Fracture analysis in plane piezoelectric media using hybrid finite element model

Hui Wang[1], Qing-Hua Qin[2,3*],

[1]Department of Engineering Mechanics, Henan University of Technology, Zhengzhou 450001, China
[2]Department of Mechanics, Tianjin University, Tianjin, 300072, China
[3]Research School of Engineering, Australian National University, ACT 0200, Australia
[*] Corresponding author: qinghua.qin@anu.edu.au

Abstract A two-dimensional electroelastic fracture analysis is performed on a plane piezoelectric material by the finite element model based on fundamental solution approximation. In the present element model, the fundamental solution of the piezoelectric problem is employed to construct the intra-element fields to make the final stiffness equation containing element boundary integrals only. Solving the stiffness equation can yield nodal displacement and electric potential, which are in turn used to evaluate the stress intensity factor and electric intensity factor by way of extrapolation techniques. Numerical results are provided to show the accuracy of the present method.

Keywords Piezoelectric materials, Crack, Stress intensity factor, Hybrid finite element method, Fundamental solution.

1. Introduction

Fracture analysis of piezoelectric materials is important for enhancing our understanding on the effect of the coupling properties on their fracture or damage behavior [1,2]. Because of complicated inherent coupling between electric and mechanical behaviors in piezoelectric solids, numerical simulation techniques have been widely employed for the fracture analysis of piezoelectric materials under complicated electric and mechanical load conditions. For example, finite element method [3, 4], boundary element method [5, 6], hybrid Trefftz finite element method [7,8], meshless methods [9], and virtual boundary integral method [10] have been developed for solving crack problems of piezoelectric materials.

In the paper, the hybrid finite element method developed recently by Wang and Qin [11-14] is extended to the case of cracked piezoelectricity. Different to other hybrid methods like the Trefftz finite element method [15-18], the basic idea of the proposed finite element model is the use of the novel interpolation kernels composed of fundamental solutions (or Green's functions) inside the multi-edge element to achieve the purpose of analytical satisfaction of governing equations of the problems of interest, and then the element stiffness equation which includes element boundary integrals only are formed for solving the electroelastic behavior.

In this work, the hybrid finite element method is extended for solving the piezoelectric problems. Using the fundamental solutions of piezoelectric problems, the intra-element displacement and electric potential fields are constructed. A modified variational functional is introduced to produce the linkage between intra-element and boundary fields. The minimization of the functional yields the stiffness matrix equation. As a result, the nodal displacements and electric potential can be determined by solving the linear stiffness equation. In the fracture analysis, the present hybrid finite element model is used to determine the displacement and electric potential distribution in the vicinity of the crack tip and then the stress and electric intensity factors can be obtained using the extrapolation technique [19].

2. Basic equations

For the transversely isotropic piezoelectric material, if the *x*-*y* plane is considered as the isotropic plane and the polarizing direction is assumed along the *z*-direction, one can employ either *x*-*z* or *y*-*z* plane to study the piezoelectric behavior. Here a plane piezoelectric media in the *x*-*z* plane is considered. In the absence of body forces and body electric charges, the equilibrium equations are given by [20]

$$\sigma_{ij,j} = 0, \qquad D_{i,i} = 0 \tag{1}$$

where $\sigma_{ij}(i, j = x, z)$ and D_i are stress tensor and electric displacement in the *i*-direction, respectively.

With plane strain assumption ($\varepsilon_{yy} = \varepsilon_{xy} = \varepsilon_{yz} = E_y = 0$), the constitutive equations in the *xz* system can be expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \end{Bmatrix} - \begin{bmatrix} 0 & e_{31} \\ 0 & e_{33} \\ e_{15} & 0 \end{bmatrix} \begin{Bmatrix} E_x \\ E_z \end{Bmatrix}$$

$$\begin{Bmatrix} D_x \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & e_{15} \\ e_{31} & e_{33} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \end{Bmatrix} + \begin{bmatrix} \kappa_{11} & 0 \\ 0 & \kappa_{33} \end{bmatrix} \begin{Bmatrix} E_x \\ E_z \end{Bmatrix} \tag{2}$$

where ε_{ij} and E_i denote the strain tensor and electric field in the *i*-direction, respectively. c_{ij}, e_{ij} and κ_{ij} stand for two dimensional material elastic, piezoelectric and dielectric coefficients. For the case of plane stress, the constitutive equation can be obtained similarly [20].

The remaining elastic strain-displacement and electric field-potential relations are given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad E_i = -\phi_{,i} \tag{3}$$

where u_i and ϕ are elastic displacement in the *i*-direction and electric potential, respectively.

