Study of Ductile-Brittle transition in Fiber Bundle Model Under Mean-Field Approximation

Subhadeep Roy[1*], Purusattam Ray[1]

[1] The Institute of Mathematical Sciences, Taramani, CIT Campus, Chennai 600113, Tamil Nadu, India
* Corresponding author : sroy@imsc.res.in

Abstract Defects play crucial role in the process of nucleation and propagation of fracture. We study fracture as a non-equilibrium statistical mechanical system with a threshold activated extremal dynamics. The dispersion δ in the breaking thresholds shows a transition at $\delta = \delta_c$. For $\delta < \delta_c$, the fracture or failure takes place sharply at a certain applied stress or voltage. For $\delta > \delta_c$, the fracture or failure takes place gradually with increasing stress or voltage, leading to a non-linear region in the stress-strain or current-voltage curve prior to failure. We discuss the transition point For δ_c from the point of view of ductile-brittle transition. We present analytical results on fiber bundle model and numerical results on two dimensional random resistor network model.

Keywords Ductile-brittle transition, Threshold activated dynamics, Fiber bundle model, Random resistor network model.

1. Introduction

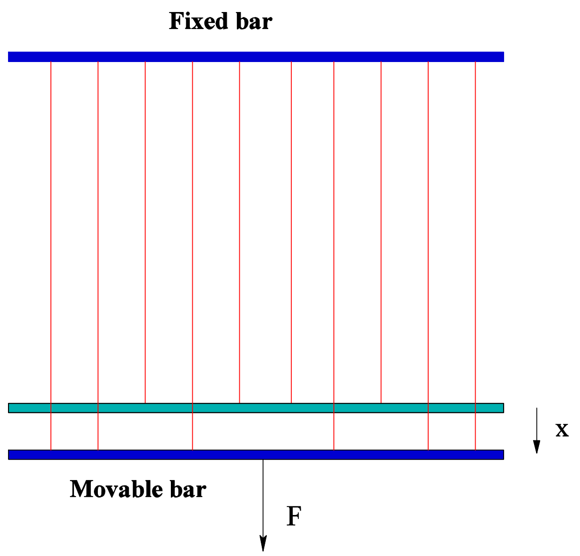
In material science and engineering ductile and brittle behaviors have received lots of attention over the past years. It is widely known that most ductile materials become brittle when the temperature or pressure is lowered or by radiation bombardment and atmospheric reaction. This phenomenon, known as ductile-brittle transition, has been extensively studied by engineers, material scientists and physicists. Disorder is known to play a crucial role in fracture. It is now known that topological disorders like dislocations and their cooperative motion give rise to ductile fracture. Brittle fracture on the other hand originates from a single micro-crack, the most vulnerable one. Thus ductile fracture can be thought of as a co-operative process of defects whereas brittle fracture is determined by extreme events [1]. Though several aspects of fracture are now understood, the mechanism and the nature of ductile-brittle transition is not clear yet. In this paper, we study the effect of dispersion (δ) in the local strengths in a material on the nucleation and propagation of fracture. We show for low δ the system behaves like a brittle material where failure happens sharply at a certain value of applied stress or voltage and the response of the system remains linear up to the failure point. For high δ the breakdown of the system happens gradually through local failures till the entire system breaks. The stress-strain curve in the elastic system or the voltage-current characteristics in electrical system shows non-linear region prior to failure. We argue for a transition point δ_c separating the low and high δ regions.

We have studied two models: fiber bundle model and random resistor network model. The fiber bundle model is solved analytically under mean field approximation. We have provided the numerical results for both the models. The models perceive fracture as a threshold activated extremal dynamical system. The disorder in these models is introduced in terms of random threshold stress of individual fibers or current in resistors.

We have found a transition point of threshold distribution ($\delta = \delta_c$) which clearly separates two different regions in the model. For $\delta < \delta_c$, the fracture phenomena is very sharp and similar to that of brittle fracture. The sharpness of this failure is expressed by the sudden jump of the fraction of unbroken bonds in fiber bundle model and conductance in random resistor network model, at the fracture point. Where for $\delta > \delta_c$, the bonds or resistors break in a correlated manner producing avalanches of all sizes and the model shows the features of ductility. We have also studied the mechanical and electrical response of the model. For δ less then δ_c the response curve shows a perfectly linear brittle like nature where as for δ greater than δ_c, the response curve contains a ductile like non-linear or plastic region.

2. Fiber Bundle Model

The review of modern physics written by S. Pradhan, A.Hansen and B.K.Chakrabarti has a very good overview on fiber bundle model [2]. The model consist of two parallel bars between which rigid fibers are attached (Fig 1a). One of the two bars is fixed and an amount of force F is applied on the other one. If there are N number of fibers then force per fiber i.e stress applied on the system is, f_0 =F/N.



