COMPUTATIONAL AND THEORETICAL STUDIES ON DYNAMIC FRACTURE MECHANICS AND THREE-DIMENSIONAL CRACK PROBLEMS

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ABSTRACT

This paper presents a brief summary of the recent and current work of the authors in the following selective areas: (I) dynamic fracture mechanics and (II) three-dimensional crack problems. In Part I, the following topics are addressed: (1) a moving singular element procedure which gives highly accurate solutions to elastodynamic problems of fast crack propagation in finite bodies, (2) numerical simulation of dynamic crack propagation and arrest in various fracture specimens using the above analysis procedure, and (3) path-independent integrals that characterize the severity of the tip of an elastodynamically propagating crack. In Part II, we present: (1) the general analytical solution for an embedded elliptical crack in an infinite solid which is subjected to arbitrary tractions on the crack surface and (2) an improved finite element alternating method for analyses of surface-flawed three-dimensional structures, which employs the above solution, and its application to various problems.

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I. DYNAMIC FRACTURE MECHANICS

1. Moving Singular Element Procedure

First, we consider a crack propagating with an instantaneous crack velocity $C(t)$ in a linear elastic solid. Nilsson [1] has shown that, in the vicinity of the crack tip, both the differential equations and the boundary conditions for an arbitrary moving crack coincide with those for the problem of steady state crack propagation with the constant crack velocity C. The general solutions (eigen solutions) for the stress and displacement of the near-tip field under an unsteady as well as a steady condition were obtained in [2] in a unified fashion for all the three fracture modes:

$$\sigma_{xn} = \frac{K_n^o B_I(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ (1+2\beta_1^2-\beta_2^2) r_1^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\theta_1 - 2h(n) r_2^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\theta_2 \right\}$$
$$+ \frac{K_n^* B_{II}(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ (1+2\beta_1^2-\beta_2^2) r_1^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\theta_1 - 2h(\bar{n}) r_2^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\theta_2 \right\} \tag{1.a}$$

$$\sigma_{yn} = \frac{K_n^o B_I(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ -(1+\beta_2^2) r_1^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\theta_1 + 2h(n) r_2^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\theta_2 \right\}$$
$$+ \frac{K_n^* B_{II}(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ -(1+\beta_2^2) r_1^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\theta_1 + 2h(\bar{n}) r_2^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\theta_2 \right\} \tag{1.b}$$

$$\sigma_{xyn} = \frac{K_n^o B_I(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ -2\beta_1 r_1^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\theta_1 + \frac{(1+\beta_2^2)}{\beta_2} h(n) r_2^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\theta_2 \right\}$$
$$+ \frac{K_n^* B_{II}(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ +2\beta_1 r_1^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\theta_1 - \frac{(1+\beta_2^2)}{\beta_2} h(\bar{n}) r_2^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\theta_2 \right\} \tag{1.c}$$

$$\sigma_{xzn} = \frac{K_n^+ B_{III}(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} r_2^{\frac{n}{2}-1} \begin{cases} \sin(\tfrac{n}{2}-1)\theta_2 & : n \text{ odd} \\ \cos(\tfrac{n}{2}-1)\theta_2 & : n \text{ even} \end{cases} \tag{1.d}$$

$$\sigma_{yzn} = \frac{K_n^+ B_{III}(C)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \beta_2 r_2^{\frac{n}{2}-1} \begin{cases} \cos(\tfrac{n}{2}-1)\theta_2 & : n \text{ odd} \\ -\sin(\tfrac{n}{2}-1)\theta_2 & : n \text{ even} \end{cases} \tag{1.e}$$

$$\sigma_{zn} = \begin{cases} 0 & : \text{plane stress} \\ \nu(\sigma_{xn}+\sigma_{yn}) & : \text{plane strain} \end{cases} \tag{1.f}$$

$$u_n = \frac{K_n^o B_I(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ r_1^{\frac{n}{2}} \cos\tfrac{n}{2}\theta_1 - h(n) r_2^{\frac{n}{2}} \cos\tfrac{n}{2}\theta_2 \right\}$$
$$+ \frac{K_n^* B_{II}(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ r_1^{\frac{n}{2}} \sin\tfrac{n}{2}\theta_1 - h(\bar{n}) r_2^{\frac{n}{2}} \sin\tfrac{n}{2}\theta_2 \right\} \tag{2.a}$$

