THE ANALYSIS FOR THE CORNER REGION AND CRACK TIP OF PLATE WITH TRANSVERCE SHEAR DEFORMATION

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INTRODUCTION

For the analysis of the bending problem for the plates containing corners and cracks, the high-order theory of plate should be used. The singular field for the cracked plates are investigated by eigen-expansion technique in the Ref. [1][2]. The full-field solutions of stresses and displacements for the cracked plates are obtained by asymptotic approach in Ref. [3][4]. In this paper, based on Reissner's theory, which takes into account the transverse shear deformation, the stress-displacement fields around the corner region are obtained for various combinations of boundary conditions. The full-field solution of cracked plates and the singular field of the bending plate containing flat rigid-inclusions without thickness are investigated.

THE CHARACTERISTICS OF DEFORMATION AND THE BASIC EQUATIONS FOR BENDING PLATES

A thin plate containing corner regions or crack tips subjected to bending and twisting is shown in Fig. 1. When h/L is small (L being the characteristic length of the plate), the deformation of Reissner's plate has the character of a boundary layer, as is pointed out in Ref.[3]. A bending plate of any shape can be divided into three regions according to the character of stress--state; in region I the inner solution is dominant; region II is the ordinary boundary layer region and region III is the extraordinary boundary layer region. Among the various cases of region III, III_a corresponds to the crack tip region, III_b the corner region (when $\alpha \to 2\pi$, region III_b becomes region III_a) and region III_c the region with steep variation of stress induced by the discontinuity of the boundary condition. The solutions for the region I, II and III_a have been obtained in the Ref. [3] and [4] respectively.

Let us consider region III_b. One of our aims is to obtain the stress and displacement fields around the corner region. The plate is subjected

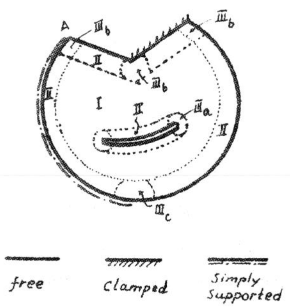

Fig.1 (a)

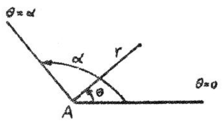

Fig. 1(b)

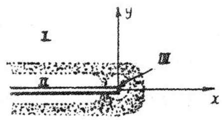

Fig.2

to remote loading and the boundary conditions for two sides of the corner are homogeneous, being combination of the free, simply supported I, II and clamped cases. No surface load is assumed on the plate. The basic equations can be expressed as

$$Q_\xi = C(\beta_\xi + w,_\xi)$$

$$M_{\xi\eta} = D[\frac{1-\nu}{2}(\beta_\xi|_\eta + \beta_\eta|_\xi) + \nu\beta_\lambda|_\lambda\delta_{\xi\eta}]$$

$$w|_{\xi\xi} + \beta_\xi|_\xi = 0$$

$$(1-\nu)\beta_\xi|_{\eta\eta} + (1+\nu)\beta_\eta|_{\xi\eta} - \frac{10}{h^2}(1-\nu)(\beta_\xi + w,_\xi) = 0 \qquad (2.1)$$

$$\xi,\eta,\lambda = 1,2$$

where symbol "|" denotes the covariant differentiation, $M_{\xi\eta}$, Q_ξ are internal resultant moments and forces, β_ξ, W are the rotation and deflection of middle surface. $D=\frac{Eh^3}{12(1-\nu^2)}$, $C=Eh/12(1+\nu)$ and h is the thickness of the plate.

The boundary conditions in polar coordinates can be written as

$$Q_\theta(r,\theta_0) = M_\theta(r,\theta_0) = M_{r\theta}(r,\theta_0) = 0 \qquad \text{(free)}$$

$$w(r,\theta_0) = M_{\theta\theta}(r,\theta_0) = M_{r\theta}(r,\theta_0) = 0 \quad \text{(simply supported I)}$$

$$w(r,\theta_0) = M_{\theta\theta}(r,\theta_0) = \beta_r(r,\theta_0) = 0 \quad \text{(simply supported II)}$$

$$w(r,\theta_0) = \beta_\theta(r,\theta_0) = \beta_r(r,\theta_0) = 0 \quad \text{(clamped)} \qquad (2.2)$$

where $\theta_0 = 0,\alpha$.

