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Asymmetric Crack Problem in Transversely Isotropic Bodies

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ABSTRACT: An exact solution is proposed for general boundary value problems (displacement, traction, mixed) in transversely isotropic medium. The geometry of the problem considered is symmetric but the load is arbitrary. In cylindrical coordinate system Fourier expansion and Hankel integral transforms are applied with respect to circumferential and radial coordinates respectively. As an example, a penny-shaped crack in an infinite transversely isotropic body is considered with arbitrary normal tractions on both sides of the crack. Author assume that the crack surface loading function $p(r,\theta)$ *is an even function of* θ. *The closed form expressions in terms of functions describing the tractions on the crack surface are given under assumption that the functions involved may be expanded in Fourier series with respect to circumferential coordinate. Some numerical tests using ABAQUS system are presented and compared with theoretical values.*

Notation

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{\theta z}, \sigma_{rz}, \sigma_{r\theta}$ - components of the stress tensor

$\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{r\theta}, \varepsilon_{rz}, \varepsilon_{\theta z}$ - components of the strain tensor

u_r, u_θ, u_z - components of the displacement vector

$c_{11}, c_{12}, c_{13}, c_{33}$ and c_{44} - elastic constants

Introduction

Theoretical idealizations adopted to analyse of variety of problems encountered in geomechanics, fibre-reinforced composite and in micro-mechanics defects in solids often reduced to the solution of a boundary-value problem involving a transversely isotropic semi-infinite or infinite elastic media. The geometries relevant to many practical applications are axisymmetric but involve loading and boundary conditions which do not preserve axial symmetry. For isotropic bodies this was studied especially by Muki [1].

Some previous papers treated of these problems [2] - [12] were restricted to the derivation of general solutions for transversely isotropic media in terms of potential functions and the application of these solution to axisymmetric problems of elasticity.
Some quantities of obtained solutions are compute using MapleV program very useful either in numeric calculations or in symbolic conversions [13], [14].

Parallel to symbolic investigations some numeric computations based on FEM model using in ABAQUS system [15] - [17] is introduced. This program offers its users a wide spectrum of numerical tools for both linear and nonlinear analyses. It is possible to use in the finite element analysis various types of the elements. All these capabilities, together with description of computation J-integral and fracture problems model, the set of input parameters, physical model of foundation and its geometry, are also included in the paper. Results of the numerical experiments performed and especially their convergence with the theoretical solutions confirm usefulness of ABAQUS system for computational analysis of solid mechanics. FEM programs in the nearest future may become very useful tools in fracture mechanics [18].

Theoretical solutions

We shall use the notation (r, θ, z) to denote cylindrical coordinates. Consider the elastic bodies possessing transverse isotropy. If we take the z-axis as the axis of elastic symmetry, then the stress-strain relations in cylindrical components are:

$$\sigma_{rr} = c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz},$$
$$\sigma_{\theta\theta} = c_{12}\varepsilon_{rr} + c_{11}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz},$$
$$\sigma_{zz} = c_{13}\varepsilon_{rr} + c_{13}\varepsilon_{\theta\theta} + c_{33}\varepsilon_{zz},$$
$$\sigma_{r\theta} = (c_{11} - c_{12})\varepsilon_{r\theta}, \tag{1}$$
$$\sigma_{rz} = 2c_{44}\varepsilon_{rz}$$
$$\sigma_{\theta z} = 2c_{44}\varepsilon_{\theta z},$$

Strain components are given by Cauchy relations:

$$
\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\
\varepsilon_{\theta\theta} &= \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}, \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\
\varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \\
\varepsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_z}{r\partial\theta} + \frac{\partial u_\theta}{\partial z}\right).
\end{aligned} \tag{2}
$$

The equations of equilibrium take the forms:

$$
\begin{gathered}
\left[\frac{1}{2}(c_{11} - c_{12})\nabla^2 + c_{44}\frac{\partial^2}{\partial z^2}\right]u_r + \frac{\partial}{\partial r}\left[\frac{1}{2}(c_{11} + c_{12})\Delta_H + (c_{13} + c_{44})\frac{\partial u_z}{\partial z}\right] \\
-\frac{1}{2}(c_{11} - c_{12})\frac{1}{r}\left(\frac{u_r}{r} + 2\frac{\partial u_\theta}{r\partial\theta}\right) = 0, \\
\left[\frac{1}{2}(c_{11} - c_{12})\nabla^2 + c_{44}\frac{\partial^2}{\partial z^2}\right]u_\theta + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{2}(c_{11} + c_{12})\Delta_H + (c_{13} + c_{44})\frac{\partial u_z}{\partial z}\right] \\
-\frac{1}{2}(c_{11} - c_{12})\frac{1}{r}\left(\frac{u_\theta}{r} - 2\frac{\partial u_r}{r\partial\theta}\right) = 0, \\
\left[c_{44}\nabla^2 + c_{33}\frac{\partial^2}{\partial z^2}\right]u_z + (c_{13} + c_{44})\frac{\partial}{\partial z}\Delta_H = 0
\end{gathered} \tag{3}
$$

where

$$
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \\
\Delta_H &= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}
\end{aligned} \tag{4}
$$

