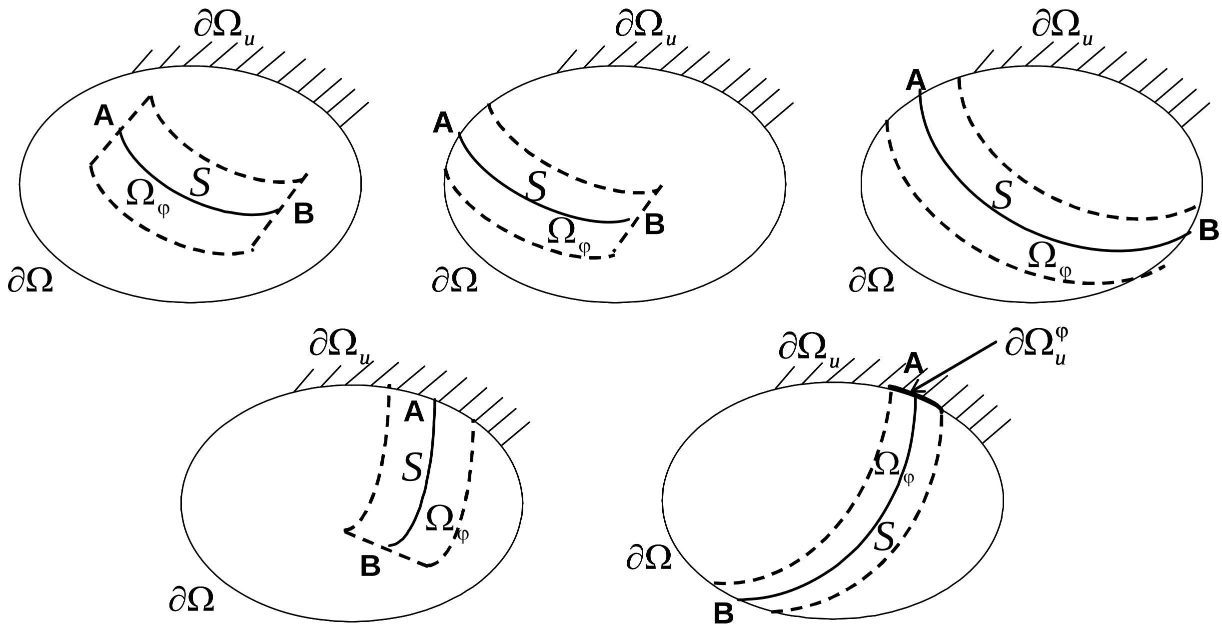

Figure 2: Possible configuration of the interface S inside the domain. S intercepts or not the restrained boundary.

Weak formulation
The starting point of the weak formulation is the mixed multi-fields Hu-Washizu functional

$$\Pi^{HW}(\sigma, \chi, \zeta, \mathbf{t}_S, \chi_S, \zeta_{S,} \hat{\mathbf{u}}, \tilde{\mathbf{u}}, \varepsilon, \alpha_e, \omega_e, \dot{\varepsilon}_p, \dot{\alpha}_p, \dot{\omega}_p, [\![\dot{\mathbf{u}}]\!]_{S_p}, \dot{\alpha}_{S_p}, \dot{\omega}_{s_p})$$

in which the enhanced displacement field $\tilde{\mathbf{u}}$, the internal plastic variables α, α_S and their conjugated internal forces χ, χ_S in the continuum and on the interfaces respectively are introduced. Damage is described by the kinematic and dual variables (ω, ζ), (ω_S, ζ_S) respectively. The displacement jump is denoted by $[\![\mathbf{u}]\!]$ and the conjugated traction on the discontinuity surface S is $\mathbf{t}_S$. Using the previously defined kinematics of the SDA, functional Π^{HW} takes the expression:

$$\begin{aligned}
\Pi^{HW} = & \int_{\Omega/S} \sigma \cdot (\nabla^S \hat{\mathbf{u}} + \nabla^S \tilde{\mathbf{u}} - \varepsilon_e - \varepsilon_{p_0} - \dot{\varepsilon}_p \Delta t) d\Omega \\
& - \int_{\Omega/S} \chi \cdot (\alpha_e + \alpha_{p_0} + \dot{\alpha}_p \Delta t)\, d\Omega - \int_{\Omega/S} \zeta \cdot (\omega_e + \omega_{p_0} + \dot{\omega}_p \Delta t)\, d\Omega \\
& - \int_S \chi_S \cdot \left(\alpha_{S_e} + \alpha_{S_{p0}} + \dot{\alpha}_{S_p} \Delta t\right) dS - \int_S \zeta_S \cdot \left(\omega_{S_e} + \omega_{S_{p0}} + \dot{\omega}_{S_p} \Delta t\right) dS \\
& + \int_S \mathbf{t}_S \cdot \left(\Delta \tilde{\mathbf{u}} - [\![\mathbf{u}]\!]_{S_e} - [\![\mathbf{u}]\!]_{S_{p0}} - [\![\dot{\mathbf{u}}]\!]_{S_p} \Delta t\right) dS \\
& + \int_{\Omega/S} \phi(\varepsilon_e, \alpha_e, \omega_e) d\Omega + \int_{\Omega/S} d(\dot{\varepsilon}_p, \dot{\alpha}_p, \dot{\omega}_p) \Delta t d\Omega \\
& + \int_S \phi_S([\![\mathbf{u}]\!]_{S_e}, \alpha_{S_e}, \omega_{S_e}) dS + \int_S d_S([\![\dot{\mathbf{u}}]\!]_{S_p}, \dot{\alpha}_{S_p}, \dot{\omega}_{S_p}) \Delta t dS \\
& - \int_{\Omega/S} \mathbf{b} \cdot \mathbf{u} d\Omega - \int_{\partial(\Omega/S)_q} \mathbf{q} \cdot \mathbf{u} ds - \int_{\partial(\Omega/S)_u} \mathbf{r} \cdot (\mathbf{u} - \bar{\mathbf{u}})\, ds
\end{aligned}$$

where the following additive decomposition for the kinematic variables has been assumed:

$$\begin{aligned}
\varepsilon &= \varepsilon_e + \varepsilon_p = \varepsilon_e + \varepsilon_{p0} + \dot{\varepsilon}_p \Delta t \\
\alpha &= \alpha_e + \alpha_p = \alpha_e + \alpha_{p0} + \dot{\alpha}_p \Delta t \\
\omega &= \omega_e + \omega_p = \omega_e + \omega_{p0} + \dot{\omega}_p \Delta t
\end{aligned}$$

$$[\![\mathbf{u}]\!]_S = [\![\mathbf{u}]\!]_{S_e} + [\![\mathbf{u}]\!]_{S_{p0}} + [\![\dot{\mathbf{u}}]\!]_{S_p} \Delta t$$

$$\begin{aligned}
\alpha_S &= \alpha_{S_e} + \alpha_{S_p} = \alpha_{S_e} + \alpha_{S_{p0}} + \dot{\alpha}_{S_p} \Delta t \\
\omega_S &= \omega_{S_e} + \omega_{S_p} = \omega_{S_e} + \omega_{S_{p0}} + \dot{\omega}_{S_p} \Delta t
\end{aligned}$$

ϕ and d denote respectively the elastic and dissipation functional ruling the reversible and irreversible behaviour of the material in the bulk and on the interface S.
By eliminating strain and kinematic internal variables and performing appropriate Legendre transformations, the generalized Hellinger-Reissner functional Π^{HR} is obtained:

$$\Pi^{HR}(\hat{\mathbf{u}}, \tilde{\mathbf{u}}, \sigma, \chi, \zeta, \chi_S, \zeta_S, \mathbf{t}_S)$$