The generalizd displacement around the corner can be estimated as

$$w \sim r^{\lambda_1} \quad , \beta_\xi \sim r^{\lambda_2}$$

When the condition

$$|\lambda_1 - \mathrm{Re}\,\lambda_2| < 1 \qquad (2.3)$$

is satisfied, we can show that Reissner plate equations can be decomposed around the corner into two uncoupled problems in the sense of asymptotic expansion;

1) Shear Problem A

$$\begin{cases} Q_\xi = Cw,_\xi \\ w|_{\xi\xi} = 0 \end{cases} \qquad \begin{cases} Q_\theta(r,\theta_0) = 0 \quad \text{(free)} \\ w(r,\theta_0) = 0 \text{ (S.S. I,II, Clamped)} \end{cases} \qquad (2.4)$$

where $\theta_0=0,\alpha$. The solution corresponding to the first term of the expansion is: $\lambda_1=\pi/\alpha$

$$w \simeq r^{\pi/\alpha}(A_3 \cos\frac{\pi}{\alpha}\theta + B_3 \sin\frac{\pi}{\alpha}\theta)$$

$$\begin{Bmatrix} Q_r \\ Q_\theta \end{Bmatrix} \simeq C\,\frac{\pi}{\alpha} r^{\frac{\pi}{\alpha}-1} \left\{ A_3 \begin{bmatrix} \cos\frac{\pi}{\alpha}\theta \\ -\sin\frac{\pi}{\alpha}\theta \end{bmatrix} + B_3 \begin{bmatrix} \sin\frac{\pi}{\alpha}\theta \\ \cos\frac{\pi}{\alpha}\theta \end{bmatrix} \right\}$$

At the boundary $\theta=0$, $B_3=0$ (free edge) and $A_3=0$ (Simply supported and clamped edge).

2) Quasi-Plane Problem B

$$M_{\xi\eta} = D[\frac{1-\nu}{2}(\beta_\xi|_\eta + \beta_\eta|_\xi) + \nu\beta_\lambda|_\lambda\ \delta_{\xi\eta}]$$

$$(1-\nu)\beta_\xi|_{\eta\eta} + (1+\nu)\ \beta_\eta|_{\xi\eta} = 0$$

$$M_{\theta\theta}(r,\theta_0) = M_{r\theta}(r,\theta_0) = 0 \qquad \text{(free, S.S.I)} \tag{2.5}$$

$$M_{\theta\theta}(r,\theta_0) = \beta_r(r,\theta_0) = 0 \qquad \text{(S.S.II)}$$

$$\beta_\theta(r,\theta_0) = \beta_r(r,\theta_0) = 0 \qquad \text{(Clamped)}$$

where $\theta_0=0,\alpha$. With the complex potential functions $\phi(z)$, $\psi(z)$ the combinations of internal moments and rotations can be expressed as

$$M_{rr} + M_{\theta\theta} = \frac{h^3}{3}\ \mathrm{Re}\ [\phi'(z)]$$

$$M_{\theta\theta} + iM_{r\theta} = \frac{h^3}{12}\left\{\phi'(z) + \overline{\phi'(z)} + e^{2i\theta}\ [\bar{z}\phi''(z) + \psi'(z)]\right\}$$

$$\beta_r + i\beta_\theta = \frac{1}{2G}\ e^{-i\theta}[\varkappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}]\ ,\ \varkappa = \frac{3-\nu}{1+\nu} \tag{2.6}$$

The first term of the eigen-expansion for $\phi(z)$ and $\psi(z)$ will be

$$\begin{Bmatrix}\phi(z)\\ \psi(s)\end{Bmatrix} \simeq \begin{Bmatrix}a_1\\ b_1\end{Bmatrix} z^{\lambda_2} + \begin{Bmatrix}a_2\\ b_2\end{Bmatrix} z^{\overline{\lambda}_2} \tag{2.7}$$

where a_1, b_1, a_2 and b_2 are complex constants. Using the eigen-expansion method proposed by Williams[6], we obtain the eigen equations of the first term in the quasi-plane problem (Table 1).

