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Generalization of Fatigue Fracture Criteria for Multiaxial Sinusoidal Loadings in the Range of Random Loadings

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ABSTRACT A new, generalized criterion of maximum shear and normal stresses in a fracture plane has been formulated for multiaxial random loadings. From this criterion three particular forms of previous fatigue criteria for a triaxial random stress state result, together with some classical criteria for a multiaxial sinusoidal stress state. There are some formal limitations which make a generalization for the range of multiaxial random loadings impossible in the case of three other known criteria for multiaxial sinusoidal loadings, proposed by Gough–Pollard, Findley, and McDiarmid. The limitations have been analytically determined.

Introduction

Multiaxial fatigue has been investigated for about 100 years and many different mathematical models of the limit state of strength have been formulated. At present we know more than 30 criteria of fatigue strength for multiaxial loading, e.g. (1)–(5). From an analysis of these criteria it appears that some parameters are unsuitable and can create misinterpretations. Conversely, several important parameters are either neglected or misused. Misuse of parameters occurs, for example, when terms employed in static strength hypotheses are replaced by the cyclic range; e.g., by replacing static principal stresses by amplitudes of sinusoidal stresses. Changes in directions of principal stresses have been also ignored for many years. Studies concerning the influence on fatigue of phase displacement between stresses can be instigated by tests in which changes of principal directions occur during a full cycle of loading, e.g., (6)(7).

Criteria for random multiaxial stress conditions should take into account the random changes of directions of the principal stresses (8)–(10), and so positions of the principal axes at any time, t, should be determined. Using a simple static rule for stresses $\sigma_1(t) \geqslant \sigma_2(t) \geqslant \sigma_3(t)$ or, for isotropic bodies, an equivalent rule for strains $\varepsilon_1(t) \geqslant \varepsilon_2(t) \geqslant \varepsilon_3(t)$ leads to the observation that in many fatigue tests of specimens under multiaxial sinusoidal loading the directions of the principal stresses and strains change. The directions $\sigma_1(t)$ and $\varepsilon_1(t)$ also change in a sinusoidal tension–compression (uniaxial) test (9)(11).

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In papers (8)–(10) five fatigue fracture criteria are presented for a random triaxial stress state of an isotropic body in which the stress components have zero mean values. It was assumed that a random tensor of stress is a narrow-band six-dimensional stationary and ergodic Gaussian process. Stress and strain fatigue criteria were formulated on the assumption that fatigue fracture of materials is defined only by those components of the stress or strain states which were acting on an existing fatigue fracture plane (or are acting on an expected fracture plane). In the case of energy based criteria it was assumed that fatigue fracture is determined by the amount of energy equal to the specific work related to the strain in one direction, namely that connected with the fatigue fracture plane. It has been assumed also that the position of the fatigue fracture plane is described by mean values $\hat{l}_n$, $\hat{m}_n$, $\hat{n}_n$ $(n = 1, 2, 3)$ of the direction cosines of the principal axes of stress.

In this paper a new, generalized form of one of the five criteria mentioned above for random triaxial stressing is presented. Limitations are also presented that make it impossible to create a general theory for the other criteria of random loading under sinusoidal cycling.

A generalized criterion involving the maximum and normal stresses acting on a fracture plane

Let us make the following assumptions.

(1) Fatigue fracture is caused by the normal stress $\sigma_n(t)$ and shear stress $\tau_{ns}(t)$ acting in the $\bar{s}$ direction, on a fracture plane with a normal $\bar{n}$.

(2) The direction $\bar{s}$ on the fracture plane coincides with the mean direction of the maximum shear stress $\tau_{n\max}(t)$.

(3) In the limit state that conforms to the fatigue strength, the maximum value of combined $\tau_{ns}(t)$ and $\sigma_n(t)$ stresses under multiaxial random loading satisfies the following equation

$$\{B\tau_{ns}(t) + K\sigma_n(t)\}_{\max,t} = F \tag{1}$$

where B = constant for a particular form of equation (1), and K and F = material constants determined from sinusoidal fatigue tests.

The left side of equation (1) can be written as $\{W(t)\}_{\max,t}$ and should be interpreted as the 100 per cent quantile of the random variable W. If the maximum value of $\{W(t)\}_{\max,t}$ exceeds the value of F, then damage will accumulate resulting in fracture (Fig. 1). The random process $W(t)$ can be interpreted as a stochastic process of the fatigue strength of a material.

