THE DISCONTINUITY IN QUASI-STATIC PLASTIC FIELDS

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ABSTRACT

In this paper the type of the basic equations of quasi-static plastic fields is discussed for both three dimensional problem and plane strain problem, each case including perfectly plastic and strain hardening materials Therefore, the discontinuity of the solution to the quasi-static plastic problem is revealed.

INTRODUCTION

The discontinuity in plastic-fields is an important problem. For perfectly plastic, the plane strain problem was studied in detail by Gao and Hwang[1]. Unfortunately, the basic method and conclusions have not been understood well, and was criticized unjustly by [2]. For the same material, the general three dimensional case was discussed by Gao[3], and is published in Chinese, therefore not too many would know the conclusion. For the strain hardening material the type of the basic equations was discussed by Gao and Hwang[4], but the interpretation is not in detail. After this work, Drugan and Rice discussed the same essentiality of the problem[2], and obtained the same conditions $S_{YZ}= S_{YY} = S_{ZZ} = 0$, see[4] (1.14) and [2](4.16). It is very strange that from the same conditions quite different conclusions were obtained. In this paper all of the problem will be cleared.

I. THREE DIMENSIONAL PROBLEM

1. Perfectly Plastic

Let x_i denote the rectangular coordinates. $\varepsilon_{ij}, \sigma_{ij}, \lambda$ denote strain,

stress, flow factor, respectively. i, j = 1,2,3. We suppose x_1=const is the discontinuity-surface and let $x_1=\xi$, according to[3], then we can rewrite the basic equation as

$$A \frac{\partial^2}{\partial \xi^2} [{}^{\dot\sigma}_{\lambda} ij] + \dots = 0 \tag{1.1}$$

here,

$$[{}^{\sigma}_{\lambda} ij] = (\dot\sigma_{11}\ \ \dot\sigma_{22}\ \ \dot\sigma_{33}\ \ \dot\sigma_{12}\ \ \dot\sigma_{13}\ \ \dot\sigma_{23}\ \ \lambda)^T \tag{1.2}$$

the expression of matrix A can be found from[3]. It is well known the characteristic equation of the problem is

$$\det \ \| A \| = 0 \tag{1.3}$$

namely (E-Youngs' modulus, ν-Poisson's ratio)

$$\frac{1+\nu}{E^2} [\ S_{22}^2 + S_{33}^2 + 2\nu S_{22}\ S_{33} + 2(1-\nu)\ \sigma_{23}^2] = 0 \tag{1.4}$$

In order to discuss the type of the basic equations, we only need to discuss the roots of Eq. (1.4).

From (1.4) we obtain

$$S_{22} = S_{33} = \sigma_{23} = 0 \tag{1.5}$$

then we have the equivalent form:

$$\begin{cases} \sigma_{11} = \sigma_{22} = \sigma_{33} \\ \sigma_{23} = 0 \end{cases} \tag{1.6}$$

We consider a special transformation of coordinates from x_i to x'_i

$$\begin{cases} x'_1 = x_1 \\ x'_\alpha = T_\alpha^\beta\ x_\beta \end{cases} \tag{1.7}$$

where α,β take value 2 or 3. We can choose a proper transformation (1.7) so that $\sigma'_{13}= 0$ then we have

$$\begin{cases} \dot{\sigma}_{11} = \dot{\sigma}_{22} = \dot{\sigma}_{33} \\ \dot{\sigma}_{23} = \dot{\sigma}_{13} = 0 \end{cases} \tag{1.8}$$

the equivalent form is

$$\begin{cases} \dot{\sigma}_{11} = \dot{\sigma}_{22} \\ \dot{\sigma}_{33} = \frac{1}{2} (\dot{\sigma}_{11} + \dot{\sigma}_{22}) \\ \dot{\sigma}_{23} = \dot{\sigma}_{13} = 0 \end{cases} \tag{1.9}$$