$$v_n = \frac{K_n^o B_I(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ -\beta_1 r_1^{\frac{n}{2}} \sin\tfrac{n}{2}\theta_1 + \frac{h(n)}{\beta_2} r_2^{\frac{n}{2}} \sin\tfrac{n}{2}\theta_2 \right\}$$
$$+ \frac{K_n^* B_{II}(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ \beta_1 r_1^{\frac{n}{2}} \cos\tfrac{n}{2}\theta_1 - \frac{h(\bar{n})}{\beta_2} r_2^{\frac{n}{2}} \cos\tfrac{n}{2}\theta_2 \right\} \tag{2.b}$$

$$w_n = \frac{K_n^+ B_{III}(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) r_2^{\frac{n}{2}} \begin{cases} \sin\tfrac{n}{2}\theta_2 & : n \text{ odd} \\ \cos\tfrac{n}{2}\theta_2 & : n \text{ even} \end{cases} \tag{2.c}$$

where $r_j e^{i\theta_j} = x + i\beta_j y$ $(j = 1,2)$; $i = \sqrt{-1}$; $\beta_1^2 = 1 - C^2/C_d^2$; $\beta_2^2 = 1 - C^2/C_s^2$; $B_I(C) = (1+\beta_2^2)/D(C)$; $B_{II}(C) = 2\beta_2/D(C)$; $B_{III}(C) = 1/\beta_2$; $D(C) = 4\beta_1\beta_2 - (1+\beta_2^2)^2$; $h(n) = \{2\beta_1\beta_2/(1+\beta_2^2)$: n odd, $(1+\beta_2^2)/2$: n even$\}$; $\bar{n} = n + 1$; and $0 \le n < \infty$. In the above, C_d and C_s are the dilatational and shear wave velocities. The undetermined parameters K_n^o, K_n^*, K_n^+ are related to the Modes I, II, and III, respectively. The general solutions expressed by Eqs. (1) and (2) contain the zero stress and rigid body motion $(n = 0)$, the singular stresses and corresponding displacements (n = 1), the constant stresses and linear displacements (n = 2), and the higher order terms (n ≥ 3). Thus, the parameters K_1^o, K_1^*, K_1^+ are equivalent to the dynamic stress intensity factors K_I, K_{II}, and K_{III}, respectively.

In Refs. [3,4], a "moving singular element" procedure for the dynamic analysis of fast crack propagation in finite bodies was presented. In this procedure, the eigen-functions for displacement given by Eq. (2) are used as basis functions of the element which surrounds the crack-tip. The singular element may move by an arbitrary amount of crack length increment Δa in each time increment Δt of the numerical time-integration procedure. The moving singular element, within which the crack-tip always has a fixed location, retains its shape at all times; but the mesh of regular (isoparametric) finite elements, surrounding the moving singular element, deforms accordingly. To simulate large amounts of crack propagation, the mesh pattern of the regular elements is readjusted periodically.

The conditions of compatibility of displacement, velocity, and acceleration between the moving singular element and the surrounding regular elements are satisfied through a least squares technique [3]. An energy-consistent variational statement was also developed in [3], as a basis for the above method of fast fracture analysis. It has been demonstrated [4] that the above procedure leads to a direct evaluation of the dynamic stress intensity factors in as much as they are unknown parameters in the assumed basis functions for the moving singular element.

The above analysis procedures were verified through application to the problems for which analytical solutions for infinite domain cases are available, such as:

(i) self-similar, constant-velocity propagation from a finite initial

length of a central crack in a square plate subject to a uniform time-independent, tensile stresses normal to crack axis, at the edges (analogous to the problems of Broberg [5] and Rose [6]);

(ii) a stationary central crack in a rectangular plate subject to impact tensile stresses (a Heaviside step-function) at the edges (analogous to the problems of Baker [7] and Sih et al [8]);

(iii) a problem similar to (ii), except that, at time t_o, the crack starts propagating symmetrically with a constant speed (analogous to the problem studied by Freund [9]);

(iv) constant-velocity propagation of an edge crack in a finite width strip subject to prescribed displacements at the edges (analogous to the problem of Nilsson [10]).

The results obtained by the moving singular element procedure were compared with these analytical solutions. The numerical results [4] have been found to correlate well with the analytical solutions for corresponding problems in infinite domains, during the time for which these analytical solutions may be considered as valid. The computed solutions beyond these times, and knowledge of the times involved for wave-interaction in finite bodies, indicate both qualitatively and quantitatively the effects of stress-wave interactions on dynamic stress intensity factors for cracks propagating in finite bodies. For the problems (i) to account for the effect of the finite domain on the dynamic stress intensity factors, an approximate solution was developed in [4,11]. Good correlations between the numerical solutions and the approximate solutions were also noted in [4,11].