The positions of the unit vectors $\bar{n}$ and $\bar{s}$ are determined with the mean direction cosines of the principal axes of stress or strain. The proposed averaging method uses weight functions, which lead to conformity between the expected position of the fatigue fracture plane and experimental results (9)(11)(12).

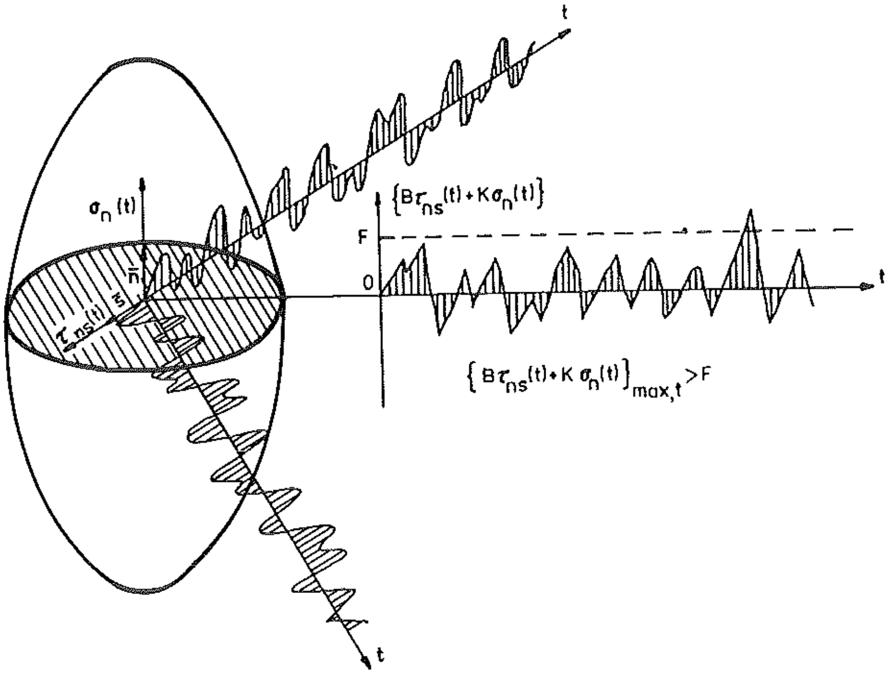

Fig 1 Random stresses in the fracture plane determining fatigue life under multiaxial loadings

Many results of fatigue tests under multiaxial sinusoidal loading show that for elasto-brittle materials the fatigue fracture plane is often perpendicular to the direction of the normal stress having the maximum amplitude σ_{a1}. For elasto-plastic materials the fracture plane is often one of two planes where shear stresses have the maximum amplitude τ_{a1}. There are also intermediate positions of fatigue fracture planes.

In some special cases we can expect that the unit vector, $\bar{n}$, coincides with the mean direction of the maximum normal stress $\sigma_1(t)$ (Fig. 2), i.e.

$$\bar{n} = \hat{l}_1\bar{i} + \hat{m}_1\bar{j} + \hat{n}_1\bar{k} \tag{2}$$

where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors of the axes x, y, z, respectively. The normal stress $\sigma_n(t)$ in the direction $\bar{n}$ is equal to

$$\sigma_n(t) = \hat{l}_1^2\sigma_{xx}(t) + \hat{m}_1^2\sigma_{yy}(t) + \hat{n}_1^2\sigma_{zz}(t) + 2\hat{l}_1\hat{m}_1\sigma_{xy}(t) + 2\hat{l}_1\hat{n}_1\sigma_{xz}(t) + 2\hat{m}_1\hat{n}_1\sigma_{yz}(t) \tag{3}$$

In other special cases we can expect that the unit vector, $\bar{s}$, coincides with the mean direction of the maximum shear stress $\tau_1(t)$, and the fracture plane is determined by the mean position of one of the two planes on which $\tau_1(t)$ acts (Fig. 3).

