

A NEW SOLUTION FOR 3D CRACK EXTENSION BASED ON LINEAR ELASTIC STRESS FIELDS

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ABSTRACT. *A solution to the 3D stress field based on the maximum tangential stress (MTS) criterion is presented in this paper. The solution allows for the estimation of the critical crack plane, the direction of growth in terms of both twist and tilt angles and the equivalent crack driving force for a given mixed-mode loading condition. It also shows the graphical relationship between the three different stress intensities for a given driving force. Initial results have shown good correlation with experimental data obtained from literature.*

INTRODUCTION

The principles of linear elastic fracture mechanics (LEFM) are founded on elastic material behaviour whereby the relationship between stress-strain is linear and only very small scale yielding occurs. Basic LEFM stress field equations derived from Cauchy stress tensors describe the stress field near a crack tip as function of location, 3D loading applied and geometry of the structure. By incorporating the stress intensity factor (K) which takes into account the loading and geometry, the equations can be simplified to describe the magnitude of the stress field in terms of K and location characterised by polar coordinates from the crack tip.

Based on the stress field equations formulated for the three fracture modes, this paper proposes an analytical solution to derive the crack path plane, growth direction and equivalent K value. This utilises the K_I , K_{II} and K_{III} stress field equations in conjunction with the maximum tangential stress (MTS) criterion [1] which states that a crack will initiate in the direction where the largest circumferential stress at a small distance from the crack tip reaches a critical tensile value. The solution uses a tensor transformation to find the critical plane and direction of crack growth which then allows for the prediction of crack growth direction for a 3D case and, in turn, the equivalent crack driving force.

The numerical results obtained for the MTS criterion show the graphical relationships between the applied stress intensities (K_I , K_{II} and K_{III}) and the crack growth direction angle, the crack plane orientation and the crack driving force. The results are compared with both experimental and numerical data from literature and they are found to work well for predicting the crack plane and the equivalent driving force K_{eq} , for materials that experience tensile mode fatigue cracking and fracture.

ANALYTICAL SOLUTION

The asymptotic expansion of the stress field around a crack tip in an isotropic linear elastic body was first proposed by Williams [2] and the first term of the expansion can be expressed as follows [3]:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{xz} = \frac{K_{III}}{\sqrt{2\pi r}} \left(-\sin \frac{\theta}{2} \right)$$

$$\sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}$$

Poisson's ratio ν , is taken as 0.3 in these calculations. The y -direction is normal to the crack, and the crack extends in the positive x -direction if it remains coplanar.

The stresses from the expansion above are then grouped together to form a modified stress tensor S , where stress is multiplied by $\sqrt{(2\pi r)}$ to model the elastic singularity.

$$S = \sqrt{2\pi r} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

The modified stress tensor S is then transformed from one set of Cartesian coordinates to another by means of rotation through two angles, to tilt and twist the crack by undergoing the matrix transformation:

$$S' = LSL^t$$

where the tensor L contains the direction cosines expressed in matrix form as follows:

$$L = \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi \cos\psi & \cos\chi \cos\psi & \sin\psi \\ \sin\psi \sin\chi & -\sin\psi \cos\chi & \cos\psi \end{bmatrix}$$

where χ = tilt angle and ψ = twist angle as illustrated in Figure 1. The transformed stress tensor can then be expressed as:

$$S' = \sqrt{2\pi r} \begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'z'} \end{bmatrix}$$

Here, y' is normal to the new crack plane and x' denotes the crack growth direction.

In the first iteration of the analytical solution described in this paper, the maximum tangential stress (MTS) criterion ($\sigma_{y'y'}$) was used as the failure criteria and the equations were solved to obtain the biggest possible value which corresponds to $\sigma_{y'y'}$ in the S' matrix. In this work, a value of $K_{eq} = 10 \text{ MPa}\sqrt{\text{m}}$ was used together with selected values of K_{II} and K_{III} from which K_I , χ and ψ were then calculated. The tilt angle χ defines the crack growth direction along x' and the twist angle ψ denotes the crack plane which is the plane of facets in a “factory roof” formation along the original crack front z . The results obtained are shown in the following section.

