TOWARDS CHAOS IN THE DYNAMIC RESPONSE OF DAMAGED STRUCTURES

Alberto Carpinteri, Nicola Pugno

Department of Structural Engineering and Geotechnics, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino
carpinteri@polito.it, pugno@polito.it

Abstract

The aim of this paper is to evaluate the transition towards chaos in the dynamic response of multicracked nonlinear structures under excitation. The developed approach permits to capture the sub-harmonic components in the structural dynamic response, describing complex phenomena like the experimentally observed Period Doubling.

Sommario

In questo lavoro ci si propone di valutare la transizione al caos deterministico della risposta dinamica forzata di strutture multi-fessurate non lineari. L'approccio sviluppato permette di cogliere le componenti sub-armoniche della risposta dinamica, descrivendo così fenomeni complessi come il Period Doubling, recentemente osservato sperimentalmente.

1. Introduction

The aim of this paper is to develop a coupled theoretical and numerical approach to evaluate the complex oscillatory behavior in multicracked nonlinear structures under excitation. In particular, we have focused our attention on a cantilever beam with several breathing transverse cracks and subjected to harmonic excitation perpendicular to its axis. The method, that is an extension of the super-harmonic analysis carried out by Pugno, Ruotolo and Surace (2000) to sub-harmonic and zero frequency components, has permitted to capture the complex behavior of the nonlinear structure, e.g., the occurrence of period doubling, as experimentally observed by Brandon (1998). The method described assumes that the cracks open and close continuously (Carpinteri Al. and Carpinteri An., 1982) instead of instantaneously, as suggested by experiments (Pugno et al., 2000). As a consequence, the cracks are not considered to be either fully open or fully closed, but the intermediate configurations with partial opening can also be taken into account. The period of the response is not assumed *a priori* equal to the period of the harmonic excitation, as classically supposed (absence of sub-harmonic components). This has permitted to capture the complex behavior of the nonlinear structure. The analysis has systematically shown the presence of an offset (zero frequency component) in the structural response. The differential nonlinear equations governing the oscillations of the structure, discretized by the Finite Element Method, have been analyzed by means of the Fourier Trigonometric Series and the Harmonic Balance Approach. This has permitted to obtain a nonlinear system of algebraic equations, easy to solve numerically. Numerical simulations complete the paper.

2. Dynamic analysis

Considering a multicracked cantilever beam, clamped at one end and subjected to a harmonic force with angular frequency ω and amplitude F, acting perpendicularly to the axis at a given position, the equation of motion, obtained by discretizing the structure with the Finite Element Method, is (Pugno et al., 2000):

$$[\mathrm{M}]\{\ddot{q}\}+[\mathrm{D}]\{\dot{q}\}+[\mathrm{K}]\{q\}+\sum_{m}\left[\Delta\mathrm{K}^{(m)}\right]f^{(m)}(\{q\})\{q\}=\{F\}, \tag{1}$$

where [M] is the mass matrix, [D] the damping matrix, $[\mathrm{K}]+\sum_m[\Delta\mathrm{K}^{(m)}]$ the stiffness matrix of the undamaged beam and $[\Delta\mathrm{K}^{(m)}]$ is half of the variation in stiffness introduced when the *m*th crack is fully open (see Pugno et al., 2000). $\{F\}$ is the vector of the applied forces and $\{q\}$ is the vector containing the generalized displacements of the nodes (translations and rotations). According to this notation, $f^{(m)}(\{q\})$ is between -1 and $+1$ and models the transition between the conditions of *m*th crack fully-open and fully-closed. Assuming that this transition is instantaneous and hence takes place discontinuously, $f^{(m)}(\{q\})$ is a step function and has the sign of the curvature of the corresponding cracked element. With this simple model of crack opening and closing, $f^{(m)}(\{q\})$ can thus only be equal to -1 or $+1$. On the other hand, in the present investigation as in the previous (Pugno et al., 2000), $f^{(m)}(\{q\})$ is assumed to be a linear function of the curvature of the corresponding cracked element. In other words, the cracks are not considered fully open or fully closed, as the intermediate configurations with partial opening are also taken into account. Thus, the stiffness varies continuously between the two extremes of undamaged or totally damaged beam (fully open cracks), rather than stepwise. The solution for the elements of the $\{q\}$ vector $\in L^2$ (i.e., $|q_i|^2$ can be integrated according to Lebesgue) can be found by means of the following trigonometric series:

$$q_i = \sum_{j=0}^{\infty}\left(A_{ij}\sin j\frac{\omega}{\Theta}t + B_{ij}\cos j\frac{\omega}{\Theta}t \right), \tag{2}$$

in which we take into account, with an *ad hoc* introduced "complexity index" Θ (positive integer), the sub-harmonic components of the dynamic response. This means that the response could have a period $\widetilde{P} = \Theta P$ that is not *a priori* coincident with the period P of the excitation. A value for Θ tending to infinite describes a transition towards a chaotic (nonperiodic) response.

It is interesting to note that, even if the trigonometric series (2) converges, it could not be a trigonometric Fourier series. In fact, the Fischer-Riesz theorem affirms that it is a Fourier series if and only if $\sum_{j=0}^{\infty}\left(|A_{ij}|^2 + |B_{ij}|^2\right)$ converges. In this case, the Parseval equation:

$$\sum_{j=0}^{\infty}\left(|A_{ij}|^2 + |B_{ij}|^2\right) = \frac{2}{\widetilde{P}}\int_{-\widetilde{P}/2}^{+\widetilde{P}/2} |\, q_i(t)|^{\,2}\mathrm{d}t\,, \tag{3}$$

obviously implies:

$$\lim_{j\to\infty} A_{ij} = \lim_{j\to\infty} B_{ij} = 0\,. \tag{4}$$

The last relationships permit to put the solution for the elements of $\{q\}$ in the following approximated form:

$$q_i = \sum_{j=0}^{N}\left(A_{ij}\sin j\frac{\omega}{\Theta}t + B_{ij}\cos j\frac{\omega}{\Theta}t \right), \tag{5}$$

where N should be large enough to provide a good approximation. The function $f^{(m)}(\{q\})$ is considered linear versus the curvature of the corresponding cracked element, i.e.,

$$f^{(m)}(\{q\}) = \frac{q_{m_k} - q_{m_h}}{\left|q_{m_k} - q_{m_h}\right|_{max}} = \Lambda_m\left(q_{m_k} - q_{m_h}\right), \tag{6}$$

where the numerator represents the difference in the rotations at the ends of the corresponding cracked element and the denominator is the maximum absolute value that can be reached by this difference: consequently, the generic component of function $\{g^{(m)}\} = f^{(m)}(\{q\})\{q\}$ (that appears in eq. (1)) can be expressed as:

$$g_i^{(m)} = \Lambda_m\left(q_{m_k} - q_{m_h}\right) q_i\,. \tag{7}$$

The same concepts argued for the q_i components can be now applied to the $g_i^{(m)}$, ensuring that they can be developed in a trigonometric Fourier series and can thus be put in the approximated form:

$$g_i^{(m)} = \sum_{j=0}^{N} \left(C_{ij}^{(m)} \sin j \frac{\omega}{\Theta} t + D_{ij}^{(m)} \cos j \frac{\omega}{\Theta} t \right), \tag{8}$$

with:

$$C_{ij}^{(m)} = \frac{2}{\tilde{P}} \int_0^{\tilde{P}} g_i^{(m)}(t) \sin(j \frac{\omega}{\Theta} t) \mathrm{d}t, \tag{9}$$

$$D_{ij}^{(m)} = \frac{2}{\tilde{P}} \int_0^{\tilde{P}} g_i^{(m)}(t) \cos(j \frac{\omega}{\Theta} t) \mathrm{d}t. \tag{10}$$

