Determination of statistical secured residual lifetime based on sensitivity analysis and stochastic crack propagation simulation

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Abstract To consider uncertainties due to the scattering of input parameters common residual lifetime calculations are deduced from conservative deterministic crack propagation simulations. Conclusions about the reliability of the residual lifetime are not possible in general. Those can be obtained from stochastic crack propagation simulations. Therefore the significant input parameters have to be identified and statistically modelled. In the present investigations quantil curves of the crack propagation data of 42CrMo4 are derived. From those the fracture mechanical material parameters were statistically modelled. After identifying the significant material parameters by performing a sensitivity analysis stochastic input vectors of the relevant input parameters are generated. Using the Monte Carlo simulation and the analytical crack propagation software NASGRO stochastic residual lifetime calculations are performed and statistically analyzed. Afterwards residual lifetimes can be related to survival probabilities.

Keywords residual lifetime calculation, sensitivity analysis, Monte Carlo method, *Forman*/*Mettu*-equation

1. Introduction

The residual lifetime of components and structures can be calculated by crack propagation simulations [1]. Those are mostly performed analytically due to the low computational effort. Therefore, deterministic simulations are commonly used. Uncertainties through the scattering of input parameters e.g. the crack propagation curves are considered by conservative appreciated material parameters and safety factors [2]. As a result of this approach, it is not possible to derive conclusions about the failure probability of the calculated residual lifetimes. Alternatively stochastic crack propagation simulations e.g. the Monte Carlo simulation [3–5] can be performed instead of deterministic ones. Therefore, the significant stochastic input parameters must be statistically characterized. From those, independent input vectors are calculated and a multiplicity of crack propagation simulations are performed. Afterwards the residual lifetimes are statistically analyzed.

To identify the significant stochastic input parameters a sensitivity analysis [6–8] was used in the present study. Therefore, only scattering of the crack propagation curve was considered. From crack propagation curves of 42CrMo4 for different stress ratios quantile curves were deduced. By fitting those curves with the *Forman/Mettu*-equation [9] its parameters and the related distribution functions of the parameters could be obtained. Hereby, stochastic input vectors were calculated and crack propagation simulations performed by use of the analytical crack propagation software NASGRO 6 [9]. The sensitivity analysis respectively the robustness analysis provided the significant input parameters of the resistance against cyclic crack propagation. Those were used to perform stochastic crack propagation simulations.

2. Theoretical background

In this chapter some basics of analytical crack propagation simulation, sensitivity analysis and Monte Carlo simulation are provided.

2.1. Analytical crack propagation simulation

The basic requirements for an analytical crack propagation simulation are the solution of the stress

intensity factor (SIF) and a relationship between the cyclic SIF ΔK and the crack propagation *da/dN* [1]. An equation to describe the crack propagation curve is the *Forman*/*Mettu*-equation

$$\frac{da}{dN} = C \cdot \left(\frac{1-\gamma}{1-R} \cdot \Delta K\right)^n \cdot \left(1 - \frac{\Delta K_{\mathrm{th}}}{\Delta K}\right)^p \cdot \left(1 - \frac{K_{\max}}{K_{\mathrm{C}}}\right)^{-q}. \tag{1}$$

This function, which is implemented in NASGRO [9], characterizes the whole crack propagation curve. The empirical coefficients C, n, p and q have to be fitted to test data. K_{C} is the fracture toughness. The R-dependence in eq. (1) is explained by the crack closure effect and considered by *Newman*'s crack closure function

$$\gamma = \frac{K_{\mathrm{op}}}{K_{\max}} = \begin{cases} \max(R, A_0 + A_1 \cdot R + A_2 \cdot R^2 + A_3 \cdot R^3) & R \geq 0 \\ A_0 + A_1 \cdot R & -2 \leq R < 0 \end{cases} \tag{2}$$

with the coefficients

$$\begin{aligned} A_0 &= (0{,}825 - 0{,}34\alpha_{\mathrm{CF}} + 0{,}05\alpha_{\mathrm{CF}}^2) \cdot \left[cos\left(\frac{\pi}{2} \cdot \frac{\sigma_{\max}}{\sigma_{\mathrm{F}}}\right)\right]^{\frac{1}{\alpha_{\mathrm{CF}}}} \\ A_1 &= (0{,}415 - 0{,}071\alpha_{\mathrm{CF}}) \cdot \frac{\sigma_{\max}}{\sigma_{\mathrm{F}}} \\ A_2 &= 1 - A_0 - A_1 - A_3 \\ A_3 &= 2A_0 + A_1 - 1. \end{aligned} \tag{3}$$