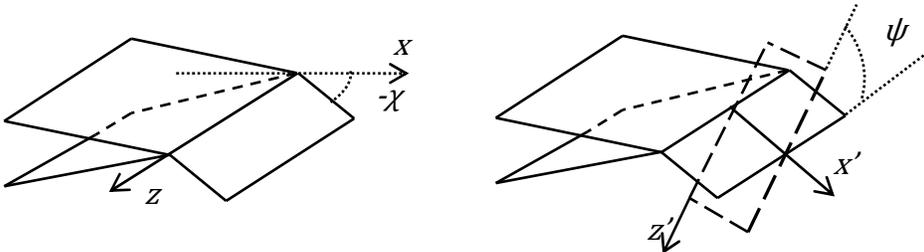


Figure 1: Schematic illustration of the negative tilt (χ) and subsequent twist (ψ) angle

RESULTS

New solution obtained from MTS criterion

The results obtained from solving the analytical solution discussed above are presented in this section. Figure 2 illustrates the three dimensional surface plot of K_I , represented in the vertical z -axis, for a given K_{II} , K_{III} and $K_{eq} = 10$. If a 2D slice is taken for K_I against K_{II} with K_{III} equal to zero, the relationship between K_I and K_{II} is approximately parabolic. If the same is done for K_I and K_{III} , the relationship is rather linear if K_{III} is the dominant term but becomes parabolic once K_I becomes the dominant term (see Figure 5).

Figure 2 shows one quarter of the dome-shape failure surface, which is symmetrical with K_{II} and K_{III} , both of which can take negative values. Only positive values of K_I have been used assuming crack closure will occur where negative K_I values are calculated.

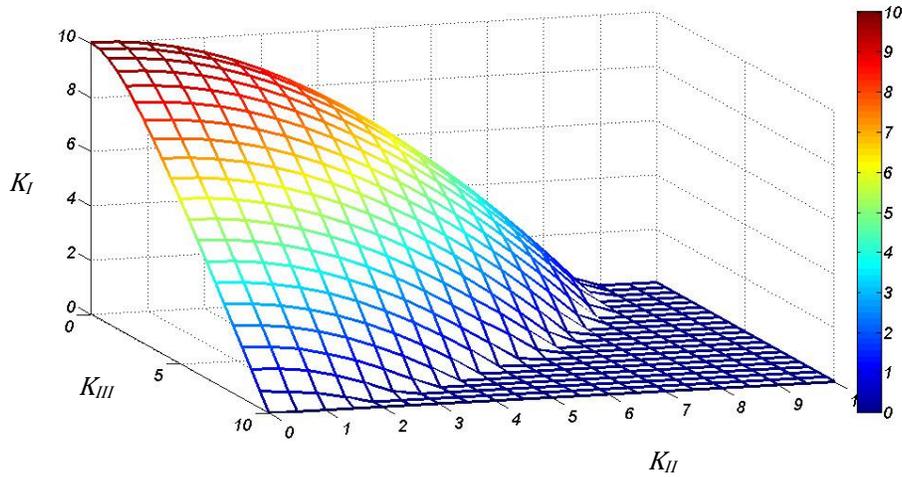


Figure 2: Illustration of the 3D stress intensity fields for $K_{eq} = 10$

Figure 3 and Figure 4 show the change in the tilt (χ) and twist (ψ) angle respectively for a given combination of K_{II} and K_{III} . The results obtained from the tilt angle show that it is predominantly influenced by the K_{II} term and the K_{III} term has only a small influence on the change in tilt angle. The opposite is true for the twist angle where the predominant influence is the K_{III} term although K_{II} does have a more significant influence on the twist angle if compared to the influence of K_{III} in the tilt angle. These surfaces are truncated where the calculated K_I values have begun to turn negative.