The procedure of the solution for the corner region for Reissner plate theory corresponding to the range of the angle α are shown in the last three columns in Table 1. The symbol $A\frown B$ shows that the problem can be solved separately for the uncoupled problems A and B, and the symbol A→B' (or B→A') expresses that we can solve problem A (or B) at first and then calculate the problem B'(or A'). The problem A'(or B') are formulated as follows

problem A'

$$Q_\xi = C(\beta_\xi + w_{,\xi}) \qquad \begin{cases} Q_\theta(r,\theta_0) = 0 & \text{(free)} \\ w(r,\theta_0) = 0 & \text{(S.S.I,II and clamped)} \end{cases} \tag{2.8}$$

$$w|_{\xi\xi} = -\beta_\xi|_\xi \qquad \theta_0 = 0,\ \alpha$$

Table 1. (Note: Boundary condition: f-free, ss-simply supported, c-clamped)

case	theory of the plate	B.C. $\theta=0$	B.C. $\theta=\alpha$	eigen-equation $Z = \lambda_2\alpha$	constant	intervalof α for real λ_2	range of angle α	procedure of the solution*	case
N_{11}	R	f ssI	f ssI	$\sin z = c_1 z$	$c_1 = \pm\frac{\sin\alpha}{\alpha}$	$\alpha \geq 146.7°$	$120° \leq \alpha \leq 360°$	$A\frown B$	N_{11}
	K	f	f		$c_1 = \pm\frac{1-\nu}{3+\nu}\frac{\sin\alpha}{\alpha}$	$0 \leq \alpha \leq 2\pi$	$\alpha < 120°$	A→B'	
N_{22}	R	c	c	$\sin z = c_2 z$	$c_2 = \pm\frac{1+\nu}{3-\nu}\frac{\sin\alpha}{\alpha}$	$\alpha \geq 115.0°$	$90° \leq \alpha \leq 360°$	$A\frown B$	N_{22}
	K	c	c		$c_2 = \pm\frac{\sin\alpha}{\alpha}$	$\alpha \geq 146.7°$	$\alpha < 90°$	B→A'	
N_{33}	R	ssII	ssII	$\sin z = c_3$	$c_3 = \pm\sin\alpha$	$0 \leq \alpha \leq 2\pi$	$120° \leq \alpha \leq 360°$	$A\frown B$	N_{33}
	K	ss	ss		$c_3 = \pm\sin\alpha$	$0 \leq \alpha \leq 2\pi$	$\alpha < 120°$	A→B'	
N_{21}	R	c	f ssI	$\sin^2 z = k_1^2 - k_2^2 z^2$	$K_1 = \pm2[(3-\nu)(1+\nu)]^{-\frac{1}{2}}$ $K_2 = \pm[\frac{1+\nu}{3-\nu}]^{\frac{1}{2}}\frac{\sin\alpha}{\alpha}$	$\alpha \leq 130.6°$	$180° \leq \alpha \leq 360°$	$A\frown B$	N_{21}
	K	c	f		$K_1 = \pm2[(3+\nu)(1-\nu)]^{-\frac{1}{2}}$ $K_2 = \pm[\frac{1-\nu}{3+\nu}]^{\frac{1}{2}}\frac{\sin\alpha}{\alpha}$	no real value	$\alpha < 180°$	B→A	
N_{31}	R	ssII	f ssI	$\sin 2z = 2c_4 z$	$c_4 = \frac{\sin 2\alpha}{2\alpha}$	$0 \leq \alpha \leq 2\pi$	$60° \leq \alpha \leq 360°$	$A\frown B$	N_{31}
	K	ss	f		$c_4 = -\frac{1-\nu}{3+\nu}\frac{\sin 2\alpha}{2\alpha}$	$0 \leq \alpha \leq 2\pi$	$\alpha < 60°$	A→B'	
N_{32}	R	ssII	c	$\sin 2z = 2c_5 z$	$c_5 = -\frac{1+\nu}{3-\nu}\frac{\sin 2\alpha}{2\alpha}$	$\alpha \geq 57.5°$	$180° \leq \alpha \leq 360°$	$A\frown B$	N_{32}
	K	ss	c		$c_5 = \frac{\sin 2\alpha}{2\alpha}$	$0 \leq \alpha \leq 2\pi$	$\alpha < 180°$	B→A'	

R — Reissner theory, K — Kirchhoff theory.

