Stress Concentration based on Non-linear Motion Equations and its Application for Non-destructive Detection of Plate

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Abstract For a plate with large deformation, where the possible fracture will be initiated is an important problem. It is true that the defects or voids of material usually are the initial positions. However, the more important problem is the local defect (fatigue-fracture) caused by the global deformation of medium, as this problem is very common in practical engineering. For this purpose, two sets of non-linear motion equations of deformation established in rational mechanics frame are used to study this problem. The stress concentration problem is defined as: for a given deformation, how the stresses are distributed to meet the motion equations under the cost of fatigue-fracture within the plate. The motion equations show that the local curvature is the main cause of fatigue-fracture. Taking the local deformation curvature as a parameter function, the stress transportation solution is obtained. The result shows that: for plate bending, the stress is varied in exponential law with the path-integral of local curvature of plate. In non-destructive detect, when the pressure wave data are recorded in an array, they can be used to inverse the intrinsic local curvature of target medium region. Then, the “inversed” local curvature can be used to predict the potential fatigue-cracking initiation region.

Keywords Nondestructive detect, Stress concentration, Stress transportation, Non-linear motion equation, Large deformation

1. Introduction

Stress concentration usually is addressed by the irregular boundary problems [1-3]. As a simple example, for a plate with small hole, the stress around hole will be increased rapidly when the in-plane stretching is increased. To express the effects of stress concentration, a factor (defined as maximum stress over name stress) is introduced. Its shortage is that the voids geometry features are required. Then, taking the voids as boundary, the motion equations are used to calculate the factor based on theoretic solutions. The crack tip field theory is fully developed along this theoretic line. Hence, it is understandable that many researchers focus on their attention upon micro-scale phenomena. However, based on experiences, we usually have no ad-hoc information about where the defects exist. What we want to know is where the fatigue-fracture will be initiated? The micro-scale structure description has little help on this topic. Generally, for uniform continuum, the stress concentration phenomena can be attributed to the cause of the defects of micro-scale structures. For a plate with arbitral deformation, the real fatigue-fracture frequently appears in the highest curvature position or the position inherited with large scale deformation or singularity. On phenomenon sense, the defects or voids of material were produced by the macro deformation in space-time domain.

Recently, how the macro deformation alters the microstructures of material is raised as an important problem for fracture mechanics. The dual scale or multiscale viewpoint is proposed to answer how the macro deformation causes the microstructure instability [4-6]. To answer this question, the first problem is returned to the question about where the stress will be concentrated. This problem may be answered by formulating related motion equations.

For this purpose, two sets of non-linear motion equations of deformation [7] established in rational mechanics frame [8] are used to study this problem. One set of equations corresponds to linear momentum conservation, and another set of equations corresponds to angular momentum conservation. Then, the fatigue-fracture is initiated by local asymmetrical stress (which is related with local curvature [9]), although it may be very small comparing with the symmetrical stress. Hence, the research is looking for the "omitted" items in classical mechanics.

In this research, the stress concentration problem is defined as: for a given deformation, how the stresses are redistributed to meet the motion equations under the cost of fatigue-fracture within the plate. The motion equations show that the local curvature is the main cause of fatigue-fracture (in classical plate stability problem, the local curvature plays the similar role in von Karman equations [10]). Taking the local deformation curvature as a parameter function, the stress transportation solution is obtained. The result shows that: for plate bending, the stress is varied in exponential law with the path-integral of local curvature of plate.

For bending with harmonic local curvature, the integral along any path tends to be small, then, there is no stress concentration potential. However, for bending with monotone local curvature, the integral along any path tends to be large. Then, referring to a given stress position (as the path-integral starting point), the stress concentration will appear at the positions with maximum of path integral. In non-destructive detect, when the pressure wave data at different frequencies are recorded in an array, they can be used to inverse the intrinsic local curvature of target medium region. Then, the "inversed" local curvature can be used to predict the potential fatigue-cracking initiation region.