To simplify the procedures in [3], a study was undertaken in [12,13] to address: (i) the effect of using only the stationary crack eigen functions in the moving singular element for dynamic crack propagation and (ii) the use of isoparametric elements with mid-side nodes shifted to the "quarter-point" so as to yield the appropriate ($r^{-1/2}$) singularity. It was found that even these simplified procedures yield acceptable accuracies for engineering purposes.

2. Numerical Simulation of Dynamic Crack Propagation and Arrest

Numerical simulation of dynamic fracture can be classified in two catagories [14]: (i) "Generation" phase simulation and (ii) "Prediction" or "Application" or "Propagation" phase simulation. In the generation study, dynamic stress intensity variation with time is determined for a specified crack-propagation history. Thus, all the examples presented earlier and the cited analytical solutions fall into this category. On the other hand, in the prediction study, crack-propagation history is determined for a specified dynamic fracture toughness versus crack-velocity relationship.

Using the moving singular element procedure, extensive work has been conducted in [15,16] concerning both the generation and the prediction studies for wedge-loaded, rectangular double cantilever beam (RDCB) specimens of a transparent epoxy resin, Araldite B. These results were found to be in excellent accord with the experimental results reported in [17]. It was also found [16] that the position of loading point in the finite element mesh has a large influence on the input energy into the specimens. Thus, the simplified loading conditions such as loadings at the edges or crack surfaces, which are often employed in numerical analyses, may result in erroneous simulations.

These studies were later extended in [18] to study fast fracture in non-transparent RDCB specimens made of structural steel. In the experiments [19] on steel, the stress intensity factors were measured by caustics reflected from the mirrored surface of the specimen. The measured

stress intensity factors in the steel specimen [19] show large oscillations during some stage of crack-propagation, whereas the data for Araldite B [17] is rather smooth. The authors of [19] attribute this oscillation to high frequency stress waves interacting with the crack. However, it was considered [18] that this oscillation may be limited to the surface of the specimen due to the 3-dimensional nature of the vibration of the specimen surface and that the stress intensity factor along the crack front inside of the specimen thickness may be expected to show a rather smooth variation. The results in Ref. [18] in fact tend to confirm this view.

Among the conclusions of the study in [18] are: (i) the dynamic crack propagation and arrest are influenced largely by the mass density, moderately by Young's modulus, and almost negligibly by Poisson's ratio; (ii) the variation of dynamic stress intensity factors is influenced by the various waves originally generated from the fast crack initiation and then reflected from the boundary of the specimen; (iii) the ratio of the maximum kinetic energy to the input energy increases with the decreasing Rayleigh wave velocity for different material properties; (iv) the crack arrest toughness, for a given crack propagation history, increases with the increasing ratio of maximum kinetic energy to the input energy or with the decreasing Rayleigh wave velocity; and (v) analysis with a realistic wedge loading condition (contact/no-contact) gives a slightly lower variation of s. i. f. than with the fixed loading condition (specimen always in contact with the wedge).

For dynamic tear test (DTT) specimen, the conclusions of [20] concerning load-rate sensitivity of dynamic fracture toughness for a propagating crack were critically examined in [21]. Numerical simulation of crack-propagation histories in four cases of dynamic tear experiments, under impact loading on 4340 steel specimens, were performed [21]. The influence of the loss of contact of the specimen at various times with either the supports or the tip or both was also critically examined. In each case, the variation of the dynamic stress intensity factor for the simulated crack propagation history was directly computed using the moving singular element procedure. The results in [21] appear to indicate that the conclusions in [20] may not be fully warranted.

Based on the numerical results in [21], a method was proposed in [22] for determining the dynamic s. i. f. directly from crack-mouth opening displacements in DTT specimens. This simple method [22] should be of value in the experimental measurement of dynamic s. i. f. for propagating cracks in (opaque) metallic specimens, rather than inferring the s. i. f. from caustics reflected from the mirrored surfaces of the specimen.

3. Path Independent Integrals

Now we consider a crack propagating at an angle θ_c measured from the X_1 axis. The energy release rate G can be written as [23]:

$$G = (C_k/C)G_k = G_1\cos\theta_c + G_2\sin\theta_c \ ;$$

$$G_k = \lim_{\Gamma_{\varepsilon\to 0}} \int_{\Gamma_\varepsilon} [(W + T)n_k - t_i u_{i,k}]dS \qquad (3)$$

where C_k denotes the component of crack velocity in the X_k direction; W and T are the strain and kinetic energy densities, respectively; n_k, the outward normal direction cosines; t_i, the traction; and $(\ )_{,k}$ denotes $\partial(\ )/\partial X_k$.