*Reissner theory

problem B'

$$M_{\xi\eta} = D[\frac{1-\nu}{2}(\beta_\xi|_\eta + \beta_\eta|_\xi) + \nu\beta_\lambda|_\lambda\,\delta_{\xi\eta}]$$

$$(1-\nu)\,\beta_\xi|_{\eta\eta} + (1+\nu)\,\beta_\eta|_{\xi\eta} = \frac{10(1-\nu)}{h^2}\,w,_\xi$$

$$M_{\theta\theta}(r,\theta_0) = M_{r\theta}(r,\theta_0) = 0 \quad \text{(free and S.S. I)}$$

$$M_{\theta\theta}(r,\theta_0) = \beta_r(r,\theta_0) = 0 \quad \text{(S.S. II)}$$

$$\beta_\theta(r,\theta_0) = \beta_r(r,\theta_0) = 0 \quad \text{(clamped)} \quad \theta_0 = 0,\ \alpha \tag{2.9}$$

Even thougth the condition (2.3) is not satisfied, we can still use the procedure of solution A→B' (or B→A'), and the calculation can obviously be simplified.

If λ_2 is complex, the relation for the coefficients in Eq.(2.7) can be deduced. The intervals of α corresponding to real λ_2 are listed in Table 1.

BENDING PROBLEMS OF PLATE CONTAINING CRACK OR RIGID-INCLUSION

As applications of above solutions, the following examples are discussed. First, an infinite cracked plate shown in Fig.2 is analysed. Corresponding to the case N_{11} (free-free boundary) in Table 1, we can solve shear problem A and quasi-plane problem B separately. The solution for stress and displacement in region I, II can be obtained as in Ref.[3] and the unknown complex constant a_1 (corresponding to complex SIF $k_I - ik_{II}$) can be determined by means of asymptotic expansion technique as in Ref. [4]. We can find the relations between SIFS k_I^R, k_{II}^R obtained by Reissner plate theory and k_I^c, k_{II}^c obtained by Kirchhoff theory for order zero approximation. The general expressions are

$$k_I^{R(0)} = \frac{1+\nu}{3+\nu}\,k_I^c\,, \qquad k_{II}^{R(0)} = k_{II}^c \tag{3.1}$$

The analytical expression of SIF for infinite cracked plate subjected to symmetrical moment $\overline{M}(t)$ along crack surface is

$$k_I^{R(0)} = \frac{1+\nu}{3+\nu}\,\frac{2}{\pi}\int_0^1 \frac{\overline{M}(t)}{1-t}\,dt \tag{3.2}$$

Then the full-field solution (decomposed into regions I, II, III) are obtained for the cracked plate.

Let us consider an infinite plate containing a flat-rigid inclusion without thickness. It corresponds to the case N_{22} in Tab. 1, with $\alpha=2\pi$ and $\lambda_2=\frac{1}{2}$. The singular field near the rigid-inclusion tip is investigated, and the singular solution for rotation and moments are