2. General Equations for Large Deformation

For large deformation, the deformation tensor (defined by displacement gradient) was introduced by Truesdell [11] and is named as a two-point tensor. Viewing that the base vector transformation for commoving dragging coordinator (natural coordinator) uniquely defines the deformation between initial configuration and current configuration, Chen Zhida [8, 12] established a new formulation for rational mechanics. This theoretic formulation is based on point-set transformation and is briefed as following. For a set of co-moving dragging coordinators defined in continuum, a material point is coordinated as (x^1, x^2, x^3). For initial configuration, the base vector is expressed as ($\vec{g}_1^0(x), \vec{g}_2^0(x), \vec{g}_3^0(x)$). For current configuration, the base vector is expressed as ($\vec{g}_1(x), \vec{g}_2(x), \vec{g}_3(x)$). Then, the differential distance vector between two material points is expressed as: $d\vec{s}_0(x) = dx^i \vec{g}_i^0(x)$ (here and after repeating index summation convention is applied for $i = 1,2,3$) for initial configuration; $d\vec{s}(x) = dx^i \vec{g}_i(x)$ for current configuration (here and after, repeat index summation convention is used).

For large deformation, the deformation tensor $F_j^i(x)$ (the coordinator dependence x will be omitted below and after) is defined by base vector transformation equation (point-sets group transformation):

$$\vec{g}_i = F_i^j \vec{g}_j^0 = (\delta_i^j + u^j\big|_i)\vec{g}_j^0 . \tag{1}$$

Where, u^j is displacement field defined in initial configuration, the covariant derivative $\big|_i$ is performed in initial gauge field.

The Cauchy strain tensor is defined as:

$$\varepsilon_i^j = F_i^j - \delta_i^j = u^j\Big|_i . \tag{2}$$

For simple idea isotropic elastic continuum, the stress tensor is defined as a mixture tensor $\sigma_i^j(\vec{g}^i \otimes \vec{g}_j^0)$ [7-9] through constitutive equations:

$$\sigma_i^j = \lambda \varepsilon_k^k \delta_i^j + 2\mu \varepsilon_i^j . \tag{3}$$

In engineering sense, the stress tensor component σ_i^j is explained as the surface force acting on current face $\vec{g}^i$ in the direction $\vec{g}_j^0$. In fact, this engineering interpretation is widely used in mechanics textbooks about stress tensor σ_{ij} (a kind of logic weakness on the sense that which one index represents surface or which one represents direction). For this mixture stress definition, the physical components of stress tensor are defined as $\tilde{\sigma}_i^j = \frac{\sqrt{g^0_{(jj)}}}{\sqrt{g_{(ii)}}}\sigma_i^j$. Surely, the stress symmetry in engineering stress sense does not mean the stress is symmetrical in intrinsic sense.

Then, as logic consequence, the stress differential for large deformation is:

$$\sigma_i^j\Big|_k = \frac{\partial \sigma_i^j}{\partial x^k} + \sigma_i^l \Gamma_{lk}^j - \sigma_l^j \tilde{\Gamma}_{ik}^l . \tag{4}$$

Where, the connection Γ_{jk}^i is defined in initial configuration; $\tilde{\Gamma}_{jk}^i$ is defined in current configuration. Without losing generality [7], taking the initial configuration in standard rectangular coordinator system to make a simplification that $\Gamma_{jk}^i = 0$, it can be simplified as:

$$\sigma_i^j\Big|_k = \frac{\partial \sigma_i^j}{\partial x^k} - \sigma_l^j \frac{\partial \varepsilon_i^l}{\partial x^k} . \tag{5}$$

It shows that the non-linear items for large deformation are mainly originated from strain gradient and large stress. In resent years, the role of strain gradient has been studied extensively.

In rational mechanics of Chen formulation [7-9], the motion equations are classified into two categories as: covariant form and anti-variant form. In deformation mechanics, they must be satisfied at the same time. Generally speaking, the anti-variant force corresponds to linear momentum conservation and the covariant force corresponds to angular momentum conservation.

For large deformation with body force $\vec{f} = f^i \vec{g}_i^0$ in local standard rectangular coordinator system, the anti-variant form of motion equation (linear momentum conservation) is:

$$\frac{\partial \sigma_j^i}{\partial x^j} - \sigma_l^i \frac{\partial \varepsilon_j^l}{\partial x^j} = f^i . \tag{6}$$

The covariant form of motion equation (angular momentum conservation) is:

$$\frac{\partial \sigma_i^j}{\partial x^j} - \sigma_l^j \frac{\partial \varepsilon_i^l}{\partial x^j} = f^j F_i^j . \tag{7}$$

In von Karman elastic shell theory the item $-\frac{\partial \varepsilon_j^l}{\partial x^j} = -\frac{\partial^2 u^l}{\partial x^j \partial x^j}$ ($l = 3, i, j = 1,2$) is related with the curvature of shell or plate. So, the Eq.6 should be viewed as von Karman equation. Following this curvature interpretation, it is concluded that: for high stress deformation, the curvature produced by deformation must be taken into consideration.