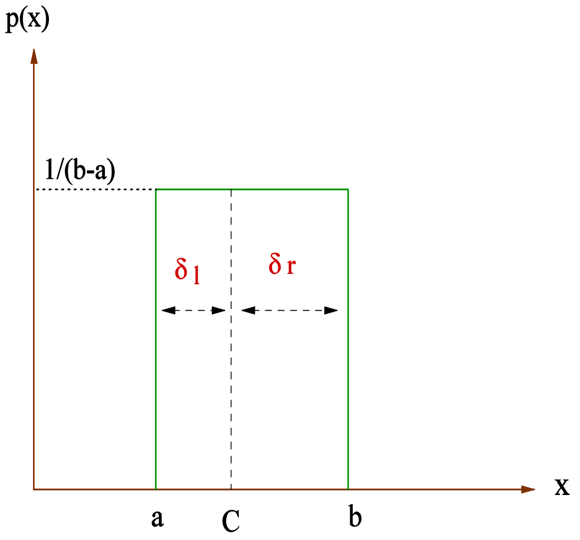


Figure 1a: Fiber Bundle Model *Figure 1b: Threshold Distribution*

Every fiber has some threshold stress value and when the external stress exceeds this value it breaks. After breaking of one fiber the stress is re-distributed among all other fibers according to global stress re-distribution scheme; which would increase the stress and may cause further breaking of fibers. For example, at a constant applied force F, the re-distributed stress will be F/(N-1) after breaking of one fiber, F/(N-2) after breaking of two fibers and so on. This process will continue until either all the fibers break or the redistributed stress is less than the next threshold value. At this stage the model comes to equilibrium. The number of remaining fibers at each equilibrium is denoted by N_{eq}. Then the fraction of fibers remained at equilibrium point is $n_{eq} = N_{eq}/N$. Next we increase the external stress ($F \rightarrow F+\Delta F$); which leads to further breaking, re-distribution and equilibration. This process goes on until all the fibers break. The threshold stress is given to the fibers in a random way, picking random numbers between 0 to 1 from an uniform distribution around any mean value C and width δ_r and δ_l respectively on right and left side of the mean value (Fig 1b). We are considering the vulnerability of the defect in the form of a breaking threshold of a fiber. The threshold distribution represents the fluctuations of these vulnerabilities in mean field limit (where the stress fluctuation on the bonds are neglected). The fiber bundle model gives us an idea of the correlations in the breaking processes in fibers. The fluctuations in the breaking threshold gives rise to the cooperative nature in the breaking processes in fibers which is the essence of ductile fracture.

2.1 Analytical Calculations

In this paper we provide the analytical calculations for the static and dynamic behavior of the model. We have compared our results with some previously calculated values depending on some specific choices of the mean and width of threshold distribution. At first we will study the distribution for giving threshold to the fibers. From figure 1b we can see that the mean is at C which is in between 0 and 1. Then the point ‘a’ and ‘b’ is at ($C-\delta_l$) and ($C+\delta_r$) respectively. Now we can take $\delta_r=\delta$ and relate this with δ_l by the relation : $\delta_l=\alpha\delta_r=\alpha\delta$. For the analytical calculations we will take $\alpha=1$. This will give us a distribution of width 2δ ranging from a to b. Then the fraction of broken bond at every equilibrium will be given by

$$1 - n_{eq} = \int_0^{f_0/n_{eq}} P(\sigma)d\sigma \qquad \ldots\ldots\ldots\ldots..1$$

We are interested in the region $a \le f_0/n_{eq} \le b$ as the other regions are quite trivial. For $f_0/n_{eq} > b$, all fibers are broken while for $f_0/n_{eq} < a$, all fibers are intact.

2.1.1 Determination of fraction of unbroken bonds at equilibrium points

Within the region $a \le f_0/n_{eq} \le b$ we can express the fraction of unbroken bonds as

$$1 - n_{eq} = \int_a^{f_0/n_{eq}} \frac{1}{2\delta} d\sigma = (1/2\delta)(f_0/n_{eq} - a) \qquad \ldots\ldots\ldots\ldots2$$

This will give us a quadratic equation of n_{eq} :

$$n_{eq}{}^2 - \left(1 + \frac{a}{2\delta}\right) n_{eq} + \frac{f_0}{2\delta} = 0 \qquad \ldots\ldots\ldots\ldots..3$$

The solution of the above equation will give us the fraction of unbroken bonds :

$$n_{eq} = \frac{1}{2}\left(1 + \frac{a}{2\delta}\right) \pm \frac{1}{2}\sqrt{\left(1 + \frac{a}{2\delta}\right)^2 - \frac{2f_0}{\delta}} \qquad \ldots\ldots\ldots..4$$