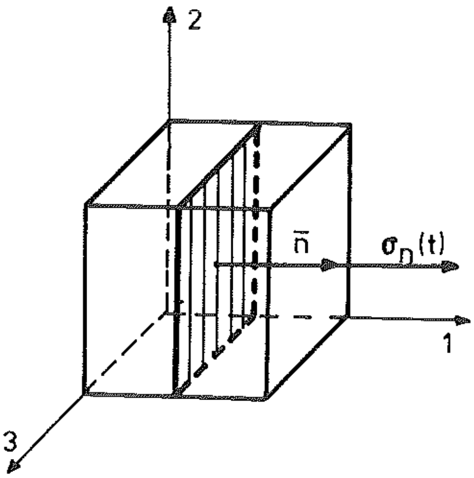

Fig 2 Direction of the unit vector, $\bar{n}$, when the fatigue fracture plane is perpendicular to the mean direction of the maximum normal stress $\sigma_1(t)$. Here 1, 2, 3 are the mean positions of the principal stress axes

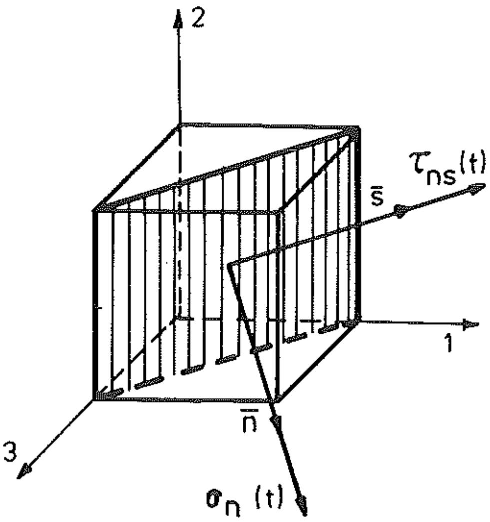

Fig 3 Direction of the unit vector, $\bar{s}$, on the fatigue fracture plane determined by the mean position of one of two planes on which the maximum shear stress $\tau_1(t)$ acts

According to Fig. 3 unit vectors $\bar{n}$ and $\bar{s}$ can be written in the following way

$$\bar{n} = \frac{\hat{l}_1 + \hat{l}_3}{\sqrt{2}}\,\bar{i} + \frac{\hat{m}_1 + \hat{m}_3}{\sqrt{2}}\,\bar{j} + \frac{\hat{n}_1 + \hat{n}_3}{\sqrt{2}}\,\bar{k} \tag{4}$$

$$\bar{s} = \frac{\hat{l}_1 - \hat{l}_3}{\sqrt{2}}\,\bar{i} + \frac{\hat{m}_1 - \hat{m}_3}{\sqrt{2}}\,\bar{j} + \frac{\hat{n}_1 - \hat{n}_3}{\sqrt{2}}\,\bar{k} \tag{5}$$

Normal stress $\sigma_n(t)$ in direction $\bar{n}$ and the stress $\tau_{ns}(t)$, in direction $\bar{s}$ perpendicular to $\bar{n}$, are equal to

$$\begin{aligned}\sigma_n(t) = {} & \frac{(\hat{l}_1 + \hat{l}_3)^2}{2}\,\sigma_{xx}(t) + \frac{(\hat{m}_1 + \hat{m}_3)^2}{2}\,\sigma_{yy}(t) + \frac{(\hat{n}_1 + \hat{n}_3)^2}{2}\,\sigma_{zz}(t) \\ & + (\hat{l}_1 + \hat{l}_3)(\hat{m}_1 + \hat{m}_3)\sigma_{xy}(t) + (\hat{l}_1 + \hat{l}_3)(\hat{n}_1 + \hat{n}_3)\sigma_{xz}(t) \\ & + (\hat{m}_1 + \hat{m}_3)(\hat{n}_1 + \hat{n}_3)\sigma_{yz}(t)\end{aligned} \tag{6}$$

Similarly

$$\begin{aligned}\tau_{ns}(t) = {} & \frac{\hat{l}_1^2 - \hat{l}_3^2}{2}\,\sigma_{xx}(t) + \frac{\hat{m}_1^2 + \hat{m}_3^2}{2}\,\sigma_{yy}(t) + \frac{\hat{n}_1^2 - \hat{n}_3^2}{2}\,\sigma_{zz}(t) \\ & + (\hat{l}_1\hat{m}_1 - \hat{l}_3\hat{m}_3)\sigma_{xy}(t) + (\hat{l}_1\hat{n}_1 - \hat{l}_3\hat{n}_3)\sigma_{xz}(t) \\ & + (\hat{m}_1\hat{n}_1 - \hat{m}_3\hat{n}_3)\sigma_{yz}(t)\end{aligned} \tag{7}$$