The path independent integral which is equivalent to the components of the energy release rate was derived in [2]:

$$J'_k = \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} [(W + T)n_k - t_i u_{i,k}]dS$$

$$= \lim_{\varepsilon\to 0}\left\{\int_{\Gamma+\Gamma_c} [(W + T)n_k - t_i u_{i,k}]dS + \int_{V-V_\varepsilon} [\rho\ddot{u}_i u_{i,k} - \rho\dot{u}_i\dot{u}_{i,k}]dV\right\} \qquad (4)$$

The above path independent integral is valid under general mixed-mode conditions for a crack propagating with non-constant velocity under unsteady-state conditions. Under a steady-state condition, it was shown [24] that the J'_1 integral can be reduced to the expression obtained by Sih [25], which is valid for only steady-state crack propagation at constant velocity. The existence of many path independent integrals, which do not have the meaning of energy release rate, was discussed in [26].

The other two types of path independent integrals derived in [23,27] are:

$$\hat{J}_k = \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} [Wn_k - t_i u_{i,k}]dS$$

$$= \lim_{\varepsilon\to 0}\left\{\int_{\Gamma+\Gamma_c} [Wn_k - t_i u_{i,k}]dS + \int_{V-V_\varepsilon} [\rho\ddot{u}_i u_{i,k}]dV\right\} \qquad (5)$$

and

$$J_k = \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} [(W - T)n_k - t_i u_{i,k}]dS$$

$$= \lim_{\varepsilon\to 0}\left\{\int_{\Gamma+\Gamma_c} [(W - T)n_k - t_i u_{i,k}]dS + \int_{V-V_\varepsilon} [\rho\ddot{u}_i u_{i,k} + \rho\dot{u}_i\dot{u}_{i,k}]dV\right\} \qquad (6)$$

Using the asymptotic solutions [2] given by Eqs. (1) and (2), the relations between the path independent integrals and the instantaneous stress intensity factors can be expressed as [2]; for $\theta_c = 0$:

$$J_1'^o = G_1^o = \frac{1}{2\mu}\left\{K_I^2 A_I(C) + K_{II}^2 A_{II}(C) + K_{III}^2 A_{III}(C)\right\} ;$$

$$J_2'^o = G_2^o = -\frac{K_I K_{II}}{\mu} A_{IV}(C) \qquad (7)$$

$$\hat{J}_1^o = \frac{1}{2\mu}\left\{K_I^2 \hat{F}_I(C) + K_{II}^2 \hat{F}_{II}(C) + K_{III}^2 \hat{F}_{III}(C)\right\} ;$$

$$\hat{J}_2^o = -\frac{K_I K_{II}}{\mu} \hat{F}_{IV}(C) \qquad (8)$$

$$J_1^o = \frac{1}{2\mu}\left\{K_I^2 F_I(C) + K_{II}^2 F_{II}(C) + K_{III}^2 F_{III}(C)\right\} ;$$

$$J_2^o = -\frac{K_I K_{II}}{\mu} F_{IV}(C) \qquad (9)$$

The detailed expressions for the crack speed functions are given in [2]. For $\theta_c \neq 0$, the components of the path independent integrals can be related to those for $\theta_c = 0$. For example, the components of the energy release rate are [28,29]:

$$G_1 = G_1^o\cos\theta_c - G_2^o\sin\theta_c \; ; \; G_2 = G_1^o\sin\theta_c + G_2^o\cos\theta_c \qquad (10)$$

Substituting Eq. (10) into Eq. (3), we obtain $G = G_1^o$ for any θ_c. The $G = G_1^o$ relation can be also obtained directly from Eq. (3) for $\theta_c = 0$. Thus, the above fact confirms that the energy release rate G is invariant while the components G_k depend on the coordinate system.

A numerical study [26] using the moving singular element procedure indicates that the path independent integrals given by Eqs. (4), (5), (6) give distinctly different numerical results, as expected from the theo-

retical point of view [2]. Although the moving singular element procedure gives highly accurate solutions, especially for detailed stress distribution near the propagating crack tip, this procedure may be difficult to be applied by general users of finite element method because of its sophistication. From this point of view, a study [28] was undertaken to use the path independent integrals with a less sophisticated finite element model such as the isoparametric elements. The combined use of the J' integral and the moving isoparametric element procedure was shown [28] to be an effective tool for the evaluation of the crack-tip parameters such as the stress intensity factors as well as energy release rate. It was also found [28] that the use of $\hat{J}$ and J integrals in the finite element model, in which the singularity in the kinetic energy density is not incorporated, led to false values of stress intensity factors.