The following boundary conditions are admissible on the boundary of the piezoelectric medium

$$\begin{aligned} u_i &= \bar{u}_i \quad \text{or} \quad t_i = \sigma_{ij} n_j = \bar{t}_i \\ \phi &= \bar{\phi} \quad \text{or} \quad D_n = D_i n_i = \bar{D}_n \end{aligned} \tag{4}$$

in which the components with an over bar denote specified values, and n_i is the *i*-component of outward unit normal vector.

3. Stress and electric intensity factors for plane piezoelectric crack

Now let us consider a crack embedded in an infinite plane piezoelectric solid, the origin of the local coordinate system is at the crack tip to be analyzed. Along the crack surface, the traction and normal electric displacement are free. According to Sosa's work [19] based on Lekhnitskii theory, the displacement components and electric potential along the upper crack surface near the crack tip can be written as

$$\begin{Bmatrix} u_x \\ u_z \\ \phi \end{Bmatrix} = \sqrt{\frac{2r}{\pi}} \,\mathrm{Re} \begin{bmatrix} \sum_{j=1}^{3} p_j \Lambda_{j1} & -\sum_{j=1}^{3} p_j \Lambda_{j2} & -\sum_{j=1}^{3} p_j \Lambda_{j3} \\ \sum_{j=1}^{3} q_j \Lambda_{j1} & -\sum_{j=1}^{3} q_j \Lambda_{j2} & -\sum_{j=1}^{3} q_j \Lambda_{j3} \\ \sum_{j=1}^{3} \lambda_j \Lambda_{j1} & -\sum_{j=1}^{3} \lambda_j \Lambda_{j2} & -\sum_{j=1}^{3} \lambda_j \Lambda_{j3} \end{bmatrix} \begin{Bmatrix} \mathrm{K_I} \\ \mathrm{K_{II}} \\ \mathrm{K_E} \end{Bmatrix} \tag{5}$$

where K_I, K_{II}, K_E are respectively stress and electric intensity factors and r is the distance from the crack tip. In Eq. (5), the coefficients p_i, q_i, λ_i and Λ_{jk} can be found in Sosa's work [19].

From Eq. (5), it can be seen that the accuracy of displacement and electric potential can affect the results of stress and electric intensity factors. Therefore, developing new numerical approaches to achieve high accuracy of displacements and electric potential is important. The obtained numerical results below demonstrate that the proposed hybrid finite element formulation can serve for this purpose.

4. Hybrid finite element formulation

Consider a 2D electro-mechanical coupling problem, the energy functional used for constructing the proposed finite element model is given by

$$\Pi_e = \frac{1}{2}\int_{\Omega_e} (\sigma_{ij}\varepsilon_{ij} + D_i E_i)\mathrm{d}\Omega - \int_{\Gamma_t} \bar{t}_i \tilde{u}_i \mathrm{d}\Gamma - \int_{\Gamma_D} \bar{D}_n \tilde{\phi} \mathrm{d}\Gamma + \int_{\Gamma_e} t_i(\tilde{u}_i - u_i)\mathrm{d}\Gamma + \int_{\Gamma_e} D_n(\tilde{\phi} - \phi)\mathrm{d}\Gamma \quad (6)$$

in which $\tilde{u}_i$, $\tilde{\phi}$ and u_i, ϕ are the displacement components and electric potential respectively defined along the element boundary and inside the element domain.

In the present hybrid finite element model, the intra-element fields like displacement and electric potential inside the element can be approximated using the combination of the fundamental solution at different source points (the fundamental solution can be derived by either Lekhnitskii's formalism [21, 22] or Stroh's formalism [5]):

$$\begin{Bmatrix} u_x \\ u_z \\ \phi \end{Bmatrix} = \mathbf{N}_e \mathbf{c}_e \quad (\mathbf{x} \in \Omega_e, \mathbf{y}_{sk} \notin \Omega_e) \quad (7)$$