In the case of a general position of the fatigue fracture plane (Fig. 4) directions of vectors $\bar{n}$ and $\bar{s}$ can be written with the mean direction cosines $\hat{\alpha}_{ni}$, $\hat{\alpha}_{nj}$ and $\hat{\beta}_{si}$,

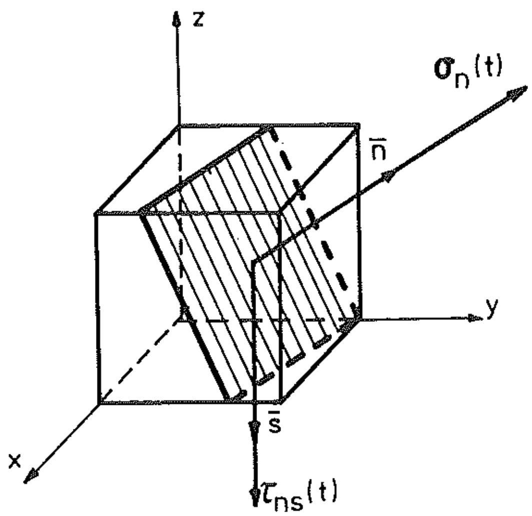

Fig 4 Directions of vectors, $\bar{n}$ and $\bar{s}$, and of stresses, $\sigma_n(t)$, $\tau_{ns}(t)$, in the case of any position of the fatigue fracture plane in relation to the system of axes x, y, z

$\hat{\beta}_{sj}$ $(i, j = x, y, z)$ in relation to the constant system of axes x, y, z in the following way

$$\bar{n} = \hat{\alpha}_{nx}\bar{i} + \hat{\alpha}_{ny}\bar{j} + \hat{\alpha}_{nz}\bar{k} \tag{8}$$

$$\bar{s} = \hat{\beta}_{sx}\bar{i} + \hat{\beta}_{sy}\bar{j} + \hat{\beta}_{sz}\bar{k} \tag{9}$$

whereupon the criterion expressed by equation (1) has the following form

$$\{B\hat{\beta}_{si}\hat{\beta}_{sj}\sigma_{ij}(t) + K\hat{\alpha}_{ni}\hat{\alpha}_{nj}\sigma_{ij}(t)\}_{\max,t} = F \tag{10}$$

Some particular cases of the generalized criterion

A choice of constants, B, K, and F, together with an assumed position of the fatigue fracture plane, leads to particular forms of equation (1). Three special cases are considered here.

(A) For $B = 0$, $K = 1$, $F = \sigma_{az}$, i.e., the fatigue limit under uniaxial sinusoidal tension–compression, and with the assumption that the fatigue fracture plane is perpendicular to the mean direction of the maximum normal stress, $\sigma_1(t)$, we obtain

$$\{\sigma_n(t)\}_{\max,t} = \sigma_{az} \tag{11}$$

i.e., the criterion of the maximum normal stress on the fracture plane (**8**)–(**9**)(**10**). It should be noted that equation (3) is included into the criterion expressed by equation (11). In order to show that in a particular case equation (11) is reduced to the classical criterion of the maximum normal stress for sinusoidal loading (**2**)(**4**)

$$\sigma_{a1} = \sigma_{az} \tag{12}$$

it is sufficient to assume that the normal stress having the maximum amplitude acts along the axis, x, i.e., $\sigma_{xx}(t) = \sigma_{a1} \sin \omega t$ and that $\hat{l}_1 = 1$. Then, according to equations (3) and (11) we obtain

$$\{\sigma_n(t)\}_{\max,t} = \{\sigma_{a1} \sin \omega t\}_{\max,t} = \sigma_{a1} = \sigma_{az}$$

(B) For $B = 2$, $K = 0$, $F = \sigma_{az}$ (or $B = 1$, $K = 0$, $F = (\sigma_{az})/2$) and with the assumption that the fatigue fracture plane is determined by the mean position of one of two planes on which the maximum shear stress $\tau_1(t)$ acts, we obtain

$$\{2\tau_{ns}(t)\}_{\max,t} = \sigma_{az} \tag{13}$$

i.e., the criterion of the maximum shear stress on the fracture plane and the criterion of strain energy of distortion in the direction of the maximum shear stress on the fracture plane (**8**)–(**10**). These criteria have the same mathematical form.