$$\begin{aligned}
\Pi^{HR} = & \int_{\Omega/S} \sigma \cdot \nabla^S \hat{\mathbf{u}} d\Omega - \int_{\Omega/S} \phi'(\sigma, \chi, \zeta) d\Omega - \int_{\Omega/S} d'(\sigma, \chi, \zeta) d\Omega \\
& - \int_S \phi'_S(\mathbf{t}_S, \chi_S, \zeta_S) dS - \int_S d'_S(\mathbf{t}_S, \chi_S, \zeta_S) dS + \int_S \mathbf{t}_S \cdot \Delta \tilde{\mathbf{u}} dS \\
& - \int_{\Omega/S} (\sigma \cdot \varepsilon_{p_0} + \chi \cdot \alpha_{p_0} + \zeta \cdot \omega_{p_0}) d\Omega - \int_S (\chi_S \cdot \alpha_{S_{p0}} + \zeta_S \cdot \omega_{S_{p0}} + \mathbf{t}_S \cdot [\![\mathbf{u}]\!]_{S_{p0}}) dS \\
& - \int_{\Omega/S} \mathbf{b} \cdot (\hat{\mathbf{u}} + \tilde{\mathbf{u}}) d\Omega - \int_{\partial(\Omega/S)_q} \mathbf{q} \cdot (\hat{\mathbf{u}} + \tilde{\mathbf{u}}) ds - \int_{\partial(\Omega/S)_u} \mathbf{r} \cdot (\hat{\mathbf{u}} + \tilde{\mathbf{u}} - \bar{\mathbf{u}}) ds
\end{aligned}$$

in which the constitutive conditions in the bulk and on *S* are already imposed.
The optimization problem is stated as:

$$\inf_{(\hat{\mathbf{u}},\tilde{\mathbf{u}})} \sup_{(\sigma,\mathbf{t}_S,\chi_S,\zeta_S)} \Pi^{HR}$$

and the equilibrium solution is given by

$$\inf_{(\hat{\mathbf{u}},\tilde{\mathbf{u}})} \{\Phi_{ep}(\hat{\mathbf{u}},\tilde{\mathbf{u}}) - \int_{\Omega/S} \mathbf{b}\cdot(\hat{\mathbf{u}}+\tilde{\mathbf{u}})d\Omega - \int_{\partial(\Omega/S)_q} \mathbf{q}\cdot(\hat{\mathbf{u}}+\tilde{\mathbf{u}})ds - \int_{\partial(\Omega/S)_u} \mathbf{r}\cdot(\hat{\mathbf{u}}+\tilde{\mathbf{u}}-\bar{\mathbf{u}})ds\}$$

$$\Phi_{ep} = \sup_{(\sigma,\mathbf{t}_S,\chi_S,\zeta_S)} \Pi^{HR}$$

The stationarity conditions of functional Π^{HR} give the relevant equations of the model. For instance, in the case of elastic medium and dissipative interfaces, vanishing the internal variables in the bulk, it follows

$$\begin{array}{lll}
\delta_{\hat{\mathbf{u}}}\Pi^{HR} & \Rightarrow & \left\{\begin{array}{lll} div\sigma + \mathbf{b} = 0 & in & \Omega/S \\ \sigma\mathbf{n} = \mathbf{q} & on & \partial(\Omega/S)_q \\ \sigma\mathbf{n} = \mathbf{r} & on & \partial(\Omega/S)_u \end{array}\right. \\
\delta_{\tilde{\mathbf{u}}}\Pi^{HR} & \Rightarrow & \mathbf{t}_S = \sigma\mathbf{n} \qquad on \quad S \\
\delta_{\sigma}\Pi^{HR} & \Rightarrow & \nabla^S\hat{\mathbf{u}} + \nabla^S\tilde{\mathbf{u}} = \nabla_\sigma\phi'(\sigma) \qquad in \quad \Omega/S \\
\delta_{\mathbf{t}_S}\Pi^{HR} & \Rightarrow & \triangle\tilde{\mathbf{u}} + \nabla_{\mathbf{t}_S}\phi'_S(\mathbf{t}_S,\chi_S) + \nabla_{\mathbf{t}_S}d'_S(\mathbf{t}_S,\chi_S) - \alpha_{S_{p0}} \qquad on \quad S \\
\delta_{\chi_S}\Pi^{HR} & \Rightarrow & -\nabla_{\chi_S}\phi'_S(\mathbf{t}_S,\chi_S) - \nabla_{\chi_S}d'_S(\mathbf{t}_S,\chi_S) - \alpha_{S_{p0}} = 0 \qquad on \quad S \\
\delta_{\mathbf{r}}\Pi^{HR} & \Rightarrow & \hat{\mathbf{u}} + \tilde{\mathbf{u}} = \bar{\mathbf{u}} \qquad on \quad \partial(\Omega/S)_u
\end{array}$$