The solution of the equilibrium equations (3) may be given in terms of three potential functions $\varphi_i(r,\theta,z)$, which satisfy the equations [9]

$$
\left(\nabla^2 + \frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\right)\varphi_i(r,\theta,z) = 0, \qquad i = 1, 2, 3 \tag{5}
$$

where s_1^2 and s_2^2 are the two roots of the quadratic

$$c_{33}c_{44}s^4 - \left[c_{11}c_{33} - c_{13}\left(c_{13} + 2c_{44}\right)\right]s^2 + c_{11}c_{44} = 0, \tag{6}$$

and

$$s_3^2 = \left(c_{11} - c_{12}\right)/2c_{44}$$

The displacement and stress components are given as:

$$\begin{aligned} u_r &= \frac{\partial}{\partial r}\left(k\varphi_1 + \varphi_2\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\varphi_3, \\ u_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}\left(k\varphi_1 + \varphi_2\right) - \frac{\partial}{\partial r}\varphi_3, \\ u_z &= \frac{\partial}{\partial z}\left(\varphi_1 + k\varphi_2\right); \end{aligned} \tag{7}$$

$$\frac{\sigma_{rr}}{c_{44}} = -(k+1)\frac{\partial^2}{\partial z^2}\left(\varphi_1 + \varphi_2\right) - 2s_3^2\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(k\varphi_1 + \varphi_2\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi_3}{\partial \theta}\right)\right],$$

$$\frac{\sigma_{\theta\theta}}{c_{44}} = -(k+1)\frac{\partial^2}{\partial z^2}\left(\varphi_1 + \varphi_2\right) - 2s_3^2\left[\frac{\partial^2}{\partial r^2}\left(k\varphi_1 + \varphi_2\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi_3}{\partial \theta}\right)\right],$$

$$\frac{\sigma_{zz}}{c_{44}} = (k+1)\frac{\partial^2}{\partial z^2}\left(s_1^{-2}\varphi_1 + s_2^{-2}\varphi_2\right), \tag{8}$$

$$\frac{\sigma_{r\theta}}{c_{44}} = 2s_3^2\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\left(k\varphi_1 + \varphi_2\right) - \frac{\partial^2 \varphi_3}{\partial r^2}\right] - \frac{\partial^2 \varphi_3}{\partial z^2},$$

$$\frac{\sigma_{rz}}{c_{44}} = (k+1)\frac{\partial^2}{\partial r \partial z}\left(\varphi_1 + \varphi_2\right) + \frac{1}{r}\frac{\partial^2 \varphi_3}{\partial \theta \partial z},$$

$$\frac{\sigma_{\theta z}}{c_{44}} = (k+1)\frac{1}{r}\frac{\partial^2}{\partial \theta \partial z}\left(\varphi_1 + \varphi_2\right) - \frac{\partial^2 \varphi_3}{\partial r \partial z},$$

where

$$k = \left(c_{33}s_1^2 - c_{44}\right)/\left(c_{13} + c_{44}\right). \tag{9}$$

In order to determine general solutions for potential functions $\varphi_i(r,\theta,z)$; $i = 1, 2, 3$, governed by eqs (5) the following representations are used:

$$\begin{aligned} \varphi_i(r,\theta,z) &= \sum_{m=0}^{\infty}\left[\phi_{im}(r,z)\cos(m\theta) + \phi_{im}^{*}(r,z)\sin(m\theta)\right], \quad i = 1, 2 \\ \varphi_3(r,\theta,z) &= \sum_{m=0}^{\infty}\left[\phi_{3m}(r,z)\sin(m\theta) - \phi_{3m}^{*}(r,z)\cos(m\theta)\right], \end{aligned} \tag{10}$$

Then, we find that eqs (5) reduce to the forms

$$\left(B_m^2 + \frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\right)\left(\phi_{im}, \phi_{im}^{\bullet}\right)(r,z) = 0, \qquad i = 1, 2, 3 \tag{11}$$

where

$$B_m^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}, \tag{12}$$

We denote the Hankel transform of $\phi_{im}(r,z)$ with respect to r, namely [9] by the symbol

$$H_m\left[\phi_{im}(r,z); \quad r \to \xi\right] = \int_0^\infty r\phi_{im}(r,z)J_m(\xi r)dr\,. \tag{13}$$

J_m denotes the Bessel function of the first kind of order m. Linear partial differential equations for ϕ_{im} and $\phi_{im}^{\bullet}$ in a domain in which $r \geq 0$ can be transformed to ordinary differential equations for the Hankel transform $\overline{\phi}_{im}(\xi,z) = H_m\left[\phi_{im}(r,z); \ r \to \xi\right]$. Once these equations have been solved to yield an expression for $\overline{\phi}_{im}(\xi,z)$, we make use of the inverse theorem to obtain the solution $\phi_{im}(r,z)$ of the original partial differential equations; this states that

$$\phi_{im}(r,z) = H_m\left[\overline{\phi}_{im}(\xi,z); \quad \xi \to r\right]. \tag{14}$$