3. Geometrical Equations for Plate Bending

For simplicity, the initial configuration of central plan is taking as standard rectangular coordinator system (x^1, x^2). Hence, the plate is described as a two dimension manifold. The thickness direction is taken as coordinator (x^3). In this research, the plate bending problem is described by unit-orthogonal transformation of base vector as:

$$\vec{g}_i = R_i^j \vec{g}_j^0 . \tag{8}$$

Where, for plate bending, based on Chen's S+R additive decomposition of deformation tensor [7-9], the tensor R_i^j is defined as:

$$R_i^j = \delta_i^j + \sin\Theta \cdot L_i^j + (1-\cos\Theta) L_k^j L_i^k . \tag{9}$$

Where, the Θ is local whole rotation in average sense (local curvature) [12], and the tensor L_i^j is the rotation direction.

The related items are expressed by the displacement fields as:

$$L_1^3 = -L_3^1 = \frac{1}{2\sin\Theta}(\frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3}) = L_2 , \tag{10-1}$$

$$L_3^2 = -L_2^3 = \frac{1}{2\sin\Theta}(\frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2}) = L_1 , \tag{10-2}$$

$$L_2^1 = -L_1^2 = \frac{1}{2\sin\Theta}(\frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}) = 0 , \tag{10-3}$$

$$\sin\Theta = \frac{1}{2}\sqrt{(\frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3})^2 + (\frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2})^2} . \tag{10-4}$$

For plate bending, as the gauge tensor is invariant, the geometrical conditions [7-9] are obtained as:

$$\frac{\partial u^1}{\partial x^1} = -(1-\cos\Theta)(L_2)^2 , \tag{11-1}$$

$$\frac{\partial u^2}{\partial x^2} = -(1-\cos\Theta)(L_1)^2 , \tag{11-2}$$

$$\frac{1}{2}(\frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1}) = -(1-\cos\Theta)(1-L_1 L_2) , \tag{11-3}$$

$$\frac{\partial u^3}{\partial x^3} = -(1-\cos\Theta) , \tag{11-4}$$

$$\frac{1}{2}(\frac{\partial u^1}{\partial x^3} + \frac{\partial u^3}{\partial x^1}) = 0 , \tag{11-5}$$

$$\frac{1}{2}(\frac{\partial u^2}{\partial x^3} + \frac{\partial u^3}{\partial x^2}) = 0 . \tag{11-6}$$

Combining Eqs.10 and Eqs.11, it is easy to find out that: all $\varepsilon_j^i = \frac{\partial u^i}{\partial x^j}$ are non-zeros, and the plate bending is completely determined by three independent quantities: $\Theta(x), L_1(x)$, and $L_2(x)$. Note that for small Θ, the $(1-\cos\Theta) \approx \frac{1}{2}\Theta^2$ is higher order smaller. The corresponding Cauchy stress components are defined by Eq.3. In engineering, the effective stresses (which contain non-linear effects) are used as a convention. In this research, the effective stress fields in the plate are studied by motion equations to study stress concentration problem.

4. Effective Stress Concentration for Plate Bending

Letting $\frac{\partial \varepsilon_j^l}{\partial x^j} = \kappa^l$ ($l = 1,2,3$) as local curvature functions, based on Eq.6, the anti-variant motion equations are rewritten as:

$$\frac{\partial \sigma^i_j}{\partial x^j} - \sigma^i_j \kappa^j = f^i \qquad (12)$$