The constraint factor α_{CF} varies between 1 for the plane-stress condition and 3 for the plane-strain condition. The constraint factor and the ratio of maximum stress $\sigma_{\max}$ and yield stress σ_{F} are also considered as fitting coefficients [9]. To describe the R-dependence of the cyclic threshold value

$$\Delta K_{th} = \Delta K_1 \cdot \sqrt{\frac{a}{a + a_0}} \cdot \left(\frac{1-R}{1-\gamma}\right)^{\left(1 + C_{th}^{+} \cdot R\right)} / (1 - A_0)^{(1-R) \cdot C_{th}^{+}} \tag{4}$$

for $R \geq 0$ and

$$\Delta K_{th} = \Delta K_1 \cdot \sqrt{\frac{a}{a + a_0}} \cdot \left(\frac{1-R}{1-\gamma}\right)^{\left(1 + C_{th}^{-} \cdot R\right)} / (1 - A_0)^{\left(C_{th}^{+} - C_{th}^{-} \cdot R\right)} \tag{5}$$

for $R < 0$ again *Newman*'s crack closure function is used. The cyclic threshold value ΔK_1 for $R \to 1$ as well as C_{th}^{+} and C_{th}^{-} are fitting coefficients. All in all ten parameters are needed to describe the resistance against cyclic crack propagation by the *Forman*/*Mettu*-equation.

2.2. Sensitivity analysis

With a sensitivity analysis the significance of input parameters X_{i} or rather factors and their effects on an output value Y is investigated [6]. The importance of a factor is denoted as sensitivity S_{i}. The effect is distinguished in main effect (first order effect) and interaction effect, which describes the effect of a factor according to the properties of another factor. Sensitivity analysis can be divided in factor screening, local and global sensitivity analysis [8]. While with factor screening only qualitative influences of factors are investigated with local and global sensitivity analysis also quantitative influences are analyzed. If the effect of a factor on an output value is investigated in a small domain of the factor with consideration of its distribution function a local sensitivity analysis or rather a robustness analysis is performed [8]. To determine sensitivities different methods e.g. the contrast method, the regression analysis or the analysis of variance (ANOVA) exists [7, 8].

The regression analysis is based on a regression model in which the dependencies between input and output factors are described analytically [8]. Therefore, linear or quadratic models are used commonly. For n_f input factors without error term a linear regression model

$$\hat{y} = b_0 + \sum_{i=1}^{n_f} b_i \cdot x_i + \sum_{i=1}^{n_f-1} \sum_{j=i+1}^{n_f} b_{ij} \cdot x_i \cdot x_j \tag{6}$$

contains one constant b_0, n_f regression coefficients b_i and $n_f \cdot (n_f - 1)/2$ independent regression coefficients b_{ij}. If the domain of the input factors is normalized on [-1 , 1], the coefficients b_i are a measure of the main effects and the coefficients b_{ij} are a measure of the interaction effects [8]. Here the scattering of the input and output factors is not considered *Saltelli* et al. [7] recommend

$$S_i = b_i \cdot \frac{s_i}{s_y} \tag{7}$$

as a measure of the sensitivity. Herein, s_i and s_y are the standard deviations of the input factor X_i and the output value Y.

To check the predictive of the regression model and thus the accuracy of the calculated sensitivities the coefficient of determination

$$R^2 = \frac{\sum_j (\hat{y}_j - \bar{y})^2}{\sum_j (y_j - \bar{y})^2} \tag{8}$$

is calculated. For a linear regression model a small coefficient of determination can be caused by non-linear effects. In such a case a transformation of the output value can lead to a better accordance. To find a valid transformation the Box-Cox-transformation [8]

$$z = \begin{cases} \dfrac{y^\lambda - 1}{\lambda \cdot g^{(\lambda-1)}} & ,\lambda \neq 0 \\ g \cdot ln(y) & ,\lambda = 0 \end{cases} \tag{9}$$

with the geometric mean value

$$g = (y_1 \cdot y_2 \cdot \ldots \cdot y_{n_r})^{\frac{1}{n_r}}. \tag{10}$$

can be performed. For different values of λ a multiplicity of transformations is realized. For each of them the coefficients of determination can be calculated to find the optimum transformation.