Using

$$H_m\left[B_m^2\phi_{im}(r,z); \quad r \to \xi\right] = -\xi^2\overline{\phi}_{im}(\xi,z) \tag{15}$$

we find that eqs (11) are equivalent to the equations

$$\left(\frac{\partial^2}{\partial z^2} - s_i^2\xi^2\right)\overline{\phi}_{im}(\xi,z) = 0 \tag{16}$$

and similarly for $\overline{\phi}^{\bullet}(\xi,z)$.

The transformed equations (16) can be solved in terms of exponentials and unknown functions of the transform variable ξ. The inverse transform is applied to recover the dependence on the r coordinate in the form of an integral equation. The solution is

$$\begin{aligned} \phi_{im}(r,z) &= \vartheta_i \int_0^\infty \xi^{-1} H_{im}(\xi, s_i z) J_m(\xi, r) d\xi, \\ \phi_{im}^{\bullet}(r,z) &= \vartheta_i \int_0^\infty \xi^{-1} H_{im}^{\bullet}(\xi, s_i z) J_m(\xi, r) d\xi, \end{aligned} \tag{17}$$

where

$$H_{im}(\xi,s_i z)=A_{im}(\xi)e^{\xi s_i z}+B_{im}(\xi)e^{-\xi s_i z},$$
$$H^{*}_{im}(\xi,s_i z)=A^{*}_{im}(\xi)e^{\xi s_i z}+B^{*}_{im}(\xi)e^{-\xi s_i z}, \tag{18}$$

$$\vartheta_1=\frac{s_2}{c_{44}(k+1)(s_1-s_2)},\ \vartheta_2=-\frac{s_1}{c_{44}(k+1)(s_1-s_2)},\ \vartheta_3=-\frac{1}{c_{44}s_3} \tag{19}$$

and where A_{im}, B_{im}, A^{*}_{im} and B^{*}_{im} are arbitrary functions of the transform variable ξ to be determined by using the given boundary conditions.

Equations (7) and (8) together with equations (10), and (17) to (19) represent the complete general solutions for equilibrium state of a transversely isotropic elastic medium. Note that in eqs (10) the first term produces deformations which are symmetric about $\theta=0$ and the second term yields deformations that are antisymmetric with respect to $\theta=0$ axis. It is noted that the solutions corresponding to $\phi^{*}_{im}(r,z)$ can be obtained immediately by making the following replacements $\phi_{im}(r,z)\to\phi^{*}_{im}(r,z)$, $A_{im}(\xi)\to A^{*}_{im}(\xi)$, $B_{im}(\xi)\to B^{*}_{im}(\xi)$, $\cos(m\theta)\to\sin(m\theta)$ and $\sin(m\theta)\to-\cos(m\theta)$. Therefore, in the ensuing analysis we consider solutions represented by only $\phi_{im}(r,z)$. In analysing half-space problem we assume that the components of the displacement vector all tend to zero as $z\to\infty$. This requires $A_{im}(\xi)=0$ and $A^{*}_{im}(\xi)=0$. The axisymmetric solution (corresponding to $m=0$) is already known.

Accordingly, the displacements u_r, u_θ and u_z and the stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{\theta z}$ and σ_{rz} can be expressed in the following form:

$$2c_{44}u_r(r,\theta,z)=-\sum_{m=0}^{\infty}\left\{\frac{1}{(k+1)(s_1-s_2)}\int_0^{\infty}\left[ks_2H_{1m}(\xi,s_1z)-s_1H_{2m}(\xi,s_2z)\right]G_{1m}(\xi r)d\xi+\frac{1}{s_3}\int_0^{\infty}H_{3m}(\xi,s_3z)G_{2m}(\xi r)d\xi\right\}\cos(m\theta),$$

$$2c_{44}u_\theta(r,\theta,z)=-\sum_{m=0}^{\infty}\left\{\frac{1}{(k+1)(s_1-s_2)}\int_0^{\infty}\left[ks_2H_{1m}(\xi,s_1z)-s_1H_{2m}(\xi,s_2z)\right]G_{2m}(\xi r)d\xi+\frac{1}{s_3}\int_0^{\infty}H_{3m}(\xi,s_3z)G_{1m}(\xi r)d\xi\right\}\sin(m\theta), \tag{20}$$

$$c_{44}u_z(r,\theta,z)=\frac{s_1s_2}{(k+1)(s_1-s_2)}\sum_{m=0}^{\infty}\int_0^{\infty}\left[\overline{H}_{1m}(\xi,s_1z)-k\overline{H}_{2m}(\xi,s_2z)\right]J_m(\xi r)d\xi\cos(m\theta);$$