Inserting relation (7), in its explicit form according to eq. (5) for the degrees of freedom i, m_h and m_k, into equations (9) and (10) and developing the integrals, gives the following expressions:

$$\begin{aligned} C_{ij}^{(m)} = &\frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 + j_2 = j} \left\{ \left(A_{m_k j_1} - A_{m_h j_1} \right) B_{ij_2} + \left(B_{m_k j_1} - B_{m_h j_1} \right) A_{ij_2} \right\} + \\ &+ \frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 - j_2 = \pm j} \pm \left\{ \left(A_{m_k j_1} - A_{m_h j_1} \right) B_{ij_2} - \left(B_{m_k j_1} - B_{m_h j_1} \right) A_{ij_2} \right\}, \end{aligned} \tag{11}$$

$$\begin{aligned} D_{ij}^{(m)} = &\frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 + j_2 = j} \left\{ - \left(A_{m_k j_1} - A_{m_h j_1} \right) A_{ij_2} + \left(B_{m_k j_1} - B_{m_h j_1} \right) B_{ij_2} \right\} + \\ &+ \frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 - j_2 = \pm j} \pm \left\{ \left(A_{m_k j_1} - A_{m_h j_1} \right) A_{ij_2} - \left(B_{m_k j_1} - B_{m_h j_1} \right) B_{ij_2} \right\}. \end{aligned} \tag{12}$$

As the nonlinearity of the components of $\{g^{(m)}\} = f^{(m)}(\{q\})\{q\}$ were expressed in a form analogous to that of the components of $\{q\}$, as indicated by equations (8), (11) and (12), it is possible at this stage to apply the Harmonic Balance Method, which leads to N different systems of nonlinear algebraic equations:

$$\begin{bmatrix} [\mathrm{K}] - \frac{j^2 \omega^2}{\Theta^2} [\mathrm{M}] & -\frac{j\omega}{\Theta} [\mathrm{D}] \\ \frac{j\omega}{\Theta} [\mathrm{D}] & [\mathrm{K}] - \frac{j^2 \omega^2}{\Theta^2} [\mathrm{M}] \end{bmatrix} \begin{Bmatrix} \{A_j\} \\ \{B_j\} \end{Bmatrix} = \{F_j\} - \sum_m \begin{bmatrix} [\Delta \mathrm{K}^{(m)}] & [0] \\ [0] & [\Delta \mathrm{K}^{(m)}] \end{bmatrix} \begin{Bmatrix} \{C_j^{(m)}\} \\ \{D_j^{(m)}\} \end{Bmatrix}, \tag{13}$$

where j=0,1,…,N and for each vector we have $\{V_j\}^T = \{V_{1j}, V_{2j}, \ldots\}^T$. In addition:

$$F_{ij} = F \delta_{j\Theta} \delta_{ip}, \tag{14}$$

p being the node position corresponding to the point where the sinusoidal force is applied.

Each system can be solved numerically using an iterative procedure interrupted by an appropriate convergence test when the relative j-error for the $\{A_j\}$ and $\{B_j\}$ vectors becomes lower than a specified value; it is a function of the kth iteration and has been defined as:

$$e_{jk} = \left\| \begin{Bmatrix} \{A_j\} \\ \{B_j\} \end{Bmatrix}_k - \begin{Bmatrix} \{A_j\} \\ \{B_j\} \end{Bmatrix}_{k-1} \right\| \Bigg/ \left\| \begin{Bmatrix} \{A_j\} \\ \{B_j\} \end{Bmatrix}_{k-1} \right\| . \tag{15}$$

The procedure consists in determining the unknowns A_{ij} and B_{ij}. It is very interesting to note that, assuming the coefficients $C_{ij}^{(m)}$, $D_{ij}^{(m)}$ to be zero at the first step, implies to force also the sub-harmonic components to be zero (see eq. (14)). So, differently from the super-harmonic analysis (Pugno et. al, 2000), we have to start with nonzero values for the coefficients $C_{ij}^{(m)}$, $D_{ij}^{(m)}$. To obtain good initial values for these coefficients, we have considered as a zero step a super-harmonic analysis ($\Theta = 1$); in this case, we can determine the unknowns A_{ij} and B_{ij} starting with zero coefficients $C_{ij}^{(m)}$, $D_{ij}^{(m)}$ and, by eqs. (11) and (12), we have their initial values for the sub-harmonic analysis. The solution thus obtained is used to determine the known vector of the right hand-side of eq. (13). The procedure is repeated until the desired precision is achieved and coefficients A_{ij} and B_{ij} are found, while equation (2) is used to determine the components of the approximate vector, which satisfies the nonlinear equation (1).