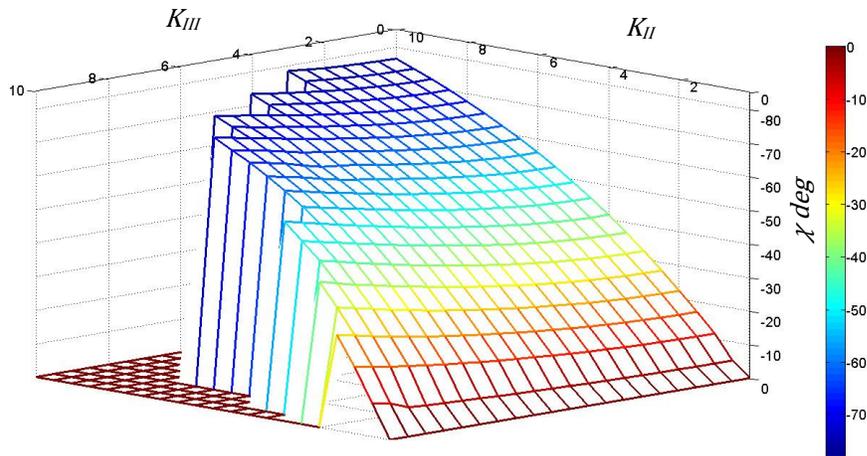


Figure 3: Change in tilt angle χ in relation to K_{II} and K_{III} for $K_{eq} = 10$

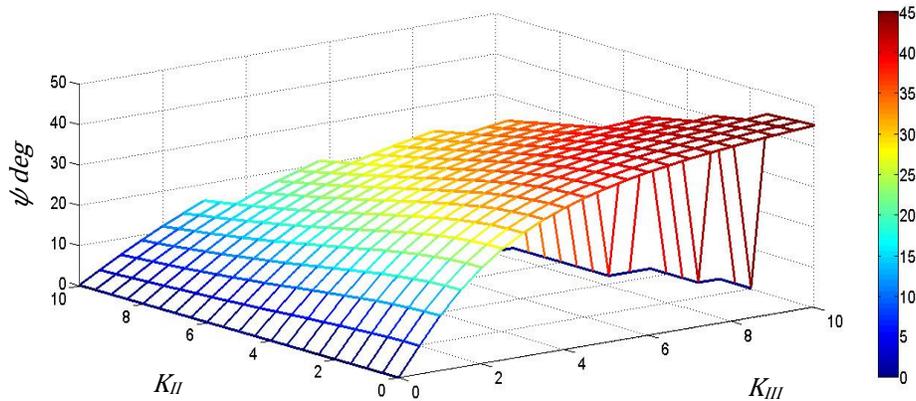


Figure 4: Change in twist angle ψ in relation to K_{II} and K_{III} for $K_{eq} = 10$

The maximum tilt angle for pure mode II was 70.5° which differs slightly from results obtained in a solution proposed by Richard *et al* [4] who found the maximum tilt angle to be approximately 70.0° using the same MTS criterion. Higher tilt angles can be achieved with negative nominal K_I if cyclic crack closure conditions generate small negative K_I values. In pure mode III, both the criterion in this paper and from literature [4] produced a maximum twist angle of 45.0° .

There are some further small discrepancies between these results and the two solutions of Richard [4] and Schollman [5] especially in the calculated tilt and twist angle. This could be due to assumptions made in the solutions proposed by Richard and Schollman which use Mohr's circle to obtain the angle required to satisfy the MTS criterion. The use of a 2D Mohr's circle provides only an approximate solution for a 3D stress field. The only assumption made in the solution in this paper is that the local stress intensity is equal to K_{eq} . Further detailed analysis is required to determine the exact regions of the discrepancies and which provides the better solution to the 3D stress fields.

Comparison with experimental data from literature

Figure 5 shows the calculated failure loci based on the MTS criterion which was used in this solution. The experimental data seen in the plot were obtained from literature [6]. If the K_I and K_{III} values given in literature are normalised by K_{Ith} , the plots show that the failure loci underestimates slightly the initiation of cracking under near threshold loading conditions. If the K_{Ith} is increased by approximately 12 percent to

account for crack closure induced by roughness in mixed mode of the fracture surface [7], the results show that both experimental data and position of the failure loci agree very well. Further data can be obtained from literature [7-9] which shows that the MTS criterion works well for correlation of failure under mixed mode I + II loading conditions.