$$\sigma_{rr}(r,\theta,z)=-\sum_{m=0}^{\infty}\left\{\frac{s_1s_2}{(s_1-s_2)}\int_0^{\infty}\xi\left[s_1H_{1m}(\xi,s_1z)-s_2H_{2m}(\xi,s_2z)\right]J_m(\xi r)d\xi\right.$$
$$-\frac{s_3^{\,2}}{(k+1)(s_1-s_2)}\frac{1}{r}\int_0^{\infty}\left[ks_2H_{1m}(\xi,s_1z)-s_1H_{2m}(\xi,s_2z)\right]\left[G_{1m}(\xi r)+mG_{2m}(\xi r)\right]d\xi$$
$$\left.-2s_3\frac{1}{r}\int_0^{\infty}H_{3m}(\xi,s_3z)J_{m+1}(\xi r)d\xi\right\}\cos(m\theta),$$

$$\sigma_{\theta\theta}(r,\theta,z)=-\sum_{m=0}^{\infty}\left\{\frac{s_1s_2}{(s_1-s_2)}\int_0^{\infty}\xi\left[s_1H_{1m}(\xi,s_1z)-s_2H_{2m}(\xi,s_2z)\right]J_m(\xi r)d\xi\right.$$
$$+\frac{2s_3^{\,2}}{(k+1)(s_1-s_2)}\frac{1}{r}\int_0^{\infty}\left[ks_2H_{1m}(\xi,s_1z)-s_1H_{2m}(\xi,s_2z)\right]$$
$$\left[G_{1m}(\xi r)+mG_{2m}(\xi r)-2\xi rJ_m(\xi r)\right]d\xi$$
$$\left.+2s_3\frac{1}{r}\int_0^{\infty}H_{3m}(\xi,s_3z)J_{m+1}(\xi r)d\xi\right\}\cos(m\theta),$$

$$\sigma_{zz}(r,\theta,z)=\frac{1}{s_1-s_2}\sum_{m=0}^{\infty}\int_0^{\infty}\xi\left[s_2H_{1m}(\xi,s_1z)-s_1H_{2m}(\xi,s_2z)\right]J_m(\xi r)d\xi\cos(m\theta), \tag{21}$$

$$\sigma_{r\theta}(r,\theta,z)=-\frac{s_3}{r}\sum_{m=0}^{\infty}\left\{\frac{2s_3}{(k+1)(s_1-s_2)}\int_0^{\infty}\left[ks_2H_{1m}(\xi,s_1z)-s_1H_{2m}(\xi,s_2z)\right]J_{m+1}(\xi r)d\xi\right.$$
$$\left.-\int_0^{\infty}H_{3m}(\xi,s_3z)\left[G_{1m}(\xi r)+mG_{2m}(\xi r)-\xi rJ_m(\xi r)\right]d\xi\right\}\sin(m\theta),$$

$$\sigma_{\theta z}(r,\theta,z)=-\sum_{m=0}^{\infty}\left\{\frac{s_1s_2}{2(s_1-s_2)}\int_0^{\infty}\xi\left[\overline{H}_{1m}(\xi,s_1z)-\overline{H}_{2m}(\xi,s_2z)\right]G_{2m}(\xi r)d\xi\right.$$
$$\left.+\frac{1}{2}\int_0^{\infty}\xi\overline{H}_{3m}(\xi,s_3z)G_{1m}(\xi r)d\xi\right\}\sin(m\theta),$$

$$\sigma_{rz}(r,\theta,z)=-\sum_{m=0}^{\infty}\left\{\frac{s_1s_2}{2(s_1-s_2)}\int_0^{\infty}\xi\left[\overline{H}_{1m}(\xi,s_1z)-\overline{H}_{2m}(\xi,s_2z)\right]G_{1m}(\xi r)d\xi\right.$$
$$\left.+\frac{1}{2}\int_0^{\infty}\xi\overline{H}_{3m}(\xi,s_3z)G_{2m}(\xi r)d\xi\right\}\cos(m\theta),$$

where

$$G_{1m}(\xi r) = J_{m+1}(\xi r) - J_{m-1}(\xi r),$$
$$G_{2m}(\xi r) = J_{m+1}(\xi r) + J_{m-1}(\xi r), \tag{22}$$

and

$$\overline{H}_{im}(\xi, s_i z) = A_{im}(\xi)e^{\xi s_i z} - B_{im}(\xi)e^{-\xi s_i z}, \qquad i = 1, 2, 3. \tag{23}$$

In many boundary value problems concerning the half-space $z \geq 0$ we have the boundary conditions

$$\sigma_{\theta z}(r,\theta,z) = 0,\ \sigma_{rz}(r,\theta,z) = 0. \tag{24}$$

From equations (21) we find that these conditions are satisfied if we take

$$B_{1m}(\xi) = B_{2m}(\xi) = \Psi_m(\xi),\ \ B_{3m}(\xi) = 0,$$
$$B^*_{1m}(\xi) = B^*_{2m}(\xi) = \Psi^*_m(\xi),\ \ B^*_{3m}(\xi) = 0 \tag{25}$$

and in addition $A_{im} = A^*_{im} = 0$ for half-space problem.