3. Numerical Analysis

The beam considered is the same as that described in the experimental analysis of Brandon (1998). It is 270mm long and has a transversal rectangular cross section of 13×5 mm^2. The material is UHMW-ethylene, with a Young's modulus of 8.61×10^8 N/m^2 and a density of 935 kg/m^3. We have assumed a modal damping of 0.002. It is discretized with 20 finite elements. For our nonlinearity we have found that a complexity index $\Theta = 4$ and N=16 give a good approximation (for larger values of Θ and N substantially coincident solutions are obtained). The first natural frequency of the undamaged structure is f_u=10.6 Hz.

As an example, some results of a numerical simulation are shown in Table 1 and Figures 1-3. It should be emphasized a strong presence of the component causing the period doubling of the response ($\omega/2$ component) as well as of an offset (zero component) describing a constant negative displacement (i.e., downwards in Fig.1) of the free end.

4. Conclusions

The proposed approach extends in a very powerful way the theory proposed by Pugno et al. (2000) to (offset and) sub-harmonic components. It has permitted to catch complex phenomena like the occurrence of a period doubling, as shown in the reported numerical example and experimentally observed by Brandon (1998). In this context, of crucial importance appears the "complexity index" Θ. For higher values of Θ we have to increase also N (e.g., $N \approx \Theta^2$), so that the complexity of the numerical simulations considerably increases. On the other hand, larger values of Θ permit to catch a transition towards deterministic chaos, i.e., towards a nonperiodic dynamic response. This approach will be used in the next future for an extensive parametric investigation.

Harmonic	Sin	Cos	Amplitude
0	0	-5,1781	5,1781
ω/4	2,5728	-2,4656	3,5635
ω/2	16,1544	5,6516	17,1144
3/4ω	0,0258	-1,6822	1,6824
ω	-44,9137	-0,2925	44,9146
5/4ω	0,487	0,4816	0,6849
3/2ω	0,0675	0,6448	0,6483
7/4ω	-0,1776	0,5622	0,5896
2ω	-0,4142	0,5074	0,655
9/4ω	-0,727	0,305	0,7884
5/2ω	-0,9927	-0,0939	0,9972
11/4ω	-1,0984	-0,8182	1,3697
3ω	-0,4109	-1,9223	1,9657
13/4ω	1,7453	-5,3399	5,6179
7/2ω	-1,8971	1,0182	2,153
15/4ω	-0,545	-0,0476	0,5471
4ω	-0,1235	0,0887	0,152

Table 1: Zero- (offset), sub- and super-harmonic sin- and cos- components [mm], *for the free end displacement (i.e., A_{20j}, B_{20j}, for j=0,1,…,16).*

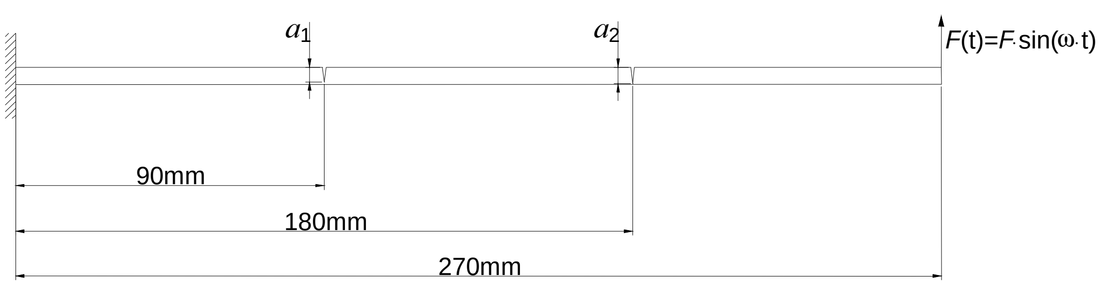

Figure 1: Damaged structure and characteristics of the excitation
(a_1=4.25mm, a_2=4.8mm, F=2N, $f=\omega/2\pi$ =18.9Hz).