As the plate bending is dominated by out-plan displacement u^3 measured by central plane coordinator (commoving dragging coordinator system), by Eqs.10 and Eqs.11, the curvature functions can be approximated as:

$$\kappa^3 = \frac{\partial^2 u^3}{\partial x^1 \partial x^1} + \frac{\partial^2 u^3}{\partial x^2 \partial x^2} = \frac{\partial (L_2 \sin\Theta)}{\partial x^1} + \frac{\partial (L_1 \sin\Theta)}{\partial x^2} \qquad (13\text{-}1)$$

$$\kappa^1 = -\frac{\partial^2 u^3}{\partial x^1 \partial x^3} = \frac{\partial (L_2 \sin\Theta)}{\partial x^3} \qquad (13\text{-}2)$$

$$\kappa^2 = -\frac{\partial^2 u^3}{\partial x^2 \partial x^3} = \frac{\partial (L_1 \sin\Theta)}{\partial x^3} \qquad (13\text{-}3)$$

By these equations, the local curvature functions are the gradient of global bending curvature. In classical plate theory, the κ^1 and κ^2 are taken as linear function about thickness and are explained as curvature variation along thickness direction.

For fatigue-fracture problems, as the deformation is given, so the effective stress is varied to meet the motion equations.

Observing Eq.3, the effective stress field $\tilde{\sigma}^i_j$ can be defined by the following equation:

$$\frac{\partial \tilde{\sigma}^i_j}{\partial x^j} = \frac{\partial \sigma^i_j}{\partial x^j} - \sigma^i_j \kappa^j = f^i \qquad (14)$$

Then the effective stresses meet classical motion equations. Omitting the derivatives of curvature functions (as first order approximation), one simple form solution for effective stress is:

$$\tilde{\sigma}^i_j(x) = \sigma^i_j(x) \cdot \exp(-\int_{x_0}^{x} \kappa^l dx^l) \qquad (15)$$

Where, the x_0 is a reference point waiting to be determined by boundary conditions and loads. This solution means that: local effective stress is redistributed by the path-integral of global curvature functions. Therefore, stress concentration may appear somewhere. Based on Eq.3, for the effective stress, the constitutive equation is:

$$\tilde{\sigma}^j_i = (\lambda \varepsilon^k_k \delta^j_i + 2\mu \varepsilon^j_i) \cdot \exp(-\int_{x_0}^{x} \kappa^l dx^l) \qquad (16)$$

It says that: for effective stress, the elasticity parameters are exponentially varied with the path-integral of curvature functions. In engineering mechanics, the increased elasticity is named as hardening and the decreased elasticity is named as softening. Then, by the above equations, the effective elasticity of bending plate has both effects. In engineering sense, the effective elasticity is varied by bending deformation significantly.

To make its meaning clear, the thickness effects can be expressed as:

$$\tilde{\sigma}^i_j(x)\Big|_{x^3} = \tilde{\sigma}^i_j(x)\Big|_{x^3=0} \cdot \exp(-\int_0^{x^3} \kappa^3 dx^3) \qquad (17)$$

It shows that the effective stresses are exponentially distributed on thickness direction. In classical plate theory, the linear approximation is assumed.

It shows that: the effective stress on central plan parallel surface is exponentially varied with path-integral of curvature along thickness direction. So, the effective stress will concentrated on one surface parallel to central plan. If the local curvature is big enough, the fatigue-fracture may be initiated as surface cracking parallel to central plan (surface sliding or buckling). In some researches [13-15], the thickness direction stress concentration effects are studied under the terms distension or post-buckling). The above equation shows that, for given curvature, the plate thickness is limited by stress concentration effects along thickness direction.

Similarly, taking the effective stress at some potions as reference, the scale effects can be expressed as effective stress transportation along plan directions as:

$$\tilde{\sigma}^i_j(x)\Big|_{x^1} = \tilde{\sigma}^i_j(x)\Big|_{x^1_0} \cdot \exp(-\int_{x^1_0}^{x^1} \kappa^1 dx^1) \tag{18-1}$$

$$\tilde{\sigma}^i_j(x)\Big|_{x^2} = \tilde{\sigma}^i_j(x)\Big|_{x^2_0} \cdot \exp(-\int_{x^2_0}^{x^2} \kappa^2 dx^2) \tag{18-2}$$

They show that: the effective stresses on central plan parallel surface are exponentially varied with path-integral of curvature along scale directions. So, the effective stress will concentrated on one side. If the local curvature is big enough, the fatigue-fracture may be initiated as surface cracking line (surface fracture). Therefore, for given curvatures, the plate scale is limited by stress concentration effects along plan directions.