The solution corresponding to these forms of the arbitrary function leads to the expressions

$$w(r,\theta) = \frac{1}{G_z C}\sum_{m=0}^{\infty}\left\{H_m\left[\xi^{-1}\Psi_m(\xi);r\right]\cos(m\theta) + H_m\left[\xi^{-1}\Psi^*_m(\xi);r\right]\sin(m\theta)\right\}, \tag{26}$$

$$p(r,\theta) = \sum_{m=0}^{\infty}\left\{H_m\left[\Psi_m(\xi);r\right]\cos(m\theta) + H_m\left[\Psi^*_m(\xi);r\right]\sin(m\theta)\right\}, \tag{27}$$

where we have written w(r, θ) for $u_z(r, \theta, 0)$ and p(r, θ) for $-\sigma_{zz}(r, \theta, 0)$ and $G_z = c_{44}$, $C = (k+1)(s_1 - s_2)/\left[(k-1)s_1 s_2\right]$, and

$$H_m\left[\Psi_m(\xi);r\right] = \int_0^{\infty}\xi\Psi_m(\xi)J_m(\xi r)d\xi. \tag{28}$$

Assume an infinite media with the penny-shaped crack $\{z = 0,\ 0 \leq r \leq a\}$ opened up under a pressure $f(r,\theta)$. As the solution of this problem we take half-space with additional boundary conditions on the edge. So besides eqs (24) the unknown function $\Psi_m(\xi)$ must hold also the following conditions:

$$p(r,\theta) = f(r,\theta), \quad 0 \leq r \leq a$$
$$w(r,\theta) = 0, \qquad r > a. \tag{29}$$

Substituting equations (26) and (27) and expanding the function f(r,θ) into Fourier series result is in the following pair of dual integral equations

$$\sum_{m=0}^{\infty}\left\{H_m\left[\Psi_m(\xi);r\right]\cos(m\theta)+H_m\left[\Psi_m^*(\xi);r\right]\sin(m\theta)\right\}=$$
$$=\sum_{m=0}^{\infty}\left[f_m(r)\cos(m\theta)+f_m^*(r)\sin(m\theta)\right]; \qquad 0\le r\le a \tag{30}$$

$$\sum_{m=0}^{\infty}\left\{H_m\left[\xi^{-1}\Psi_m(\xi);r\right]\cos(m\theta)+H_m\left[\xi^{-1}\Psi_m^*(\xi);r\right]\sin(m\theta)\right\}=0, \qquad r>a \tag{31}$$

where

$$f_0(r)=\frac{1}{2\pi}\int_0^{2\pi}f(r,\theta)d\theta\,,$$
$$f_m(r)=\frac{1}{\pi}\int_0^{2\pi}f(r,\theta)\cos(m\theta)d\theta\,, \tag{32}$$
$$f_m^*(r)=\frac{1}{\pi}\int_0^{2\pi}f(r,\theta)\sin(m\theta)d\theta\,.$$

Equations (30) and (31) must hold for all values of θ ($0\le\theta\le2\pi$). If we equate the coefficients of cos(mθ) and sin(mθ) in both sides of eqs (30) and (31) we obtain

$$H_m\left[\Psi_m(\xi);r\right]=f_m(r), \qquad 0\le r\le a$$
$$H_m\left[\xi^{-1}\Psi_m(\xi);r\right]=0, \qquad r>a \tag{33}$$

and

$$H_m\left[\Psi_m^*(\xi);r\right]=f_m^*(r), \qquad 0\le r\le a$$
$$H_m\left[\xi^{-1}\Psi_m^*(\xi);r\right]=0, \qquad r>a \tag{34}$$

Consider integral equations (33). If we take the representation

$$\Psi_m(\xi)=\xi^{1/2}\int_0^a g_m(t)J_{m+1/2}(\xi t)dt \tag{35}$$

or integrating by parts:

$$\Psi_m(\xi)=\xi^{-1/2}\int_0^a G_m(t)J_{m-1/2}(\xi t)dt-\xi^{-1/2}g_m(a)J_{m-1/2}(\xi a)\,, \tag{36}$$

where, with the assumption that $t^{m-1/2}g_m(t) \to 0$ as $t \to 0^+$

$$G_m(t) = t^{1/2-m}\frac{d}{dt}\left[t^{m-1/2}g_m(t)\right] \tag{37}$$

and making use of the integral

$$\int_0^\infty t^{1-\lambda+\mu}J_\lambda(at)J_\mu(bt)dt = \frac{b^\mu\left(a^2-b^2\right)^{\lambda-\mu-1}}{2^{\lambda-\mu-1}a^\lambda\Gamma(\lambda-\mu)}H(a-b), \tag{38}$$

where H() denotes Heaviside's unit function and Γ denotes Γamma function [20] we find that if $r > a$ $H_m\left[\xi^{-1}\Psi_m(\xi);\, r\right] = 0$ while if $0 \le r \le a$

$$H_m\left[\Psi_m(\xi); r\right] = \sqrt{\frac{2}{\pi}}r^{-m}\int_0^r \frac{t^{m-1/2}G_m(t)}{\sqrt{r^2-t^2}}dt\,. \tag{39}$$

We have a similar result if in (37) and (38) we replace Ψ_m, g_m, G_m by Ψ_m^*, g_m^*, G_m^*.