2.1.2 Determination of critical stress of the model

At critical point all the bonds break on application of an external stress, f_c, known as critical stress. The number of fractional bonds remaining just before the breakdown is nc. Thus at $f_0=f_c$, n_{eq} has only one value. This implies that the term in the square root in Eq.(4) should be zero; which in turn gives us,

$$\frac{2f_c}{\delta} = \left(1+\frac{a}{2\delta}\right)^2$$

Or, $f_c = \frac{\delta}{2}(1 + a/2\delta)^2$ ………..5

2.1.3 Fraction of unbroken bonds at critical stress

We can get the fraction of unbroken bond at critical stress by inserting the value of f_c from Eq.(5) in Eq.(4). Then we will get

$$n_c = n_{eq}|_{f_0=f_c}$$

Or, $n_c = \frac{1}{2}(1 + a/2\delta)$ ………..6

We find from Eq.(6), n_c starts increasing from a value 0.5 at $\delta = 0.5$ and reaches a value 1 at $\delta = \delta_c$. For $\delta < \delta_c$, n_c remains at 1 since it is at its maximum point and can't attain a higher value.

2.1.4 Study of Ductile-Brittle transition point

We have the expression of the critical stress given by Eq.(5). Now for $f_0/n_{eq} < a$, all bonds are intact and at the critical limit we can write

$$\frac{f_0}{n_{eq}}|_{f_0=f_c,n_{eq}=n_c=1} = a \qquad \text{………7}$$

Replacing the value of f_c from Eq.(5) in Eq.(7) we get the critical value of $\delta(\approx \delta_c)$ as :

$$\frac{\delta}{2}\left(1+\frac{a}{2\delta}\right)^2|_{\delta=\delta_c} = a$$

Or, $\delta_c = a/2$ ………8

2.2 Numerical Results

For numerical results we will set C=0.5 and α=1. This gives a threshold distribution with a mean at 0.5 and width δ on both side of the mean. The system size is taken to be 10^6. All simulated results are over 100 configuration averages.

2.2.1 Variation of fraction of unbroken bonds (n_{eq}) with applied stress (f_0)

We have plotted the fraction of unbroken bonds at each equilibrium (n_{eq}) with applied stress (f_0) for a series of δ's between 0.1 and 0.5. The results are given by figure 2. As δ increases the critical applied stress(f_c) at which the total model breaks decreases from 0.5 and reaches 0.25 at δ=0.5.

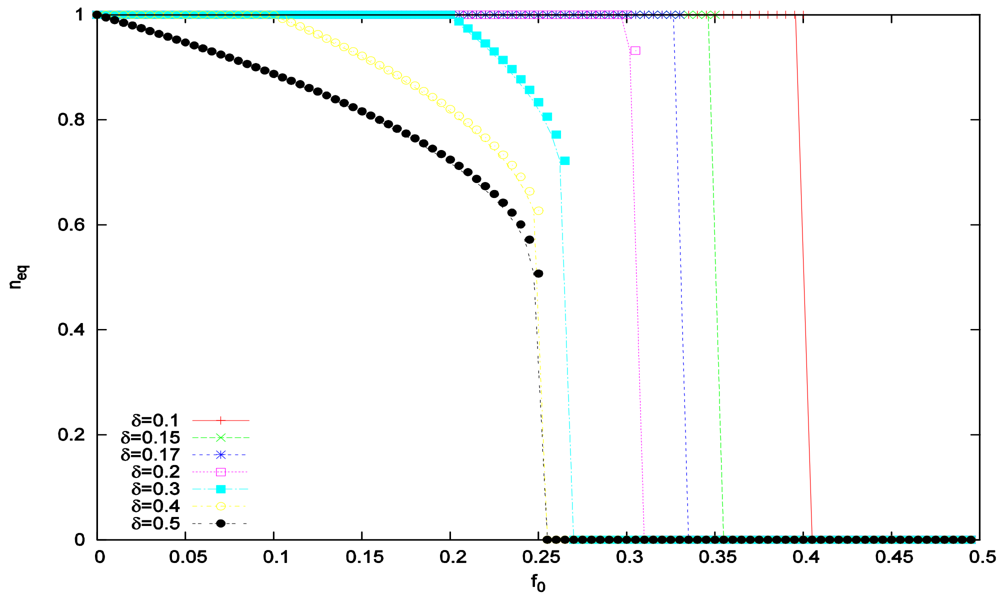


Figure 2: Fraction of unbroken bonds(n_{eq}) with applied stress(f_0)

Moreover up to a certain value δ≈0.17, the fraction of unbroken bonds at critical point (n_c) remains 1. Beyond this value, n_c starts decreasing and comes to a value 0.5 at δ=0.5. The point δ≈0.17 is like a critical point. This is quite consistent with the value we obtained analytically. At one side of the this point the model behaves like ductile materials showing a fracture through a number of equilibrium configurations at a low critical stress; where on the other side it behaves like brittle materials showing an abrupt fracture at a higher critical stress. We have studied this critical point in detail in our other simulated results.