For spatial harmonic bending with spatial frequencies (k_1, k_2, k_3), $\kappa^i = \kappa^i_0 \sin(k_1 x^1 + k_2 x^2 + k_3 x^3)$, the path-integral values depends its corresponding spatial scales $(1/k_1, 1/k_2, 1/k_3)$. Therefore, multi-scale effects are very significant [4-6] for stress concentration phenomenon or fracture problems. This explains why microstructure analysis is always dominating the theoretic development about fatigue-fracture mechanism.

For forward problems of non-destructive detect, the central plan bending functions $\Theta(x)$, $L_1(x)$,and $L_2(x)$ are measured directly. Then, the curvature functions can be calculated by Eqs.13. In this case, the stress concentration effects can be evaluated by the Eqs.18.

For inverse problems of non-destructive detect, when the effective stress data can be acquitted, the Eqs.18 can be used to estimate the curvature functions. By the estimated curvatures, the potential fatigue-fracture positions can be predicted.

For supersonic wave methods, a lot of technology to get effective elasticity is available. By Eq.16, the effective elasticity for wave (incremental deformation [16]) is determined as:

$$\Delta\tilde{\sigma}^j_i = \lambda(\Delta\varepsilon^k_k)\delta^j_i + 2\mu(\Delta\varepsilon^j_i)] \cdot \exp(-\int_{x_0}^{x} \kappa^l dx^l) \tag{19}$$

So, as the original elasticity is known parameter, the curvature functions can be inversed. Therefore, stress concentration effects can be predicted.

5. Stress Concentration Caused by Load

Usually, the typical plate bending is produced normal loading (defined by $f^1 = f^2 = 0$). By subtracting Eq.6 and Eq.7, the asymmetrical stress motion equations are obtained as:

$$\frac{\partial(\sigma^3_1 - \sigma^1_3)}{\partial x^3} = f^3 \sin\Theta \cdot L_2 + (\sigma^j_l \frac{\partial \varepsilon^l_1}{\partial x^j} - \sigma^1_l \frac{\partial \varepsilon^l_j}{\partial x^j}), \tag{20-1}$$

$$\frac{\partial(\sigma^3_2 - \sigma^2_3)}{\partial x^3} = -f^3 \sin\Theta \cdot L_1 + (\sigma^j_l \frac{\partial \varepsilon^l_2}{\partial x^j} - \sigma^2_l \frac{\partial \varepsilon^l_j}{\partial x^j}), \tag{20-2}$$

$$\frac{\partial(\sigma^1_3 - \sigma^3_1)}{\partial x^1} + \frac{\partial(\sigma^2_3 - \sigma^3_2)}{\partial x^2} = (\sigma^j_l \frac{\partial \varepsilon^l_3}{\partial x^j} - \sigma^3_l \frac{\partial \varepsilon^l_j}{\partial x^j}). \tag{20-3}$$

Where, the approximation $R^j_i \approx \delta^j_i + \sin\Theta \cdot L^j_i$ is applied. By Eq.3 and Eqs.11, the non-symmetrical stress components are σ^3_1, σ^1_3, σ^3_2, and σ^2_3. They are expressed as:

$$\sigma^3_1 = -\sigma^1_3 = 2\mu L_2 \sin\Theta, \tag{21-1}$$

$$\sigma^2_3 = -\sigma^3_2 = 2\mu L_1 \sin\Theta. \tag{21-2}$$

So, letting: $w_i = (\sigma^j_l \frac{\partial \varepsilon^l_i}{\partial x^j} - \sigma^i_l \frac{\partial \varepsilon^l_j}{\partial x^j})$, the asymmetrical stress motion equations are rewritten as:

$$4\mu\frac{\partial(L_2\sin\Theta)}{\partial x^3}=f^3\sin\Theta\cdot L_2+w_1 \tag{22-1}$$

$$-4\mu\frac{\partial(L_1\sin\Theta)}{\partial x^3}=-f^3\sin\Theta\cdot L_1+w_2 \tag{22-2}$$

$$-4\mu\frac{\partial(L_2\sin\Theta)}{\partial x^1}+4\mu\frac{\partial(L_1\sin\Theta)}{\partial x^2}=w_3 \tag{22-3}$$