It follows therefore that equations (33) and (34) are satisfied if $\Psi_m(\xi)$ and $\Psi_m^*(\xi)$ are given by representations of the type (35) $\left(\text{for } \Psi_m^*(\xi),\ g_m(t) \to g_m^*(t)\right)$ with

$$\begin{aligned}\sqrt{\frac{2}{\pi}}r^{-m}\int_0^r \frac{t^{m-1/2}G_m(t)}{\sqrt{r^2-t^2}}dt &= f_m(r),\\ \sqrt{\frac{2}{\pi}}r^{-m}\int_0^r \frac{t^{m-1/2}G_m^*(t)}{\sqrt{r^2-t^2}}dt &= f_m^*(r).\end{aligned} \tag{40}$$

The integral equations (40), which are of Abel type, are readily shown to have the solutions

$$\begin{aligned}t^{m-\frac{1}{2}}g_m(t) &= \sqrt{\frac{2}{\pi}}\int_0^t \frac{r^{m+1}f_m(r)}{\sqrt{t^2-r^2}}dr,\\ t^{m-\frac{1}{2}}g_m^*(t) &= \sqrt{\frac{2}{\pi}}\int_0^t \frac{r^{m+1}f_m^*(r)}{\sqrt{t^2-r^2}}dr,\end{aligned} \tag{41}$$

so that $\Psi_m(\xi)$ is given by (35) or (36) with g_m, G_m given by $(41)_1$, (37) and (32). Making use of (35) and (38) we find that, if $0 \le r \le a$,

$$H_m\left[\xi^{-1}\Psi_m(\xi); r\right] = \sqrt{\frac{2}{\pi}}r^m\int_r^a \frac{t^{-m-1/2}g_m(t)}{\sqrt{t^2-r^2}}dt \tag{42}$$

and similarly for $\Psi_m^*(\xi)$. From (42) it follows that the shape of the crack is given by the formula:

$$w(r,\theta)=\frac{1}{G_zC}\sqrt{\frac{2}{\pi}}\sum_{m=0}^{\infty}\left[r^m\int_r^a\frac{t^{-m-1/2}g_m(t)dt}{\sqrt{t^2-r^2}}\cos(m\theta)+r^m\int_r^a\frac{t^{-m-1/2}g_m^*(t)dt}{\sqrt{t^2-r^2}}\sin(m\theta)\right]. \tag{43}$$

Similarly, by making use of (36) and (38) it follows that

$$\lim_{r\to a^+}[2\pi(r-a)]^{1/2}H_m[\Psi_m(\xi);r]=-\frac{\sqrt{2}}{a}g_m(a). \tag{44}$$

The basic quantities of fracture mechanics are total crack energy:

$$W=\int_0^{2\pi}d\theta\int_0^a rw(r,\theta)f(r,\theta)dr, \tag{45}$$

J-integral:

$$J=\frac{1}{2\pi a}\frac{\partial W}{\partial a} \tag{46}$$

and the stress intensity factor [21]:

$$K_I=\lim_{r\to a^+}[2\pi(r-a)]^{1/2}[p(r,\theta)]_{r>a}. \tag{47}$$

From (43) and (44) we find

$$W=\frac{2\pi}{G_zC}\int_0^a t^{-1}\chi(t)dt, \tag{48}$$

where

$$\chi(t)=[g_0(t)]^2+\frac{1}{2}\sum_{m=1}^{\infty}\left\{[g_m(t)]^2+[g_m^*(t)]^2\right\} \tag{49}$$

and

$$K_I=\frac{\sqrt{2}}{a}\left\{g_0(a)+\sum_{m=1}^{\infty}[g_m(a)\cos(m\theta)+g_m^*(a)\sin(m\theta)]\right\}. \tag{50}$$

If we consider the special case of a penny-shaped crack subjected to axisymmetric normal load $p(r,\theta)=p_0=\text{const}$, then basic expressions yield:

$$f_0(r)=p_0,\quad g_0(t)=\sqrt{\frac{2}{\pi}}p_0t^{3/2},\quad g_m(t)=g_m^*(t)=0, \tag{51}$$

$$w(r)=\frac{2p_0}{\pi G_zC}\sqrt{a^2-r^2},\ r<a \tag{52}$$

$$W=\frac{4p_0^2a^3}{3G_zC},$$

$$J=\frac{2p_0^2a}{\pi G_zC}, \tag{53}$$

$$K_I = \frac{2p_0 a^{1/2}}{\sqrt{\pi}}.$$

Example results of eqs (53) together with the numerical tests are presented in chapter Numerical results.