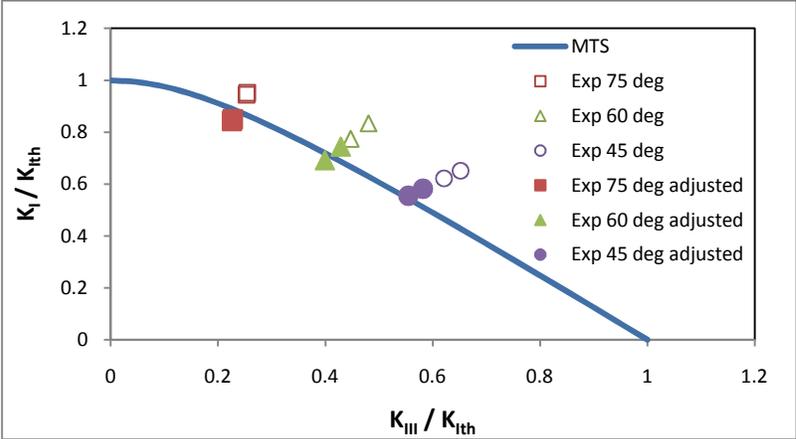


Figure 5: Failure loci for mixed mode (I + III) loading for tests of Pook [10] on En 3 steel

Further analysis of the results shows that the twist angle predicted for mixed mode $K_I + K_{III}$ loading correlates with the experimental data from literature [6] used in this paper. Figure 6 shows the comparison between the predicted twist angle needed for crack propagation and the initial slit angle used in the experiments. The discrepancies in the estimates are in the region of $\pm 5.0^\circ$ but further work is needed on both the experimental and analytical front to determine where the errors originate. More precise readings of the facet angles for individual cracks are needed to relate to ψ . Crack angles for χ , for mixed mode I + II have been reported in various papers in literature [7-9] indicating good correlation to the MTS criterion.

Calculations of crack driving force, propagation plane and growth direction are required in the analysis of failure of rails under rolling contact for high speed train lines. The non-proportional loading history experienced introduces modes I, II and III to the problem. Similar non-proportional stresses are found in automotive suspension and drivetrain components due to the complexities of service load histories.

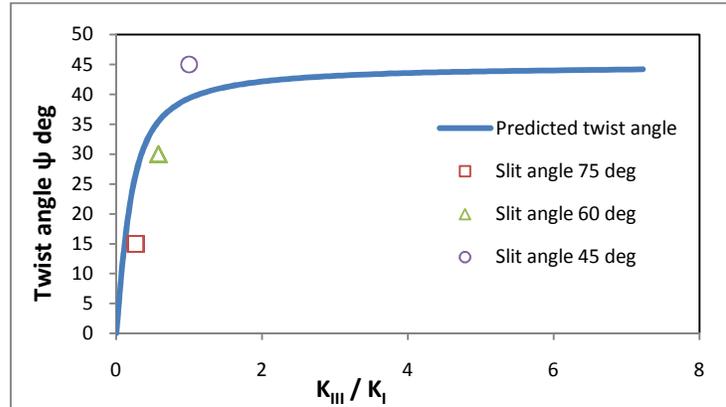


Figure 6: Comparison between predicted and experimental twist angle ψ for mixed mode (I + III) loading

FINAL COMMENTS

An analytical solution for predicting the parameters which describe 3D crack extension under mixed-mode loading is presented in this paper. The solution allows for the calculation of crack growth direction for a 3D case and in turn the equivalent crack driving force. At the moment, the solution is based on the MTS criterion and comparison of the results with experimental data from literature has shown good correlation. Further work is underway for more detailed mixed mode I + II + III results and further comparisons with experimental data. The solution is also being extended for a different failure criterion of maximum shear stress to correlate shear mode cracks.

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