Where, the third motion equation is used to determine the whole bending along plan direction. Referring to the solutions $(L_2\sin\Theta)\big|_{f^3=0}$ at central plan for $f^3=0$, the body force caused asymmetrical stresses concentration in thickness direction are obtained as:

$$\sigma_1^3\big|_{f^3}=[\sigma_1^3\big|_{f^3=0}+\frac{1}{2}\int_0^{x^3}w_1\exp(-\int_0^{x^3}\frac{f^3}{4\mu}dx^3)dx^3]\cdot\exp(\int_0^{x^3}\frac{f^3}{4\mu}dx^3) \tag{23-1}$$

$$\sigma_2^3\big|_{f^3}=[\sigma_2^3\big|_{f^3=0}-\frac{1}{2}\int_0^{x^3}w_2\exp(-\int_0^{x^3}\frac{f^3}{4\mu}dx^3)dx^3]\cdot\exp(\int_0^{x^3}\frac{f^3}{4\mu}dx^3) \tag{23-2}$$

It shows that: the local body force (load) will cause local asymmetrical stress concentration along thickness direction with exponential law about thickness. If the local curvature at free load is big enough, the load-caused stress concentration will be significant for thick plate. Then, fatigue-fracture may be initiated.

It also shows that, at local load position, the asymmetry stress and non-linear effects will be very significant for thick plate bending, where wrinkling [17] may be produced.

6. Stress Concentration for Kirchhoff Approximation

The bending momentum motion equations, as the results of Kirchhoff approximation theory of plate, are widely used in engineering. How to estimate the stress redistribution caused by thickness and global curvature is a practical problems. As many linear results are well-known, how the non-linear behaves will be exposed in this section.

Using the approximation $R_i^j\approx\delta_i^j+\sin\Theta\cdot L_i^j$, for the typical plate bending produced by normal loading (defined by $f^1=f^2=0$), the Eq.12 will be used to obtain the bending momentum motion equations in Kirchhoff approximation. Letting $\sigma_{ij}=\frac{1}{2}(\sigma_j^i+\sigma_i^j)$, the plate bending momentums M_{ij} ($i,j=1,2$) in Kirchhoff plate theory are defined as:

$$M_{ij}=\int_{-D/2}^{D/2}\sigma_{ij}x^3dx^3 \tag{24-1}$$

Where, D is plate thickness. Noting that, by Eqs.11 and Eq.3, $\sigma_{13}=\sigma_{31}=0$ and $\sigma_{23}=\sigma_{32}=0$, so, the Kirchhoff assumption is automatically satisfied. The load vector components are defined as:

$$Q_1=\frac{1}{2}\int_{-D/2}^{D/2}f^3\sin\Theta\cdot L_2\cdot x^3dx^3 \tag{24-2}$$

$$Q_2=\frac{1}{2}\int_{-D/2}^{D/2}f^3\sin\Theta\cdot L_1\cdot x^3dx^3 \tag{24-3}$$

$$q_3=\frac{1}{2}\int_{-D/2}^{D/2}f^3\cdot x^3dx^3 \tag{24-4}$$

By Eqs.20, omitting the non-linear items, the following approximations are obtained:

$$\frac{1}{2}\frac{\partial(\sigma_1^3-\sigma_3^1)}{\partial x^3}=\frac{1}{2}f^3\sin\Theta\cdot L_2\,, \tag{25-1}$$

$$\frac{1}{2}\frac{\partial(\sigma_2^3-\sigma_3^2)}{\partial x^3}=-\frac{1}{2}f^3\sin\Theta\cdot L_1, \tag{25-2}$$

$$\frac{1}{2}\frac{\partial(\sigma_3^1-\sigma_1^3)}{\partial x^1}+\frac{1}{2}\frac{\partial(\sigma_3^2-\sigma_2^3)}{\partial x^2}=0\,. \tag{25-3}$$

By Eqs.21, it is easy to identify that the above equations can be rewritten as:

$$Q_1=\int_{-D/2}^{D/2}\frac{\partial\sigma_1^3}{\partial x^3}x^3dx^3=-\int_{-D/2}^{D/2}\frac{\partial\sigma_3^1}{\partial x^3}x^3dx^3 \tag{26-1}$$

$$Q_2=\int_{-D/2}^{D/2}\frac{\partial\sigma_3^2}{\partial x^3}x^3dx^3=-\int_{-D/2}^{D/2}\frac{\partial\sigma_2^3}{\partial x^3}x^3dx^3 \tag{26-2}$$