where

$$\mathbf{N}_e = \begin{bmatrix} u^*_{x1}(\mathbf{x},\mathbf{y}_{s1}) & u^*_{x2}(\mathbf{x},\mathbf{y}_{s1}) & u^*_{x3}(\mathbf{x},\mathbf{y}_{s1}) & \ldots & u^*_{x1}(\mathbf{x},\mathbf{y}_{sn_s}) & u^*_{x2}(\mathbf{x},\mathbf{y}_{sn_s}) & u^*_{x3}(\mathbf{x},\mathbf{y}_{sn_s}) \\ u^*_{z1}(\mathbf{x},\mathbf{y}_{s1}) & u^*_{z2}(\mathbf{x},\mathbf{y}_{s1}) & u^*_{z3}(\mathbf{x},\mathbf{y}_{s1}) & \ldots & u^*_{z1}(\mathbf{x},\mathbf{y}_{sn_s}) & u^*_{z2}(\mathbf{x},\mathbf{y}_{sn_s}) & u^*_{z3}(\mathbf{x},\mathbf{y}_{sn_s}) \\ \phi^*_1(\mathbf{x},\mathbf{y}_{s1}) & \phi^*_2(\mathbf{x},\mathbf{y}_{s1}) & \phi^*_3(\mathbf{x},\mathbf{y}_{s1}) & \ldots & \phi^*_1(\mathbf{x},\mathbf{y}_{sn_s}) & \phi^*_2(\mathbf{x},\mathbf{y}_{sn_s}) & \phi^*_3(\mathbf{x},\mathbf{y}_{sn_s}) \end{bmatrix} \quad (8)$$

is the interpolation matrix and

$$\mathbf{c}_e = [c_{11} \quad c_{21} \quad c_{31} \quad \ldots \quad c_{1n_s} \quad c_{2n_s} \quad c_{3n_s}]^{\mathrm{T}} \quad (9)$$

is the unknown coefficient vector. $u^*_{ij}(\mathbf{x},\mathbf{y}_{sk})$ and $\phi^*_j(\mathbf{x},\mathbf{y}_{sk})$ are induced displacement fundamental solutions at field point $\mathbf{x}$ due to a unit concentrated point load applied in the j-direction ($j = 1,2$) and unit electric charge ($j = 3$) at source point $\mathbf{y}_{sk}$ $(i = x, z;\ k = 1, 2, \cdots, n_s)$.

Whilst, the frame displacement and electric potential over the element boundary can be defined by

$$\begin{Bmatrix} \tilde{u}_x \\ \tilde{u}_z \\ \tilde{\phi} \end{Bmatrix} = \tilde{\mathbf{N}}_e \mathbf{d}_e \quad (10)$$

where $\tilde{\mathbf{N}}_e$ is the matrix of shape functions which is that same as those used in the conventional finite element method and boundary element method. $\mathbf{d}_e$ stands for the vector of the nodal displacements and electric potential.

Applying the Gauss theorem to the functional (6) and making use of the intra-element fields (7) and frame fields (10) yields

$$\Pi_e = -\frac{1}{2}\mathbf{c}_e{}^{\mathrm{T}}\mathbf{H}_e\mathbf{c}_e + \mathbf{c}_e{}^{\mathrm{T}}\mathbf{G}_e\mathbf{d}_e - \mathbf{d}_e{}^{\mathrm{T}}\mathbf{g}_e \tag{11}$$

where

$$\begin{aligned}\mathbf{H}_e &= \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}}\mathbf{N}_e \mathrm{d}\Gamma \\ \mathbf{G}_e &= \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}}\tilde{\mathbf{N}}_e \mathrm{d}\Gamma \\ \mathbf{g}_e &= \int_{\Gamma_t} \tilde{\mathbf{N}}_e^{\mathrm{T}}\bar{\mathbf{t}} \mathrm{d}\Gamma + \int_{\Gamma_D} \tilde{\mathbf{N}}_e^{\mathrm{T}}\bar{\mathbf{D}} \mathrm{d}\Gamma\end{aligned} \tag{12}$$

In Eq. (12), the matrix $\mathbf{Q}_e$ is a coefficient matrix in terms of intra-element traction and electric displacement induced by intra-element fields, that is

$$\begin{Bmatrix} t_x \\ t_z \\ D_n \end{Bmatrix} = \mathbf{Q_e c_e} \tag{13}$$

which can be derived from the assumed displacement and electric potential (7).

With the stationary condition of the functional (11), we finally have

$$\mathbf{G}_e{}^{\mathbf{T}}\mathbf{H}_e{}^{-1}\mathbf{G}_e\mathbf{d}_e = \mathbf{g}_e \tag{14}$$

and

$$\mathbf{c}_e = \mathbf{H}_e^{-1}\mathbf{G}_e\mathbf{d}_e \tag{15}$$

5. Numerical example

In this section, to investigate the performance of the present hybrid finite element formulation for the simulation of piezoelectric fracture behavior, a plane piezoelectric prism under uniform tension is considered with an embedded central crack and traction- and electric charge-free conditions are assumed along the crack surface (see Figure 1). Due to symmetry of the problem, only one quarter of the prism modeled, i.e. the shaded domain in Figure 1. The proper symmetrical mechanical boundary conditions should be applied along the symmetric lines x=0 and z=0 and the electric potential along the symmetric line z=0 is assumed to be zero. The plane strain case is assumed here and the PZT-4 material used in the computation has the following material constants [22]