2.2.2 Variation of redistributed stress ($f=f_0/n_{eq}$) with applied stress (f_0) for different δ's

Since the fibers are totally rigid, there is no strain in the model in true sense. We will treat the re-distributed stress as strain since the non-linearity is introduced in the model through this re-distributed stress.

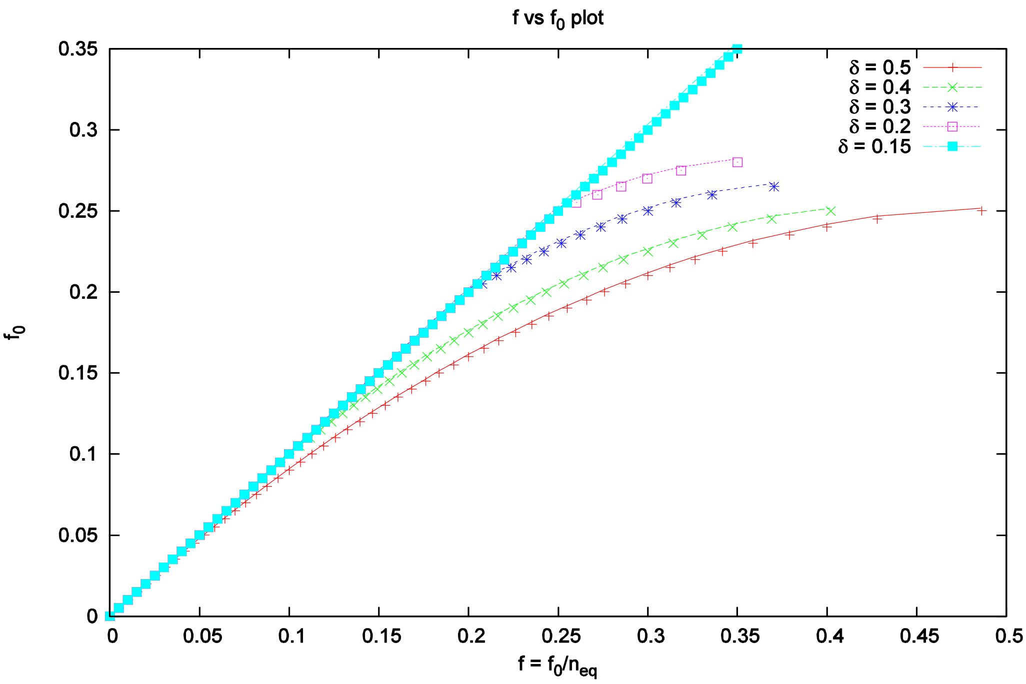


Figure 3: Stress-strain curve (f_0 vs f_0/n_{eq}) for the model

This is very similar to the response curve of mechanical systems. We have plotted f_0 with f_0/n_{eq} for different δ's with in 0.1 and 0.5. Figure 3 shows that for $\delta \leq 0.15$, the model behaves like brittle materials while for $\delta \geq 0.2$ the model shows ductility. There must be a critical value of threshold distribution width (δ_c) within the range $0.15 < \delta_c < 0.2$, where this ductile-brittle like transition occurs.

2.2.3 Variation of critical stress (f_c) with threshold distribution width (δ)

We have already shown that the critical stress to create fracture in the model increases with decreasing δ values. Figure 4 gives us an idea how this critical stress decreases with a continuous increase in threshold distribution width. Figure 4 shows that, the critical stress (f_c) decreases with increasing δ and falls from 0.4 to 0.25 within the range between 0.1 and 0.25 in δ value. Though the transition point is not quite evident from the plot, we can estimate it as the average of the above mentioned δ region. So, from above study we get $\delta_c \approx 0.17$. This agrees with the value we obtained analytically and other simulated results. The variation of critical stress (f_c) with δ and the nature of response curve is obtained before in two dimensional fiber bundle model which consists of two square plates in which fibers are intact [3]. We have reproduced the same result in a relatively easier model.