$$\begin{Bmatrix} M_{rr} \\ M_{\theta\theta} \\ M_{r\theta} \end{Bmatrix} = \frac{h^3}{12}\,r^{-\frac{1}{2}}\left\{a_{01}\begin{bmatrix} \frac{5}{4}\cos\frac{\theta}{2} - \frac{2\varkappa-1}{4}\cos\frac{3\theta}{2} \\ \frac{3}{4}\cos\frac{\theta}{2} + \frac{2\varkappa-1}{4}\cos\frac{3\theta}{2} \\ \frac{1}{4}\sin\frac{\theta}{2} + \frac{2\varkappa-1}{4}\sin\frac{3\theta}{2} \end{bmatrix} + a_{02}\begin{bmatrix} \frac{5}{4}\sin\frac{\theta}{2} - \frac{2\varkappa+1}{4}\sin\frac{3\theta}{2} \\ \frac{3}{4}\sin\frac{\theta}{2} + \frac{2\varkappa+1}{4}\sin\frac{3\theta}{2} \\ \frac{-1}{4}\cos\frac{\theta}{2} - \frac{2\varkappa+1}{4}\cos\frac{3\theta}{2} \end{bmatrix}\right\} + O(1)$$

$$\begin{Bmatrix} \beta_r \\ \beta_\theta \end{Bmatrix} = \frac{r^{\frac{1}{2}}}{2G}\left\{a_{01}\begin{bmatrix} (\varkappa-\frac{1}{2})\cos\frac{\theta}{2} - (\varkappa-\frac{1}{2})\cos\frac{3\theta}{2} \\ -(\varkappa+\frac{1}{2})\sin\frac{\theta}{2} + (\varkappa-\frac{1}{2})\sin\frac{3\theta}{2} \end{bmatrix} + a_{02}\begin{bmatrix} (\varkappa-\frac{1}{2})\sin\frac{\theta}{2} - (\varkappa+\frac{1}{2})\sin\frac{3\theta}{2} \\ (\varkappa+\frac{1}{2})\cos\frac{\theta}{2} - (\varkappa+\frac{1}{2})\cos\frac{3\theta}{2} \end{bmatrix}\right\} + O(r) \tag{3.3}$$

We now consider another mixed boundary value problem (clamped at $\theta=0$ and free at $\theta=\alpha=\pi$), It corresponds to the case N_{21} in Tab.1 with $\lambda_2=\frac{1}{2}-i\varepsilon$, $\varepsilon = \frac{1}{2\pi}\ln\varkappa$. Complex SIF for this problem is defined as

$$k = k_1 - ik_2 = 2\sqrt{2}\,\frac{h^3}{12}\,e^{\varepsilon\pi}\lim_{z\to 0} z^{\overline{\lambda}_2}\phi'(z) = 2\sqrt{2}\,\frac{h^3}{12}\,e^{\varepsilon\pi}\lambda_2 a_1 \tag{3.4}$$

The expressions for moments M_{rr}, $M_{\theta\theta}$, $M_{r\theta}$ can be obtained through calculation. We find that the solution of the oscillatory type with $\frac{\sin}{\cos}(\ln r)$ will occur if λ_2 is complex. The oscillatory solution for plane problems has been reviewed by Atkinson[8].

CONCLUSIONS

The following conclusions are drawn from the present analysis. In the general case, the stress-displacement field around the corner of the plate can be obtained through the procedure described in Tab. 1. When $|\lambda_1 - \mathrm{Re}\lambda_2| < 1$, Reissner plate equations can be decomposed into two uncoupled problems: shear problem and quasi-plane problem. Accordingly the solution for corner problem is simplified. By the local analysis, we can deduce eigen-exponent equations and stress-displacement distributions around the corners for various combinations of boundary conditions. In general, at most three unknown constants are needed to determine completely the near-field solution around the corners. Then we consider three problems. The full-field solution for cracked infinite plate and the relation between k_I^R, k_{II}^R and k_I^c, k_{II}^c are obtained. The singular solutions for flat rigid-inclusion without thickness are deduced for bending problems of plate, and it is pointed out that for certain combinations of boundary conditions and in some interval of corner angle α, the solution of the oscillatory type with $\begin{matrix}\sin\\ \cos\end{matrix}(\ln r)$ will occur. The results in this paper can be used in the finite element analysis in the corner region of the plate. Using the analysis in Ref.[7], we can extend the results of this paper to the analysis of stress-displacement in the corner region for the shallow shells. The full-field solutions for the corner region of plates and shallow-shells with transverse shear deformation can be deduced.

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