The last one of (1.9) means that axis $\dot{x}_3$ is a principle axis of stress, so $\dot{x}_1$ and $\dot{x}_2$ are included in a principle plane. Furthermore, from the first one of (1.9) we can know that $\acute{x}_1$ and $\acute{x}_2$ are just the directions of maximum shear stress. Finally, we obtain the conclusion: if there exists a discontinuity surface, then it should include a principle axis, and the other two principle axes should form 45° angles with the surface. Evidently, these requirements can be satisfied if we choose a proper orientation for the x_1 plane. On the other hand, we have not yet used the second one of (1.9), and generally this additional condition cannot be satisfied. Therefore, no characteristic surfaces exist. Then the basic equations are elliptic, the solution will be sufficiently smooth inside the plastic domain.

2. Hardening Material

For the strain hardening material we can use the similar method as in the previous section. We only need to cut out the yield condition and take λ as the following

$$\lambda = \mu \cdot h(\sigma)\, \dot{\sigma} \tag{1.10}$$

$$\mu = \begin{cases} 1 & \text{for plastic and loading} \\ 0 & \text{for elastic or unloading} \end{cases} \tag{1.11}$$

here

$$\sigma = \left(\frac{3}{2} S_{ij}\, S^{ij}\right)^{1/2} \tag{1.12}$$

and $h(\sigma)$ is a function depending on the property of material. By similar procedure as in the perfectly plastic case, we can obtain a set of equations

$$B \frac{\partial^2}{\partial \xi^2}(\dot{\sigma}_{ij}) + \dots = 0 \tag{1.13}$$

the expression of matrix B is omitted here, and we have

$$\det \|B\| = \frac{1+\nu}{E^2} \left\{\frac{1-\nu^2}{E} + \frac{3h}{2\sigma}[S_{22}^2 + S_{33}^2 + 2\nu\, S_{22}\, S_{33} + 2(1-\nu)\sigma_{23}^2]\right\} \tag{1.14}$$

From (1.14) we can see the condition $\det \|B\| = 0$ for hardening material can never be satisfied, in contrast to the case of perfectly plastic material for which the condition (1.6) could be satisfied. Only when the elastic deformation is omitted, (1.14) can reduce to (1.6). Therefore, the hardening material is more difficult than perfectly plastic to have some discontinuities. It should be pointed out that the condition (1.14) had been obtained by Gao and Hwang in[4], but the discussion is not in detail.

II. PLANE STRAIN CASE

1) Perfectly Plastic

For the plane strain case, let $x_1 = x$, $x_2 = y$, then we have the stress function ϕ so that

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{2.1}$$

From the constitutive relation and the restraint condition $\varepsilon_z = 0$ we can obtain

$$\sigma_z = \nu(\sigma_x + \sigma_y) + \varepsilon\, \sigma_p \tag{2.2}$$

here

$$\varepsilon = \frac{1}{2} - \nu \tag{2.3}$$

$$\begin{cases} \sigma_p = \frac{2E}{3} \exp[\frac{-2E}{3} \Lambda] \int_{t_0}^{t} \lambda\, [\sigma_x + \sigma_y] \exp[\frac{2E}{3} \Lambda]\, dt \\ \Lambda = \int_{t_0}^{t} \lambda dt \end{cases} \tag{2.4}$$

in which t measures time, t_0 is the initial yield time. For convenience, we introduce the maximum shear stress τ

$$\tau = [\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \sigma_{xy}^2]^{1/2} \tag{2.5}$$

Then using (2.2), the yield condition can be written as

$$\tau^2 + \frac{\varepsilon^2}{3}[\sigma_x + \sigma_y - \sigma_p]^2 = k^2 \tag{2.6}$$

We assume $\sigma_x + \sigma_y - \sigma_p \neq 0$ then from (2.4) and (2.6) we have

$$\lambda = \frac{3}{2E}\left[\pm \frac{\varepsilon}{\sqrt{3}} \frac{\dot{\sigma}_x + \dot{\sigma}_y}{(k^2-\tau^2)^{1/2}} + \frac{\tau\,\dot{\tau}}{k^2-\tau^2}\right] \tag{2.7}$$