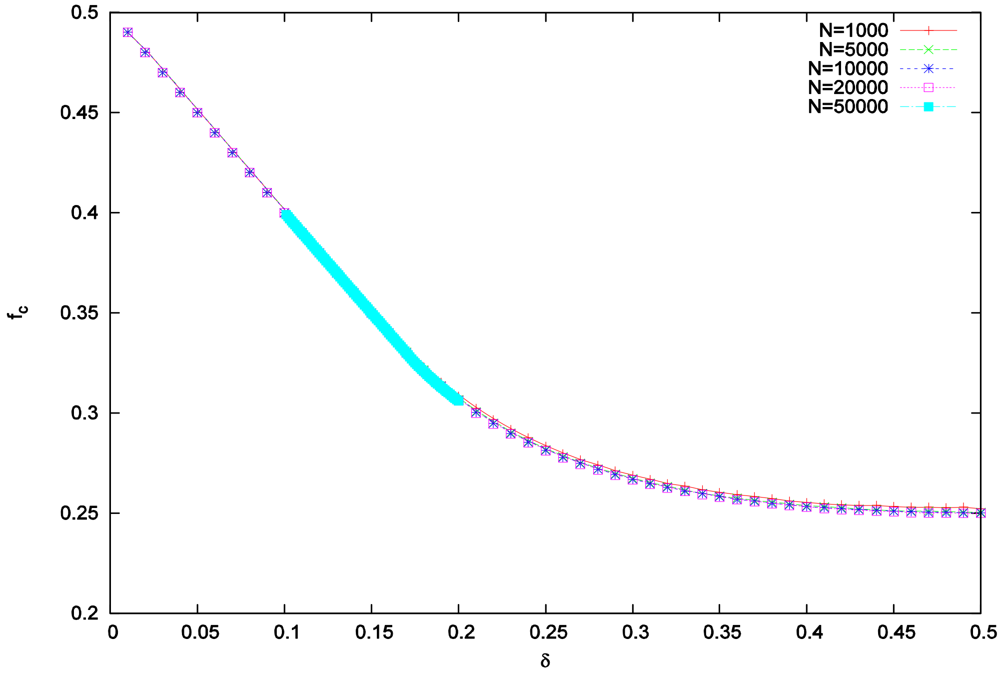


Figure 4: Variation of critical stress (f_c) with δ

2.2.4 Fraction of unbroken bonds (n_c) at critical stress with threshold distribution width (δ)

Above study of critical stress is unable to give us a clear distinction between ductile and brittle region in the model. This distinction can be drawn from the study of fraction of unbroken bonds at critical point (n_c) with varying δ value.

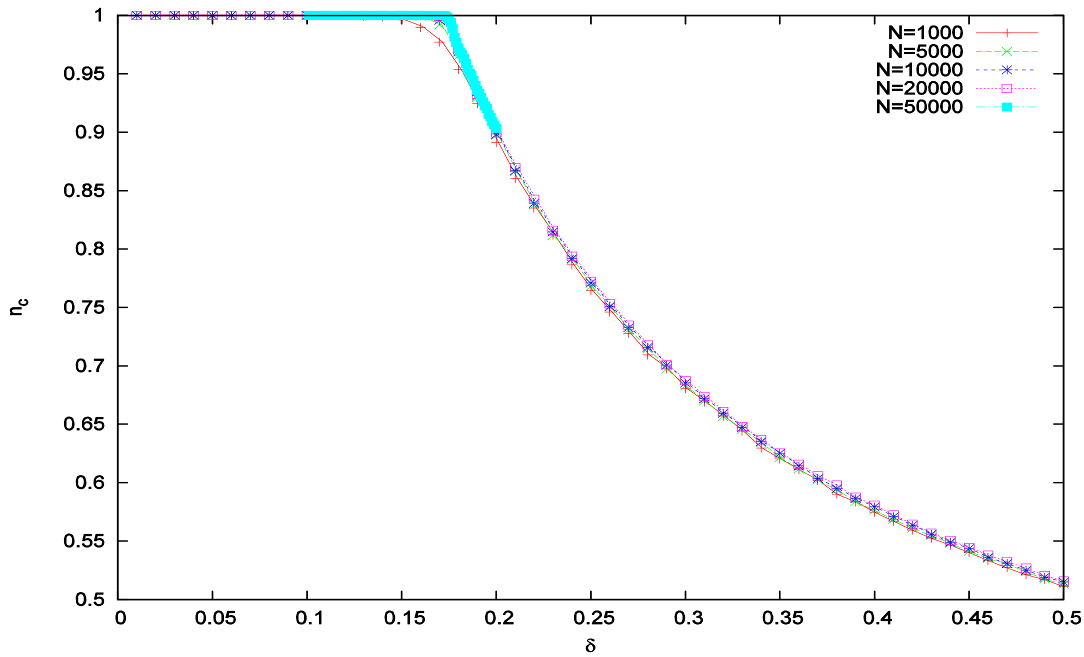


Figure 5: Variation of n_c with δ

One of the strong points that helps us to denote low δ region as brittle is, the value n_c remains 1 up to a certain low value of δ characterizing a abrupt fracture in the model. Figure 5 reflects the same fact. Up to a value $\delta_c \approx 0.17$, n_c remains 1. This is the brittle region for the model. If we increase δ beyond this n_c starts to fall and goes to 0.5 at δ=0.5. This is the ductile region. The above critical point ($\delta_c \approx 0.17$) separates these two regions very clearly.