The weak formulation is obtained discretizing the displacement fields, resolving the internal variables at constitutive level and assuming linear elastic constitutive equations for the continuum. It allows an effective numerical implementation of the interface model able to predict both the occurrence of the discontinuity and its direction; no tracking algorithm is introduced.
Among the many possible algorithmic frameworks, the one recently proposed in [5], based on the formal analogy between the enriched continuum and the theory of classical plasticity, has been implemented [6] in the FEAP code [7] and tested on benchmark numerical tests.
The Finite Element implementation of the algorithm is based on the Elements with Embedded Discontinuities [3]. Usually the equilibrium condition at the interface is satisfied in a weak sense, leading to the classical equations of Statical Kinematical Optimal Nonsymmetric formulation of SDA [8], obtained under the hypotheses that the jump field $[\![\mathbf{u}]\!]$ be constant and that a Petrov-Galerkin approximation of the incompatible strain is used in the orthogonality condition between stresses and enhanced strain. It can be shown that, at least in principle, using the present formulation in the complementary form given by the Hellinger-Reissenr functional, a standard Galerkin discretisation can be used, as opposed to the Petrov Galerkin approach usually adopted. However in this way the equilibrium condition at the interface *S* is valid only globally and not element-wise and the interface traction can be obtained from a weighted average of the stress field on *S*.

Boundary conditions

Differently from the classical equations of Statical Kinematical Optimal Nonsymmetric formulation, a general form of the enhanced displacement field is proposed that allows to satisfy the Dirichelet boundary conditions.
The displacement field defined on the continuum Ω has to satisfy the Dirichlet essential conditions on the constrained portion of the boundary $\partial\Omega_u$. These conditions are usually fulfilled only at the boundary nodes which belong to the element defining the domain Ω_φ, where the classical function $\tilde{\mathbf{u}} = M_S[\![\mathbf{u}]\!]$ vanishes. In all the internal points of the element side laying on the border the essential conditions are not met, as it is shown in Fig. 3.
In order to enforce natural BC's, an enhanced displacement field has been introduced as shown in section *Kinematics* by means of function $\overline{M}_S$. It can be shown that a proper definition of $\overline{M}_S$ can be obtained in the form $\overline{M}_S = M_S N_S$, being N_S a function depending on the standard shape functions of the element. For instance, in Fig. 4 it is shown the

definition of the enhanced enrichment function for the 4-nodes element. In the first case $N_S = N_1 + N_2 + N_3 + N_4 = 1$, in the second $N_S = N_1 + N_4$, in the third case $N_S = N_5$.

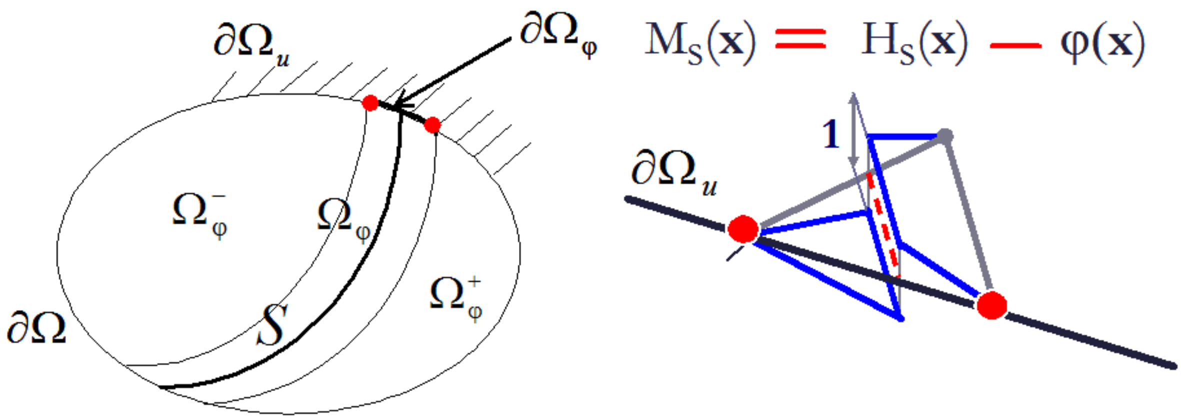

Figure 3: Enriched displacement field on the restrained boundary.