In the particular case of sinusoidal loading when the normal stress has the maximum amplitude acting along the x axis and the normal stress with the minimum amplitude acting along the z axis, i.e.

$$\sigma_{xx}(t) = \sigma_{a1} \sin \omega t, \quad \sigma_{zz}(t) = \sigma_{a3} \sin \omega t$$

and when

$$\hat{l}_1 = \hat{n}_3 = 1$$

then, according to equations (7) and (13) we obtain

$$\{2\tau_{ns}(t)\}_{max,t} = \{\sigma_{a1} \sin \omega t - \sigma_{a3} \sin \omega t\}_{max,t} = \sigma_{a1} - \sigma_{a3} = \sigma_{az} \quad (14)$$

In such a way the classical criterion of the maximum shear stresses for sinusoidal loading is obtained (**2**)(**4**).

(*C*) For $B = 1$ we obtain

$$\{\tau_{ns}(t) + K\sigma_n(t)\}_{max,t} = F \quad (15)$$

i.e., the criterion of the maximum shear and normal stresses on the fracture plane (**8**)–(**10**). From equation (15) some other criteria for sinusoidal loading result. For example, taking

$$\tau_{ns}(t) = \tau_{an} \sin \omega t, \quad \sigma_n(t) = \sigma_{an} \sin \omega t, \quad K = \frac{2\tau_{az}}{\sigma_{az}} - 1, \quad F = \tau_{az}$$

(fatigue limit under sinusoidal torsion) we obtain

$$\left\{\tau_{an} \sin \omega t + \left(\frac{2\tau_{az}}{\sigma_{az}} - 1\right)\sigma_{an} \sin \omega t\right\}_{max,t} = \tau_{an} + \left(\frac{2\tau_{az}}{\sigma_{az}} - 1\right)\sigma_{an} \quad (16a)$$

or, after transformation

$$\frac{\tau_{an}}{\sigma_{az}} + \left(2 - \frac{\sigma_{az}}{\tau_{az}}\right)\frac{\sigma_{an}}{\sigma_{az}} = 1 \quad (16b)$$

Thus we obtained the criterion formulated by Matake (**13**). He assumed that the fracture plane is the plane where the shear stress having the maximum amplitude, τ_{a1}, acts. The Stanfield criterion is also obtained when the same fracture plane is assumed (**14**). Equation (16) can be reduced, after transformation to the 'ellipse arc' criterion formulated by Gough (**15**). After substituting

$$\tau_{ns}(t) = \left[\sqrt{\left\{\left(\frac{\sigma_a}{2}\right)^2 + \tau_a^2\right\}} \sin 2\theta\right] \sin \omega t = \tau_{an} \sin \omega t$$

$$\sigma_n(t) = \left[\frac{\sigma_a}{2} + \sqrt{\left\{\left(\frac{\sigma_a}{2}\right)^2 + \tau_a^2\right\}} \cos 2\theta\right] \sin \omega t = \sigma_{an} \sin \omega t$$

$$K = k, \; F = \frac{\sigma_{az}}{2}\{k + \sqrt{(1 + k^2)}\} = \tau_{az}\sqrt{(1 + k^2)}$$

into equation (15) we obtain the relationship formulated by Findley *et al.* (**16**) and Stulen and Cummings (**17**). They assumed that the normal of the critical shear plane (fracture plane) $\bar{n}$ formed an angle θ with the direction of the normal stress having the maximum amplitude σ_{a1}, and also that a coefficient k can be calculated from

$$\frac{\sigma_{az}}{\tau_{az}} = \frac{2}{1 + \{1/\sqrt{(1 + k^2)}\}} \qquad \text{or } \tan 2\theta = 1/k$$

Limitations of other criteria

In papers (**8**) and (**9**) it has been shown analytically that scalar quantities of:

(i) total specific strain energy;
(ii) specific strain energy of distortion;
(iii) octahedral normal and shear stresses;
(iv) modulus of shear stress;

cannot determine fatigue fracture under multiaxial random loading when the directions of the principal stresses change.

It is worth investigating if the known criteria for multiaxial sinusoidal loading, based on quantities different from those mentioned above, can be generalized for random loading.