Then, the Eqs.12 is rewritten as:

$$\frac{\partial M_{11}}{\partial x^1}+\frac{\partial M_{12}}{\partial x^2}-Q_1-\frac{1}{2}\int_{-D/2}^{D/2}\sigma_j^1\kappa^j x^3dx^3=0 \tag{27-1}$$

$$\frac{\partial M_{21}}{\partial x^1}+\frac{\partial M_{22}}{\partial x^2}+Q_2-\frac{1}{2}\int_{-D/2}^{D/2}\sigma_j^2\kappa^j x^3dx^3=0 \tag{27-2}$$

$$\frac{\partial Q_1}{\partial x^1}-\frac{\partial Q_2}{\partial x^2}+\frac{1}{2}\int_{-D/2}^{D/2}\frac{\partial\sigma_3^3}{\partial x^3}x^3dx^3-\frac{1}{2}\int_{-D/2}^{D/2}\sigma_j^3\kappa^j x^3dx^3=q_3 \tag{27-3}$$

Omitting the non-linear items and letting $\sigma_{33}=0$, they can be combined as the classical linear bending momentum equation [10]:

$$\frac{\partial^2 M_{11}}{\partial x^1\partial x^1}+2\frac{\partial^2 M_{12}}{\partial x^1\partial x^2}+\frac{\partial^2 M_{22}}{\partial x^2\partial x^2}=q_3 \tag{28}$$

This research shows that the classical linear bending momentum equation is the logic results of two sets of motion equations for unit orthogonal deformation (defined by Eq.8). It makes this research soundness.

For plate bending, the non-linear bending momentum equation is:

$$\begin{aligned}&\frac{\partial^2 M_{11}}{\partial x^1\partial x^1}+2\frac{\partial^2 M_{12}}{\partial x^1\partial x^2}+\frac{\partial^2 M_{22}}{\partial x^2\partial x^2}\\&-\frac{1}{2}\frac{\partial}{\partial x^1}\left(\int_{-D/2}^{D/2}\sigma_j^1\kappa^j x^3dx^3\right)-\frac{1}{2}\frac{\partial}{\partial x^2}\left(\int_{-D/2}^{D/2}\sigma_j^2\kappa^j x^3dx^3\right)\\&+\frac{1}{2}\int_{-D/2}^{D/2}\frac{\partial\sigma_3^3}{\partial x^3}x^3dx^3-\frac{1}{2}\int_{-D/2}^{D/2}\sigma_j^3\kappa^j x^3dx^3=q_3\end{aligned} \tag{29}$$

Only taking the effects of the asymmetrical stress components σ_1^3, σ_3^1, σ_2^3, and σ_3^2 into consideration, the Eq.29 is approximated as:

$$\begin{aligned}&\frac{\partial^2 M_{11}}{\partial x^1\partial x^1}+2\frac{\partial^2 M_{12}}{\partial x^1\partial x^2}+\frac{\partial^2 M_{22}}{\partial x^2\partial x^2}\\&-\frac{1}{2}\frac{\partial}{\partial x^1}\left(\int_{-D/2}^{D/2}\sigma_3^1\kappa^3 x^3dx^3\right)-\frac{1}{2}\frac{\partial}{\partial x^2}\left(\int_{-D/2}^{D/2}\sigma_3^2\kappa^3 x^3dx^3\right)\\&-\frac{1}{2}\int_{-D/2}^{D/2}\sigma_1^3\kappa^1 x^3dx^3-\frac{1}{2}\int_{-D/2}^{D/2}\sigma_2^3\kappa^2 x^3dx^3=q_3\end{aligned} \tag{30}$$

In non-destructive detect, the Eq.28 is assumed as precondition determined by manufacture or working condition. Therefore, the stress redistribution caused by non-linear items can be studied by introducing the effective load, which is defined as:

$$\tilde{q}_3 = q_3 + \frac{1}{2}\frac{\partial}{\partial x^1}(\int_{-D/2}^{D/2} \sigma_3^1 \kappa^3 x^3 dx^3) + \frac{1}{2}\frac{\partial}{\partial x^2}(\int_{-D/2}^{D/2} \sigma_3^2 \kappa^3 x^3 dx^3) + \frac{1}{2}\int_{-D/2}^{D/2} \sigma_1^3 \kappa^1 x^3 dx^3 + \frac{1}{2}\int_{-D/2}^{D/2} \sigma_2^3 \kappa^2 x^3 dx^3 \quad (31)$$