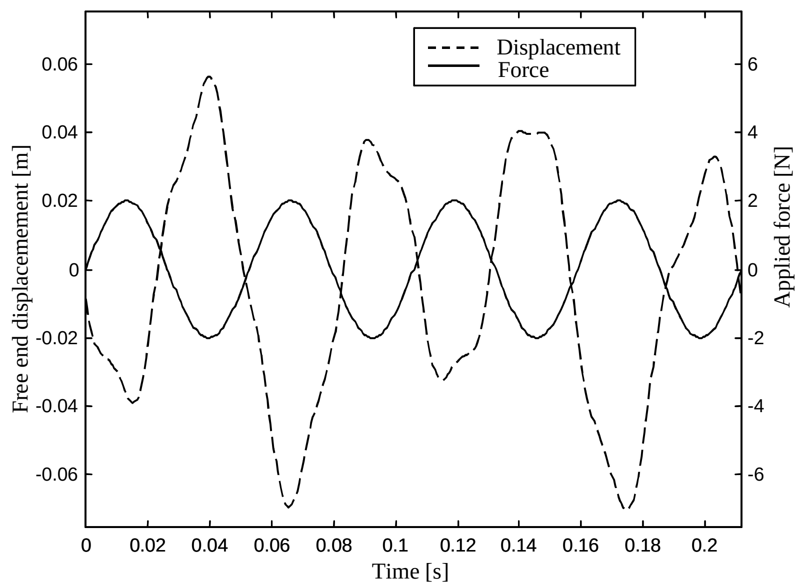

Figure 2: Time history of the free end displacement and of the applied force.

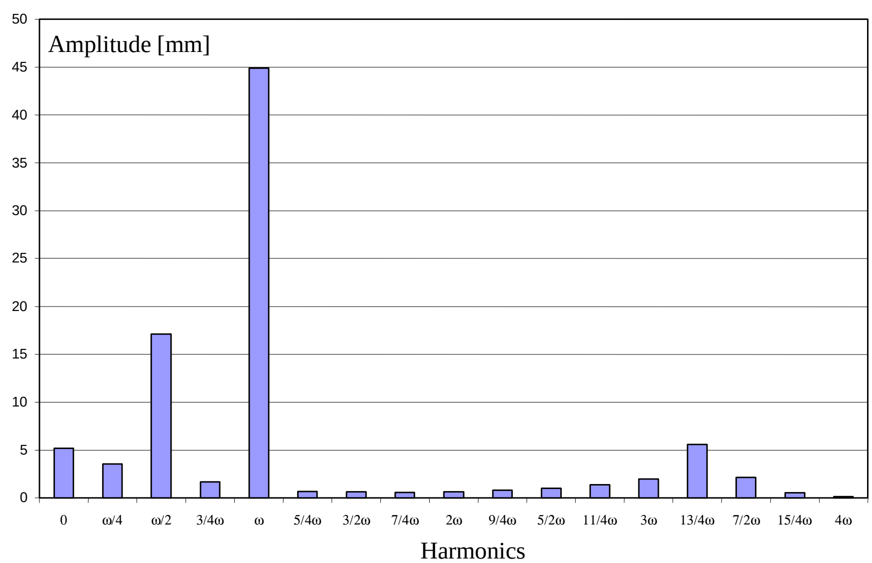

Figure 3: Zero- (offset), sub- and super-harmonic components for the free end displacement (i.e., $\sqrt{A_{20j}^2 + B_{20j}^2}$ for j=0,1,…,16).

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