$$c_{11} = 12.6\times10^{10}\,\mathrm{Nm}^{-2},\ c_{12} = 7.78\times10^{10}\,\mathrm{Nm}^{-2},\ c_{13} = 7.43\times10^{10}\,\mathrm{Nm}^{-2}$$

$$c_{33} = 11.5\times10^{10}\,\mathrm{Nm}^{-2},\ c_{44} = 2.56\times10^{10}\,\mathrm{Nm}^{-2}$$

$$e_{15} = 12.7\mathrm{Cm}^{-2},\ e_{31} = -5.2\mathrm{Cm}^{-2},\ e_{33} = 15.1\mathrm{Cm}^{-2}$$

$$\kappa_{11} = 6.464\times10^{-9}\,\mathrm{C/Nm},\ \kappa_{33} = 5.622\times10^{-9}\,\mathrm{C/Nm}$$

Firstly, to verify the present algorithm and show the accuracy of displacements and electric potential, the case without the crack is taken into consideration and the analytical results can be found in [22]. In the computation, thirty 8-node quadrilateral elements are used for the present hybrid finite element model (see Figure 2a). The results of displacements and electric potential at specific points are tabulated in Table 1, from which it’s found that the numerical results from the proposed algorithm are in good agreement with the analytical results. Thus, the developed hybrid finite model can produce highly accurate displacement and electric potential results, which are important for the evaluation of stress intensity factors by displacement and electric potential extrapolation technique in the following analysis.

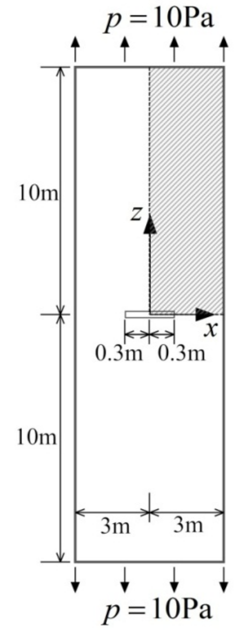
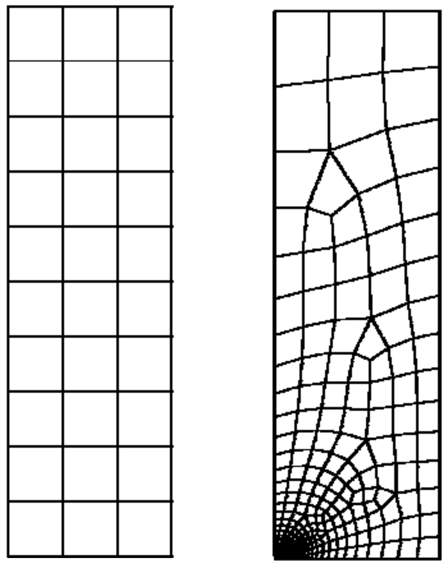


Figure 1 Geometry and boundary conditions of the piezoelectric prism with a central crack



(a) (b)

Figure 2 Mesh configurations: (a) without central crack (b) with central crack

Table 1 Comparison of the numerical results by the present algorithm and the analytical solutions

Position	Analytical solutions			Numerical results		
	u_x	u_z	ϕ	u_x	u_z	ϕ
(2, 0)	-0.7222E-10	0	0	-0.7222E-10	0	0
(3, 0)	-1.0832E-10	0	0	-1.0832E-10	0	0
(0, 5)	0	3.915E-10	1.2184	0	3.915E-10	1.2184
(0,10)	0	7.829E-10	2.4367	0	7.829E-10	2.4367

Next, fracture analysis of the prism is considered and the corresponding mesh configuration is shown in Figure 2b, in which total 287 piezoelectric quadrilateral elements are used. The numerical results for crack is tabulated in Table 2, from we can find the displacement variation along the upper surface of the crack, also the corresponding electric displacement stress intensity factors $\mathrm{K_E} \times 10^{-6} (\mathrm{Cm}^{-3/2})$ can be evaluated by Eq. (5). The average value of it is 28.753.