For mixed-mode crack propagation, a procedure was presented in [29] for determining the mixed-mode stress intensity factors from the J_k' integral. Practicality of this procedure was shown in the numerical examples of [29]. The following relations for the components of the path independent integrals were also found in [29]:

$$|G_1^o| \geq (\sqrt{A_I A_{II}}/A_{IV}) \cdot |G_2^o| \; ; \; |J_1'^o| \geq (\sqrt{A_I A_{II}}/A_{IV}) \cdot |J_2'^o| \; ;$$

$$|\hat{J}_1^o| \geq (\sqrt{\hat{F}_I \hat{F}_{II}}/\hat{F}_{IV}) \cdot |\hat{J}_2^o| \; ; \; |J_1^o| \geq (\sqrt{F_I F_{II}}/F_{IV}) \cdot |J_2^o| \, .$$

For a stationary crack ($C = 0$), these relations can be reduced to $|G_1^o| \geq |G_2^o|$, etc.

II. THREE-DIMENSIONAL CRACK PROBLEMS

1. General Analytical Solution for an Embedded Elliptical Crack in an Infinite Solid

In the alternating method [30], the analytical solution for an embedded elliptical crack in an infinite elastic medium, which is a basic solution required in the alternating technique, has been limited to a cubic variation of normal pressure on the crack surface [31]. This limitation is thought to be one of the major reasons for the relative inaccuracy of the alternating method as compared to hybrid finite element procedures or the boundary-integral equation procedures.

Recently a general solution procedure has been derived in [32] for the problem of an infinite elastic medium with an embedded elliptical crack, the faces of which are subject to arbitrary variations of normal as well as shear tractions. Later a more detailed solution, as well as a general procedure for the evaluation of the required elliptic integrals, was obtained in [33].

Since 1971 no work has appeared in literature to generalize the solution in [31] to an arbitrary pressure variation on the crack surface due to the seemingly insurmountable mathematical and algebraic difficulties. While the analytical solution [32,33] can be reduced to a closed-form solution for a relatively simple loading such as constant or linear variation of the tractions, for a high order polynomial variation of the tractions, the solution procedure requires a digital computer. To obtain the stress components at a given point by using a computer, a general evaluation procedure [33] for obtaining the partial derivatives of the potential functions used in the formulation is also one of the key algebraic steps in the successful application of the present analytical solution.

2. Finite Element Alternating Method

Recently a major improvement of the alternating method has been made in [33,34]. In the new alternating method [33,34], the complete, general analytical solution [32,33] for an elliptical crack explained earlier was implemented in conjunction with the finite element method.

The major steps required in the finite element alternating method are given in the following: (i) Solve the uncracked body under the given external load by using the finite element method. To save computation time in solving the finite element equation for multiple right hand sides, a special solution technique was implemented [33]; (ii) Using the finite element solution, compute the residual stresses at the location of the original crack in the uncracked solid; (iii) Compare the residual stresses calculated in step (ii) with a permissible stress magnitude; (iv) To satisfy the stress boundary condition on the crack surface, reverse the residual stresses. Then determine the analytical solution for the crack subjected to these reversed residual stresses; (v) Evaluate the stress intensity factors in the analytical solution for the current iteration; (vi) calculate the residual stresses on external surfaces of the body due to the applied stresses on the crack surface in step (iv). To satisfy the stress boundary condition on the external surfaces of the body, reverse the residual stresses and calculate the equivalent nodal forces; and (vii) Consider these nodal forces as external applied loads acting on the uncracked body. Repeat all steps in the iteration procedure until the residual stresses on the crack surface become negligible (step iii). To obtain the final solution, add the stress intensity factors of all iterations.

In the above, several novel computational techniques were also implemented to save the computation time and to improve the convergency and accuracy of the present finite element alternating method [33,34,35,36]. Since a very coarse mesh can be used to analyse the uncracked body, the alternating method becomes a very inexpensive procedure for routine evaluation of accurate stress intensity factors for flaws in structures. It was found [33] that this new alternating method is at least an order of magnitude inexpensive compared to the earlier hybrid-element procedure [37].

The present alternating method has so far been applied to solve the following problems: (i) embedded elliptical cracks in finite bar and plate [33], (ii) the benchmark problems, semi-elliptical surface cracks in finite-thickness plates [33], (iii) outer and inner semi-elliptical surface cracks in pressure vessels [35], (iv) a quarter-elliptical corner crack in a brick [36], (v) quarter-elliptical corner cracks emanating a hole in finite-thickness plate [36], and (vi) quarter-elliptical corner cracks emanating from a pin hole in aircraft attachment lugs [36].

Several studies are now underway to use the alternating method for multiple semi-elliptical cracks in pressure vessels and for thermal shock analysis of surface flaws in pressure vessels.

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