2.2.5 Variation of (f_c/n_c) with threshold distribution width (δ)

A plot of f_0/n_{eq} with δ gives us a clear idea about the point at which the ductile to brittle like transition occurs. The point with coordinate (f_0/n_{eq}=0.32, $\delta_c \approx 0.17$) in the plot denotes the transition point from brittle to ductile region and vice verse.

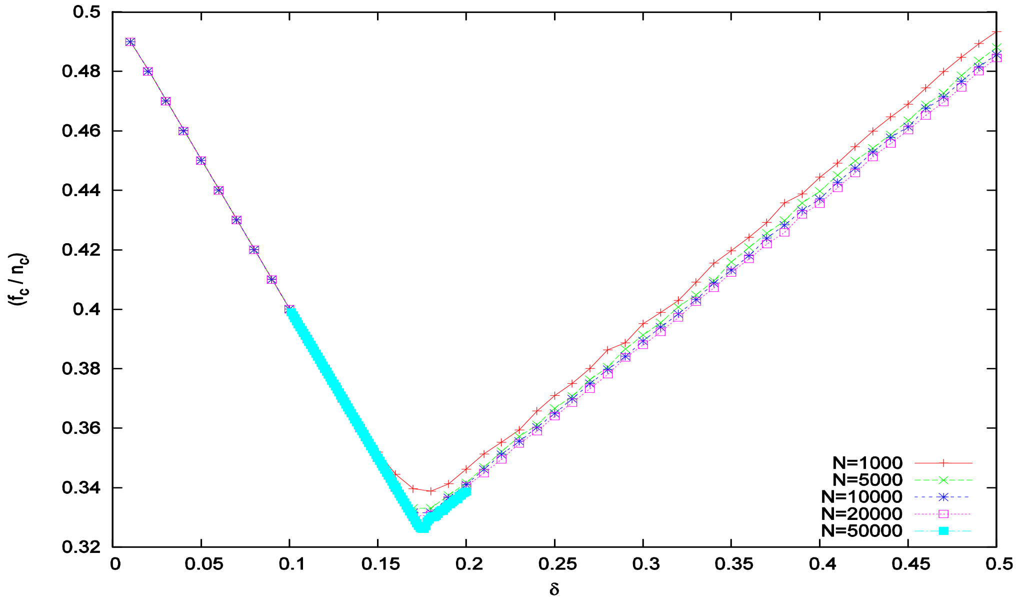


Figure 6: Variation of f_c/n_c with δ

The obtained value of δ at which the transition occurs is $\delta_c \approx 0.17$; which is very similar to the values obtained from other numerical studies and analytical calculation.

2.2.7 Phase diagram for the model

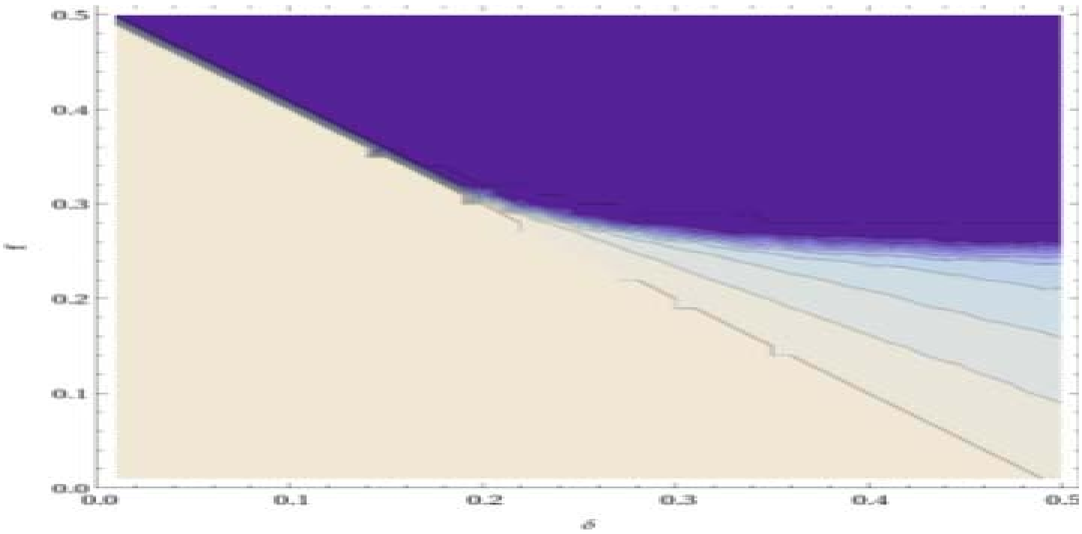
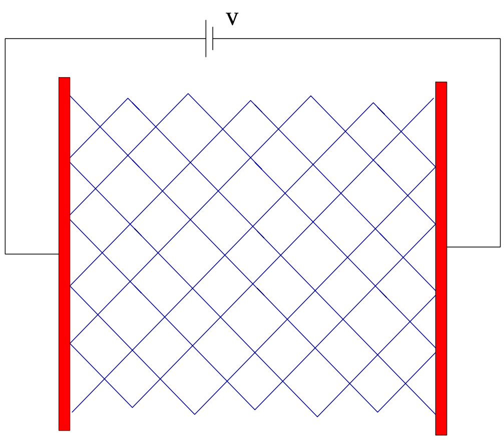