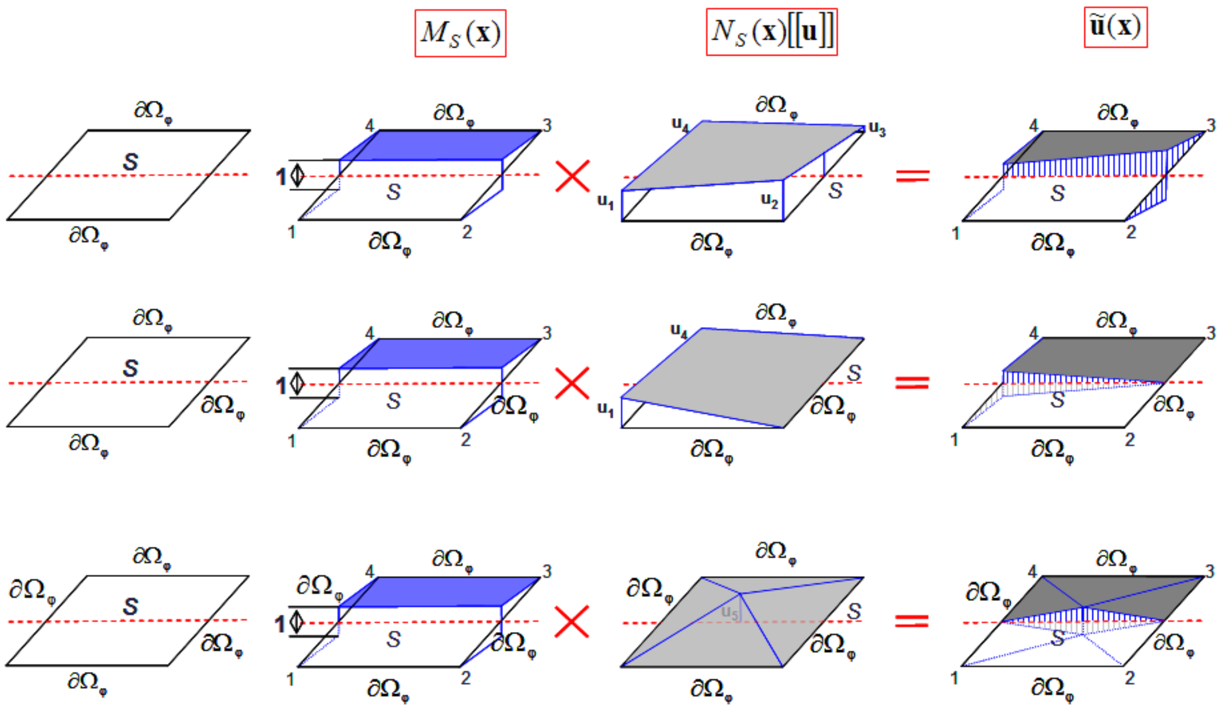

Figure 4: Enhancement enrichment function for different boundary conditions in a four-nodes element.

Conclusions

In the paper a general variational principle based on an enhanced displacement field for modelling strong discontinuities in an elastic-plastic damaging medium has been presented. From the principle all the equations of the problem can be obtained. They represent the Euler-Lagrange stationarity conditions of a generalised Hu-Washizu functional. Moreover an enhancement in the definition of the enriched displacement field of the SDA has been proposed allowing for the satisfaction of the imposed constraint all over the restrained boundary and not only at the nodal point of the discretized domain. In the present context the orthogonality condition between the stress field and the deformations obtained from the enhanced displacements can be exactly satisfied by a compatible kinematics in a global sense, but in general not locally for each element.

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