Comments on the Gough–Pollard criterion

The empirical 'ellipse quadrant' criterion for elasto-plastic materials

$$\left(\frac{\sigma_a}{\sigma_{az}}\right)^2 + \left(\frac{\tau_a}{\tau_{az}}\right)^2 = 1 \tag{17}$$

where σ_a and τ_a are amplitudes of the normal and shear stresses, respectively. This can be generalized for the range of random loading in the following way

$$\left\{\left(\frac{\sigma_b(t)}{\sigma_{az}}\right)^2 + \left(\frac{\tau_t(t)}{\tau_{az}}\right)^2\right\}_{\max,t} = 1 \tag{18}$$

where $\sigma_b(t)$ and $\tau_t(t)$ are random normal and shear stresses, respectively.

These stresses are caused by bending and torsion. Expression (18) can be useful when bending and torsion are caused by one random force, i.e., when the equality $\sigma_b(t) = a\tau_t(t)$ occurs; a is a constant factor of proportionality.

McDiarmid (**18**) rearranged equation (17) and obtained the following form

$$\tau_{an}^2 + \left(\frac{4\tau_{az}^2}{\sigma_{az}^2} - 1\right)\sigma_{an}^2 = \tau_{az}^2 \tag{19}$$

McDiarmid showed that fatigue fracture is determined by amplitudes of shear stress, τ_{an}, and normal stress, σ_{an}, on the plane of the maximum shear stress with a normal, $\bar{n}$. He thereby showed an interesting physical sense of the Gough–Pollard criterion.

After rearranging, equation (19) can be written in the following way

$$\sqrt{\left\{\left(4 - \frac{\sigma_{az}^2}{\tau_{az}^2}\right)\left(\sigma_{an}^2 + \frac{\sigma_{az}^2}{4\tau_{az}^2 - \sigma_{az}^2}\,\tau_{an}^2\right)\right\}} = \sigma_{az} \tag{20}$$

It follows that

$$\left[\sqrt{\left\{\left(4 - \frac{\sigma_{az}^2}{\tau_{az}^2}\right)\left(\sigma_n^2(t) + \frac{\sigma_{az}^2}{4\tau_{az}^2 - \sigma_{az}^2}\,\tau_{ns}^2(t)\right)\right\}}\right]_{\max,t} = \sigma_{az} \tag{21}$$

where $\sigma_n(t)$ is a random normal stress on a fracture plane – equation (6) – and $\tau_{ns}(t)$ is a random shear stress in the mean direction, $\bar{s}$, of the maximum shear stress on the plane – equation (7) – and the fracture plane is determined by the mean position of one of two planes of the maximum shear stress $\tau_1(t)$ (Fig. 3). Hence, equation (20) can be a generalization for random loading.

A reduced stress $\sigma_{red}(t)$ can be calculated according to equation (21). For this purpose let us compare the stochastic processes of fatigue strength, $W(t)$, under a triaxial stress state with that of the uniaxial state at any time, t. Let us denote a stress under the uniaxial stress state by $\sigma_{red}(t) = \sigma_{xx}(t)$ and let us assume that under this stress state $l_1 = 1$. According to equations (6), (7), and (21) a stochastic process of fatigue strength, under a uniaxial stress state, is equal to

$$W(t) = \sqrt{\left(\left(4 - \frac{\sigma_{az}^2}{\tau_{az}^2}\right)\left[\{\tfrac{1}{2}\sigma_{red}(t)\}^2 + \frac{\sigma_{az}^2}{4\tau_{az}^2 - \sigma_{az}^2}\{\tfrac{1}{2}\sigma_{red}(t)\}^2\right]\right)} = \sigma_{red}(t)$$

Thus

$$\sigma_{red}(t) = \sqrt{\left[\left(4 - \frac{\sigma_{az}^2}{\tau_{az}^2}\right)\left\{\sigma_n^2(t) + \frac{\sigma_{az}^2}{4\tau_{az}^2 - \sigma_{az}^2}\tau_{ns}^2(t)\right\}\right]} \tag{22}$$

From equation (22) it appears that $\sigma_{red}(t)$ depends on the components of the stress state $\sigma_{i,j}(t)$, $(i, j = x, y, z)$ in a non-linear way. It means that when a random tensor has a probability density distribution of a normal type, then the uniaxial process of the reduced stress has a probability density function different from the one of the normal type. But from the physical point of view the main fault is the fact that in the process $\sigma_{red}(t)$, a mean component is different from 0 ($\hat{\sigma}_{red} \neq 0$), although all the mean components of random tensor components are equal to 0 ($\hat{\sigma}_{ij} = 0$). This is a reason why it is not possible to assume equation (22) when calculating the fatigue life of materials. From this fact it also appears that any attempt to generalize the Gough–Pollard criterion for multiaxial random loading is difficult from the theoretical point of view.