The compatibility equation can be written as

$$\frac{\partial^2 \dot{\varepsilon}_y}{\partial x^2} + \frac{\partial^2 \dot{\varepsilon}_x}{\partial y^2} - 2\frac{\partial^2 \dot{\varepsilon}_{xy}}{\partial x \partial y} = 0 \tag{2.8}$$

Substituting (2.1), (2.2), (2.7) into the constitutive relations then into (2.8), we can obtain

$$A_{1111}\frac{\partial^4 \dot{\phi}}{\partial x^4} + A_{1112}\frac{\partial^4 \dot{\phi}}{\partial x^3 \partial y} + A_{1122}\frac{\partial^4 \dot{\phi}}{\partial x^2 \partial y^2} + \dots = 0 \tag{2.9}$$

in which

$$A_{1111} = \frac{16\epsilon^2}{9}(\sigma_x+\sigma_y-\sigma_p)^2(1-\nu^2)+[\sigma_y-\sigma_x+\frac{4\epsilon^2}{3}(\sigma_x+\sigma_y-\sigma_p)]^2 \tag{2.10}$$

The characteristic equation of (2.9) is

$$A_{1111} = 0 \tag{2.11}$$

(2.11) can be satisfied only when

$$\begin{cases} \varepsilon(\sigma_x + \sigma_y - \sigma_p) = 0 \\ \sigma_x = \sigma_y \end{cases} \tag{2.12}$$

The last one of (2.12) gives the orientation of the characteristic line. But, the first one of (2.12) requires that

$$\varepsilon = 0 \quad \text{or} \quad (\sigma_x + \sigma_y - \sigma_p) = 0 \tag{2.13}$$

It is easy to verify that when $\varepsilon=0$ the basic equations are indeed hyperbolic. Next, we should discuss the case that $\varepsilon\neq 0$, namely, $\nu< 1/2$, then $\sigma_x+\sigma_y-\sigma_p=0$ is required. This condition may be satisfied at a curve Γ. But, simultaneously, the second one of (2.12) should be satisfied at Γ. Generally, curve Γ cannot satisfy both conditions. Therefore, when $\nu<1/2$, the basic equation will be elliptic, the solution should be sufficiently smooth in the plastic domain.

It should be pointed out that the conclusion mentioned above was given by Gao and Hwang[1] before, and the jump conditions were discussed in detail for the case of $\nu=1/2$. There is nothing wrong in[1]. In[2] , it is expected that $\sigma_x+\sigma_y-\sigma_p = 0$ can be satisfied in some whole region. In fact, it can only be satisfied on some curves.

The practical important case is that $|\sigma_x+\sigma_y-\sigma_p| \ll |\sigma_x+\sigma_y|$, which appear in the crack-tip field ($\nu< 1/2$ case, ahead of crack-tip). In this case, if $\sigma_x+\sigma_y-\sigma_p$ is neglected, then the basic equations can be considered as hyperbolic , but only in the limiting meaning. It should also be pointed out that the $\sigma_x+\sigma_y-\sigma_p \not\approx 0$ case was considered and used by Gao[5] already.

2. Hardening Material

For hardening material plane strain case, the type of the basic equations has been discussed by Gao and Hwang[4]. From eq.(2.19) of[4], evidently only when the elastic deformation is neglected, and $\nu=1/2$, and $\sigma_x = \sigma_y$, then some discontinuities can exist.

III. CONCLUSION

1) In the three dimesnional problem both for perfectly plastic and hardening material, generally, the basic equations are elliptic. The solution will be sufficiently smooth inside a kind of domain. Some of discontinuities can only appear on the boundary between different domains.

2) In the plane strain problem for perfectly plastic material, the basic equations are hyperbolic only when $\nu=1/2$. Otherwise it is elliptic.

3) In the plane strain problem for hardening material, the basic equations are hyperbolic only when the elastic deformation is neglected and $\nu=1/2$, otherwise it is elliptic.

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