Table 2 Variations of displacement and Stress intensity factor near the crack tip

r	0.01	0.02	0.03	0.04	0.05
u_x	-20.326E-12	-19.262E-12	-18.442E-12	-17.817E-12	-17.031E-12
u_z	17.901E-12	25.352E-12	31.050E-12	35.792E-12	39.651E-12
ϕ	0	0	0	0	0
$\mathrm{K_E} \times 10^{-6}$	28.793	28.834	28.834	28.784	28.521

6. Conclusions

In the paper, a new hybrid finite element formulation is present for performing fracture analysis of piezoelectric media using fundamental solution approach. The hybrid piezoelectric element established in the present method contains element boundary integrals only. Numerical verification is conducted by analysing electroelastic behavior of plane piezoelectric prism without any internal crack. It is found that the developed hybrid finite model can produce relatively high accurate displacement and electric potential results. Subsequently, the fracture analysis is performed by considering a central crack in the piezoelectric prism and the corresponding electric displacement stress intensity factors is evaluated by using the extrapolation technique in the vicinity of crack tip.

References

[1] Q.H. Qin, Y.W. Mai, S.W. Yu, Effective moduli for thermopiezoelectric materials with microcracks. Int J Fracture, 91 (1998) 359-371.
[2] Q.H. Qin, Y.W. Mai, BEM for crack-hole problems in thermopiezoelectric materials. Eng Fract Mech, 69 (2002) 577-588.
[3] A. Benjeddou, Advances in piezoelectric finite element modeling of adaptive structural elements: a survey. Computers & Structures, 76 (2000) 347-363.
[4] C.Y. Cao, A. Yu, Q.H. Qin, A new hybrid finite element approach for plane piezoelectricity with defects. Acta Mechanica, 224 (2013) 41-61.
[5] E. Pan, A BEM analysis of fracture mechanics in 2D anisotropic piezoelectric solids. Engineering Analysis with Boundary Elements, 23 (1999) 67-76.
[6] Q.H. Qin, M. Lu, BEM for crack-inclusion problems of plane thermopiezoelectric solids. Int J Numer Meth Eng, 48 (2000) 1071-1088.
[7] Q.H. Qin, Solving anti-plane problems of piezoelectric materials by the Trefftz finite element approach. Computational Mechanics, 31 (2003) 461-468.
[8] H.T. Wang, K.Y. Sze, X.M. Yang, Analysis of electromechanical stress singularity in piezoelectrics by computed eigensolutions and hybrid-trefftz finite element models. Computational Mechanics, 38 (2006) 551-564.
[9] J. Sladek, V. Sladek, C. Zhang, F. Garcia-Sanchez, M. Wunsche, Meshless local Petrov-Galerkin method for plane piezoelectricity. Computers, Materials & Continua, 4 (2006) 109-117.
[10] W.A. Yao, H. Wang, Virtual boundary element integral method for 2-D piezoelectric media. Finite elements in analysis and design, 41 (2005) 875-891.
[11] H. Wang, Q.H. Qin, Hybrid FEM with Fundamental Solutions as Trial Functions for Heat Conduction Simulation. Acta Mechanica Solida Sinica, 22 (2009) 487-498.
[12] H. Wang, Q.H. Qin, Fundamental-solution-based hybrid FEM for plane elasticity with special elements. Computational Mechanics, 48 (2011) 515-528.
[13] H. Wang, Q.H. Qin, Special fiber elements for thermal analysis of fiber-reinforced composites. Engineering Computations, 28 (2011) 1079-1097.
[14] H. Wang, Q.H. Qin, Boundary integral based graded element for elastic analysis of 2D functionally graded plates. European Journal of Mechanics - A/Solids, 33 (2012) 12-23.
[15] H. Wang, Q.H. Qin, Matlab and C programming for Trefftz finite element methods, CRC Press, New York, 2008.
[16] Q.H. Qin, The Trefftz finite and boundary element method, WIT Press, Southampton, 2000.

[17] Q.H. Qin, Variational formulations for TFEM of piezoelectricity, International Journal of Solids and Structures, 40 (2003) 6335-6346.
[18] Q.H. Qin, Trefftz finite element method and its applications. Appl Mech Rev, 58 (2005) 316-337.
[19] H. Sosa, Plane problems in piezoelectric media with defects. International Journal of Solids and Structures, 28 (1991) 491-505.
[20] Q.H. Qin, Fracture mechanics of piezoelectric materials, WIT press, Southampton, 2001.
[21] X.L. Xu, R.K.N.D. Rajapakse, On a plane crack in piezoelectric solids. International Journal of Solids and Structures, 38 (2001) 7643-7658.
[22] H.J. Ding, G.Q. Wang, W.Q. Chen, A boundary integral formulation and 2D fundamental solutions for piezoelectric media. Computer Methods in Applied Mechanics and Engineering, 158 (1998) 65-80.