Figure 7: Phase diagram for fiber bundle model

We have constructed a phase diagram for the model and given by figure 7. The darkest color corresponds to n_{eq}=0 and the lightest color for n_{eq}=1. A very important observation is related to this diagram. For δ<0.166, n_{eq} fall to zero abruptly. When we cross this particular point the behaviour of n_{eq} changes. For δ>0.166, n_{eq} falls to zero after crossing a number of equilibrium points. This is the point where the transition occurs. On one side of this point the material shows ductile like fracture and on the other side the fracture resembles abrupt brittle fracture.

3. RANDOM RESISTOR NETWORK MODEL

We have constructed a tilted square lattice whose each bond is a resistor with a resistance 1. A potential difference of V volts is applied to two opposite ends of the lattice. On the other two ends periodic boundary condition is applied (Fig 8a). All resistors are given a random threshold taken from the same uniform distribution same as the case of fiber bundle model.



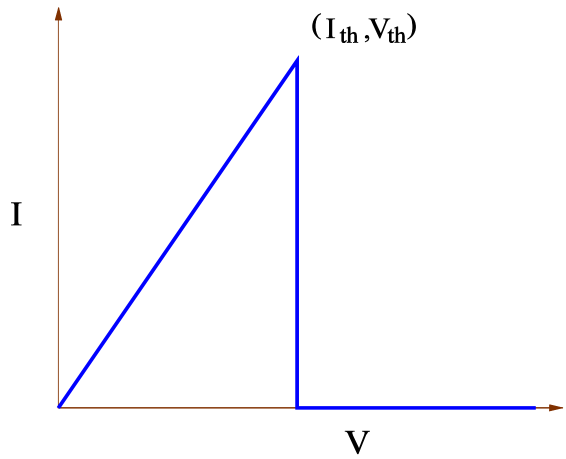


Figure 8a: Random resistor network model

Figure 8b: Response of individual resistors

The potential at each lattice point is calculated by solving ‘Kirchoff's law’ with a series of iterations. Whenever the potential drop between two lattice points becomes less than the threshold value given to the resistor joining these two points, the resistor breaks irreversibly (Fig 8b). Then we apply ‘Kirchoff's law’ again to calculate the new potentials due to the breaking of the fiber. When no more fiber breaks, we increase the applied potential and do the same thing. This process goes on until the conductance of the system becomes zero. This is the critical point for the model.

3.1 Numerical Results

For numerical simulations we have taken the same specifications like fiber bundle model. The lattice size is taken to be 16×16.

3.1.2. Variation of conductance (G) with applied voltage

Conductance is the parameter which denotes the fracture point for the model. The rate at which the conductance drops to zero is the abruptness of the fracture. We have plotted conductance with applied voltage and observed its behavior at various values of δ.