If the penny-shaped crack is subjected to normal asymmetric load $p(r,\theta) = p_0$ on a sector $0 \le r \le a$, $|\theta| \le \alpha$ (function $p(r,\theta)$ is even function of θ), then

$$f(r,\theta) = \begin{cases} \frac{1}{\pi} p_0 \left[\alpha + 2\sum_{m=1}^{\infty} \frac{\sin(m\alpha)}{m} \cos(m\theta) \right], & 0 \le r \le a \\ 0, \quad r > a \end{cases} \tag{54}$$

$$g_0(t) = \sqrt{\frac{2}{\pi}} p_0 t^{3/2} \frac{\alpha}{\pi},$$

$$g_m(t) = \sqrt{\frac{2}{\pi}} p_0 t^{3/2} \varepsilon_m \frac{\prod_{k=0}^{n}(m-2k)}{\prod_{k=0}^{n}(m-2k+1)} \frac{\sin(m\alpha)}{m}, \tag{55}$$

where $\varepsilon_m = 1$, $n = \frac{m-1}{2}$ for $m = 1, 3, 5, \ldots$ and $\varepsilon_m = \frac{2}{\pi}$, $n = \frac{m}{2} - 1$ for $m = 2, 4, 6, \ldots$.

The parameters of fracture mechanics are:

$$W = \frac{4p_0^2 a^3}{3G_z C} \left\{ \frac{\alpha^2}{\pi^2} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\varepsilon_m \frac{\prod_{k=0}^{n}(m-2k)}{\prod_{k=0}^{n}(m-2k+1)} \frac{\sin(m\alpha)}{m} \right]^2 \right\}, \tag{56}$$

and

$$K_I = \frac{2p_0 a^{1/2}}{\sqrt{\pi}} \left[\frac{\alpha}{\pi} + \sum_{m=1}^{\infty} \varepsilon_m \frac{\prod_{k=0}^{n}(m-2k)}{\prod_{k=0}^{n}(m-2k+1)} \frac{\sin(m\alpha)}{m} \cos(m\theta) \right]. \tag{57}$$

On Figure 1 is presented nondimensional parameter include in brackets of equation (57) which shows influence of magnitude of loading area on values of stress intensity factor obtained from equations 53 for load distributed on whole crack for θ coordinate.

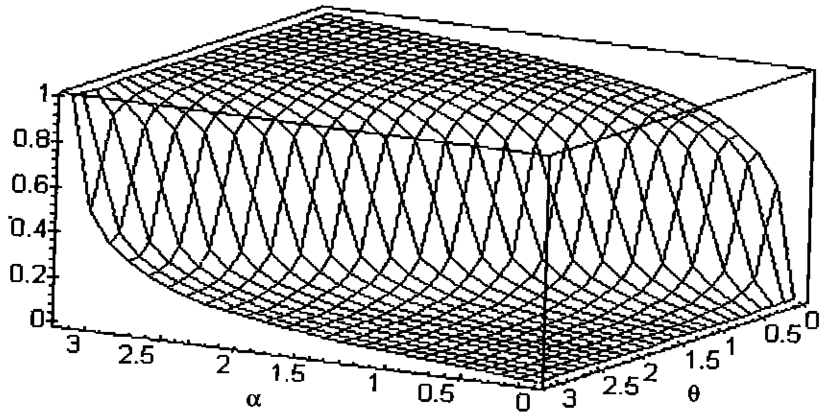

Figure 1. Influence of magnitude of a loading area on stress intensity factor K_I

Fracture mechanics with ABAQUS system.

ABAQUS offers the evaluations of two contour integrals for fracture mechanics studies: the J integral and C_t integral. The J-integral is usually used in rate-independent quasi-static fracture analysis to characterise the energy release associated with crack growth, and C_t integral can be used for time-dependent creep behaviour, where it characterises creep crack deformation under certain creep conditions, including transient crack growth.

Several contour integral evaluation at each location along the crack front may be compute. Each evaluation may be thought of as the virtual motion of a block of material (defined by contours) surrounding the crack tip.