By this way, the fatigue-fracture initiation position can be predicted by calculating the effective load. Based on previous results (Eqs.13, Eqs.21, and Eqs.31), each items of effective load can be obtained. As a first approximation, on the thickness direction, the local whole rotation angle variation can be expressed as:

$$\Theta(x^3) = \Theta + (\Theta)^2 x^3 \quad (32)$$

Then, for small Θ (defined on central plan) and small thickness ($-D/2 \le x^3 \le D/2$), the following equation can be used to simplify the asymmetrical stress in Eq.31.

$$\sin\Theta(x^3) \approx \sin\Theta + (\Theta)^2 x^3 \quad (33)$$

Furthermore, omitting the higher order infinitesimals, the following approximations are obtained:

$$\frac{1}{2}\frac{\partial}{\partial x^1}(\int_{-D/2}^{D/2} \sigma_3^1 \kappa^3 x^3 dx^3) \approx -\frac{(D)^3}{12}\cdot\mu\cdot\frac{\partial[(\Theta)^2 L_2 \kappa^3]}{\partial x^1} \quad (34\text{-}1)$$

$$\frac{1}{2}\frac{\partial}{\partial x^2}(\int_{-D/2}^{D/2} \sigma_3^2 \kappa^3 x^3 dx^3) \approx \frac{(D)^3}{12}\cdot\mu\cdot\frac{\partial[(\Theta)^2 L_1 \kappa^3]}{\partial x^2} \quad (34\text{-}2)$$

$$\frac{1}{2}\int_{-D/2}^{D/2} \sigma_1^3 \kappa^1 x^3 dx^3 \approx \frac{(D)^3}{12}\cdot\mu\cdot(\Theta)^2\cdot L_2\kappa^1 \quad (34\text{-}3)$$

$$\frac{1}{2}\int_{-D/2}^{D/2} \sigma_2^3 \kappa^2 x^3 dx^3 \approx -\frac{(D)^3}{12}\cdot\mu\cdot(\Theta)^2\cdot L_1\kappa^2 \quad (34\text{-}4)$$

As a simple example, for simple bending $\frac{\partial u^3}{\partial x^1} \approx \Theta$, $L_2 = 1$, $L_1 = 0$, $\kappa^3 \approx \frac{\partial\Theta}{\partial x^1}$, $\kappa^1 \approx (\Theta)^2$, the effective load is simplified as:

$$\tilde{q}_3 = q_3 + \frac{(D)^3}{12}\cdot\mu\cdot\{(\Theta)^4 - \frac{\partial}{\partial x^1}[(\Theta)^2\frac{\partial\Theta}{\partial x^1}]\} \quad (35)$$

It shows that the local whole rotation in average sense (local curvature) has significant effects. Here, the stress concentration is expressed by effective load variation. Hence, it can be used in forward problems to predict possible fatigue-fracture initiating positions.

6. Conclusions

For large deformation, such as plate bending, the linear momentum conservation equations and angular momentum conservation equations are used to study the stress concentration problems. The research shows that, for plate bending, the dominate Cauchy stresses are asymmetrical. The symmetrical stress components are higher order smaller. By requiring both sets of motion equations are satisfied, the classical bending momentum linear motion equation is obtained as a linear approximation.

Based on the linear momentum conservation equations, the stress concentration caused by bending curvature functions is studied by introducing effective stress transportation solutions. The results show that: the effective stresses on central plan parallel surface are exponentially varied with path-integral of curvature along scale directions. The effective stresses on thickness direction are exponentially varied with the path-integral of curvature functions. The scale effects are very significant. As the effective stresses can be detected by many non-destructive methods, the curvature functions can be estimated. For supersonic wave method, the effective elasticity can be detected and used to estimate the curvature functions. Hence, the fatigue-fracture initiating position can be predicted.

Based on the angular momentum conservation equations, the stress concentration caused by local load on thickness direction is studied. The results show that: the local body force (load) will cause local asymmetrical stress concentration along thickness direction with exponential law about thickness.

Finally, the effective load caused by local bending curvature is introduced to estimate the stress concentration as effective load variation. For simple bending, an explicit formula is given. It shows that the local rotation in average sense (local curvature) has significant effects. This equation can be easily used for non-destructive inverse problems.

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