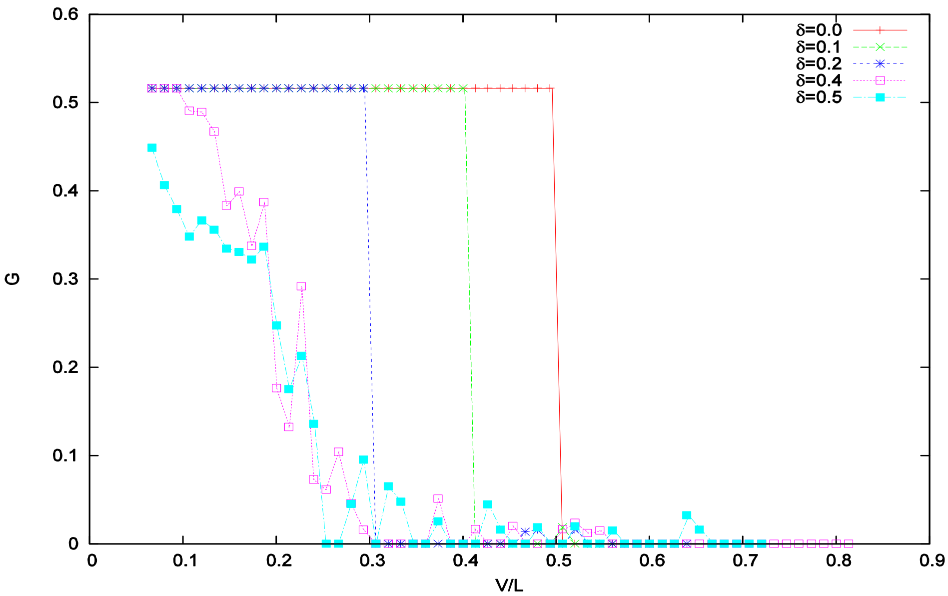


Figure 9: Variation of conductance with applied voltage

Up to δ=0.2 the conductance falls very sharply causing an abrupt fracture in the model. This behavior is analogous to sharp brittle fracture. With increasing δ this sharpness decreases. For δ ≥0.3 the fracture occurs in a co-operative manner like ductile materials. Moreover the constant initial value of conductance corresponds to a linear region in current voltage characteristic curve prior to fracture. This behavior is like brittle materials. As δ increases the conductance start decreasing from this constant value after a certain value $\delta = \delta_c$. Above this δ value current-voltage charecteristics curve stars showing non-linearity and behaves like a ductile material.

4. DISCUSSION AND CONCLUSIONS

As an outcome of this paper we can point out some important results :
1. Variation of conductance in random resistor network and fraction of unbroken bond at critical point (n_c) in fiber bundle model with δ shows that the behavior of the model in low δ is brittle like and that for high δ is ductile like.
2. The mechanical and electrical response shows a pure linear or a linear behavior with small non-linear part before some critical value of δ. Beyond that the model behaves non-linearly like ductile materials. For fiber bundle model this critical value is, $\delta_c \approx 0.16667$. For random resistor network model this critical value δ_c remains within δ value 0.2 and 0.3. More studies are required to locate the exact transition point for random resistor network model.
3. The critical stress to create fracture in the fiber bundle model decreases to lower values with increasing δ's. This signifies a decrease in hardness of the model when δ is increased.
At this stage it is difficult to draw some analogy between any exact material property that causes ductile-brittle transition and the parameter δ in our model. Still we can suggest a possible parameter similar to δ. Early studies in material science prove that ductility in materials is a consequence of cooperative motion of defects (mainly dislocation) within it [1]. The material becomes brittle if the defects within it are immobile. There are also some external parameter like temperature or pressure [4] that control this mobility of defects. Because of this characteristic, materials show ductile-brittle transition with respect to these parameters. If we can define any quantity that deals with the cooperative motion of the defects in the materials and increases with increasing mobility of defects, then that quantity will be analogous to the δ parameter. If it is possible to find some critical value of this quantity, depending on the change of external parameters (temperature, pressure or any other

parameters), which distinguishes between ductile and brittle region in the materials; then it will be similar to δ_c, the ductile-brittle transition point.

Acknowledgements

I acknowledge Soumyadeep Bhattacharya for his guidance and suggestions from time to time. I also thank my parents, grandmother and specially my brother for their support and encouragement.

References

[1] George E.Dieter, Mechanical Metallurgy, McGraw-Hill Book Company, 1928.

[2] S. Pradhan, A.Hansen and B.K.Chakrabarti, Failure processes in elastic fiber bundles, Reviews of Modern Physics 82 (2010) 499-555.

[3] Frank Raischel, Ferenc Kun and Hans J. Herrmann, Local load sharing fiber bundles with a lower cutoff of strength disorder. Physical Review E-Statistical, Nonlinear and Soft Matter Physics, 74 (2006), 035104(R).

[4] R. D. Isaac and A. V. Granato , Rate theory of dislocation motion: Thermal activation and inertial effects. Phys. Rev. B 37, 92789285 (1988).

[5] B. Kahng, G. G. Batrouni, S. Redner, Electrical breakdown in a fuse network with random distributed breaking strengths. Physical Review B, Volume 37, Number 13, 1988.