For consideration of Linear Elastic Fracture Mechanics - LEFM and the stress concentration at a sharp crack in a linear elastic infinite plate in ABAQUS we have the following relations for mode I (opening mode):

- stress σ_{zz} at the tip of the crack is:

$$\sigma_{zz}(r = a + \rho,\ z = 0) \underset{\rho \to 0}{=} \sigma_{zz}^{\infty} \frac{\sqrt{\pi a}}{\sqrt{2\pi\rho}} = \frac{1}{\sqrt{2\pi\rho}} K_I, \qquad (58)$$

- stress intensity factor K_I

$$K_I = \lim_{\rho \to 0} \sigma_{zz}(r = a + \rho,\ z = 0)\sqrt{2\pi\rho}, \qquad (59)$$

- J-integral

$$J = \int_{\Gamma} \left(Wn_1 - \frac{\partial u}{\partial x_1} \sigma n \right) ds . \tag{60}$$

Relation between mention above magnitudes for plane strain and isotropic media has the form:

$$J = \frac{1 - \nu^2}{E} K_I^2 . \tag{61}$$

The value of J-integral is independent of contour, Γ, taken around the crack tip and it means that J is path independent.

Most fracture mechanics problems can be solved satisfactorily using only small-strain analysis. Focused meshes that should normally be used for small-strain fracture mechanics evaluations is shown in Figure 2.

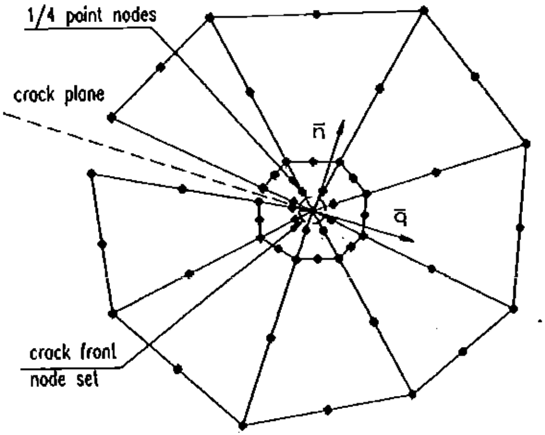

Figure 2. Typical focused mesh for fracture mechanics evaluations

The crack tip strain singularity depends on the material model used, and is introduced to ABAQUS using different types of constraints of the nodes at the crack tip of a focused mesh. For two-dimensional problems there is:

1. for linear elasticity - with a square root singularity, $\rho^{-1/2}$ (where ρ is crack tip radius), two corner nodes at the tip and the midside node between them must be tied together,
2. for perfect plasticity - with ρ^{-1} singularity, corner nodes are moved independently, midside nodes remain at midside points rather than being moved to the 1/4 points,

3. for power-law hardening - with combined $\rho^{-1/2}$ and ρ^{-1} singularity, midside nodes are moved to the quarter points, tip nodes are allowed to move independently.

All these capabilities were tested on the example.

Numerical results

Problem of penny shaped crack in transversely isotropic solid has been considered as the numerical experiment to verify the obtained results sensitivity. Radius of the crack was considered as 2.0m. and result investigations was performed to the depth of 20.0m. Distributed load on both sides of the crack is 1 MPa. The model of the problem has symmetry about plane of the crack and due to axial symmetry of the load in numerical FEM researches, it was necessary to define only a geometry of the model in the plane r-z. Such half-space was model by rather coarse mesh of elements: 131 CAX8R elements (8-node biquadratic axisymmetric elements with reduced integration) and 14 CINAX5R elements (one-way infinite axisymmetric 5-node quadratic elements, reduced integration). Three contours to evaluate J-integral have been introduced to the model (three values in Table 1 for each test). Introduced material is characterized by five independent constants (introduced to ABAQUS using ENGINEERING CONSTANTS parameter in ELASTIC option). Investigations were provided for cadmium and magnesium, the more often occurring natural anisotropic materials and isotropic media too. Some numerical results for cadmium, strongly anisotropic material, together with the theoretical obtained values are presented below.

Tests within magnesium and for isotropic material have good convergence with theoretical solution [magnesium - 50.679N/m, isotropy (E=78000Mpa, ν=0.25) - 30.607N/m.] and are omitted in the paper. Differences here are from 1% to 2%.

Numerical tests in ABAQUS with partial loaded crack are omitted here because the 3D model is very large and it cannot be run on most workstations.

Table 1. Values of J-integrals for cadmium

Method of evaluation		J-integral $\left[\frac{N}{m}\right]$
Theoretical solution from equation (53)		63.023
ABAQUS tests for linear elasticity	elements CAX8 with middle nodes	70.215 72.809 72.853
	elements CAX8R with middle nodes	69.810 72.804 72.838
	elements CAX8 with 1/4 nodes	78.115 77.095 77.275
	elements CAX8R with 1/4 nodes	78,755 77.129 77.337

Concluding remarks

Numerical tests show some information of J-integral evaluation in ABAQUS and obtained theoretical solution. Here are some of them:

a) theoretical and numerical results differs for strongly anisotropic materials but has good convergence for near isotropic or isotropic ones,

b) numerical results have nearly correct values only for linear elasticity, but not for perfect plasticity or power-law hardening model of behaviour,

c) better results obtained with the midsides nodes laying in the midside points, not 1/4 nodes,

d) differences of J-integral for contours of evaluation indicate a need for mesh refinement,

e) elements within reduced integration or without it give almost the same results.

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