Comments on the criterion of a stress state vector

This criterion was formulated and verified by Findley (2). He assumed that fatigue fracture was determined by the amplitude of a stress state vector equal to a vector sum of amplitudes of the three normal stresses (the principal stress) $\sigma_{a1}, \sigma_{a2}, \sigma_{a3}$. This criterion has the following form

$$\sqrt{(\sigma_{a1}^2 + \sigma_{a2}^2 + \sigma_{a3}^2)} = \sigma_{az} \tag{23}$$

Findley did not attribute any physical reality to his criterion; in his opinion it was no worse than other empirical criteria.

It is necessary to state that the amplitude of the total octahedral stress

$$\sigma_{ac,oct} = \sqrt{(\sigma_{aoct}^2 + \tau_{aoct}^2)} = 1/\sqrt{3}\sqrt{(\sigma_{a1}^2 + \sigma_{a2}^2 + \sigma_{a3}^2)} \tag{24}$$

where σ_{aoct} and τ_{aoct} are the amplitudes of octahedral normal and shear stress, respectively

$$\sigma_{aoct} = 1/3(\sigma_{a1} + \sigma_{a2} + \sigma_{a3})$$

and

$$\tau_{aoct} = 1/3\sqrt{\{(\sigma_{a1} - \sigma_{a2})^2 + (\sigma_{a2} - \sigma_{a3})^2 + (\sigma_{a1} - \sigma_{a3})^2\}}$$

Thus the value of $\sigma_{ac,oct}$ is equal to 0.58 of the amplitude of the stress state vector. It also appears that fatigue fracture is determined by the amplitude of the total octahedral stress, $\sigma_{ac,oct}$, and, if so, a criterion formulated on this basis is the same as equation (23).

This criterion has the same physical sense as, for example, the criterion of the maximum shear stresses and it should be called the total octahedral stress criterion. As mentioned above, scalar quantities of octahedral normal and shear stresses are not suitable for formulating a criterion for multiaxial random loading when the directions of the principal stresses change.

Comments on McDiarmid's criterion

The idea expressed in McDiarmid's criterion (**18**), namely

$$\tau_{an} + K\sigma_{an}^{r} = F \tag{25}$$

where

$$K = \frac{\tau_{az} - (\sigma_{az}/2)}{(\sigma_{az}/2)^{3/2}} \quad \text{for} \quad 1 < \frac{\sigma_{az}}{\tau_{az}} < 2$$

$$r = 3/2$$

$$F = \tau_{az}$$

is a basis for the following proposal of a criterion for multiaxial random loading

$$\{\tau_{ns}(t) + K\sigma_{n}^{r}(t)\}_{max,t} = F \tag{26}$$

Equation (26) was cited in papers (**8**)–(**10**). Its sense is the same as in the case of equation (1). An attempt to use the criterion given by equation (26) for evaluating fatigue life shows that, for $r = 3/2$, imaginary values of $\sigma_{red}(t)$ are obtained and, hence, it is not suitable from the physical point of view. Calculating $\sigma_{red}(t)$ we also obtain

$$\tfrac{1}{2}\sigma_{red}(t) + K\left\{\frac{\sigma_{red}(t)}{2}\right\}^{3/2} = \tau_{ns}(t) + K\sigma_{n}^{3/2}(t) \tag{27}$$

Thus $\sigma_{red}(t)$ becomes unreal when $\sigma_n(t) < 0$.

Conclusions

(1) From a generalized criterion involving the maximum shear and normal

stresses on a fracture plane, three particularly well known fatigue criteria result for multiaxial random loading.

(2) For particular cases, the generalized criterion coincides with:
   (i) the maximum principal stress criterion;
   (ii) the maximum shear stress criterion;
   (iii) the criteria formulated by Stanfield, Matake, Findley *et al.* and Stulen and Cummings.

(3) Formal limitations preclude a generalization for random loading of:
   (i) the 'ellipse quadrant' criterion according to Gough–Pollard;
   (ii) the criterion of a stress-state vector proposed by Findley;
   (iii) the McDiarmid criterion.

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