2.3. Monte Carlo simulation

The Monte Carlo simulation is a numerical method for the approximate solution of mathematic tasks by use of random input vectors [3]. It is e.g. used to solve multi-dimensional integrals. The generation of linear independent random vectors x_{ij} based on the input factor's distribution functions $g_i(x)$ is the main object of the Monte Carlo simulation. If $f(x)$ is a function of the factor X and $g(x)$ its' distribution function the expected value

$$E[f(x)] = \int_{-\infty}^{+\infty} f(x) \cdot g(x) dx \approx \frac{1}{N} \cdot \sum_{j=1}^{N} f(x_j) \pm \sqrt{\frac{\sigma^2}{N}} = \frac{1}{N} \cdot \sum_{j=1}^{N} f(x_j) \pm s_{\bar{F}} \tag{11}$$

of $f(x)$ can approximately calculated by the mean of the function values of the N random input values x_j [3]. The sum converges against the integral, if the standard error $s_{\bar{F}}$ of the expected value of $f(x)$ gets small. Therefore, a high number N of samples is required, depending on the variance σ^2

of the function $f(x)$. However, the accuracy of the Monte Carlo simulation is independent of the number n_f of input factors [3, 5] so it becomes attractive for systems with a multiplicity of factors.

For the basic Monte Carlo simulation the input vectors are generated by a random number generator based on the factor's distribution functions. Therefore, taking into account a statistical error of 10 %

$$N = \frac{100}{P_A} \tag{12}$$

samples are required [5]. It is obvious that the computational effort is high for small probability of failure P_A. Against this background different variants of the Monte Carlo simulation were developed to minimize the variance and therefore to increase the computational efficiency. Some of them are the Latin Hypercube sampling and the Importance sampling [5].

3. Analysis of crack propagation data

For the crack propagation simulation an analytical relationship $da/dN = f(\Delta K, R)$ is required. To obtain its coefficients this function is fitted through experimental crack propagation data. Therefore, visual criteria and a conservative approximation are commonly used [2]. An automated adaption of the *Forman*/*Mettu*-equation is not documented. This is also the case for a complete statistical analysis of its coefficients and building on a sensitivity analysis or rather a robustness analysis referred to the residual life. To consider the scattering of crack propagation data of metallic materials usually the coefficients C [2,10 – 14], ΔK_0 [2, 10, 11, 13] and K_C [2, 11] are statistically analyzed. ΔK_0 is the cyclic threshold value for $R \rightarrow 0$ which is related to ΔK_1 [9].

3.1. Automated adaption of fitting parameters

To perform an automated analysis of crack propagation data by the *Forman/Mettu*-equation a MATLAB program was developed. The adaption of the fitting coefficients on the experimental data is divided in two steps. First the threshold values against cyclic crack propagation are calculated for different stress ratios and therewith the coefficients ΔK_1, C_{th}^{+} and C_{th}^{-} of equations (4) and (5) are determined. Next the coefficients C, n, p and q of the *Forman/Mettu*-equation (1) are calculated. The remaining coefficients a, a_0, α_{CF}, σ_{max}/σ_F and K_C have to be supported for the adaption.

After loading the crack propagation data the threshold values for the different stress ratios are calculated by use of the DLR-method [15]. Therefore the user has to set an upper bound da/dN_{max} of the crack propagation data used for the analysis of the threshold value. This limit should be below the transition from the threshold to the *Paris*-domain [15]. The threshold data are analyzed in a linear scaled coordinate system, Figure 1. Every data set with R = constant is fitted by a linear function. Therewith, the threshold value is obtained for $da/dN = 0$. The crack propagation data of 42CrMo4 contains of the stress ratios $R = 0{,}1$, $R = 0{,}3$ and $R = 0{,}5$. The analyzed threshold values for each stress ratio are plotted in Figure 2.

Referring to the determination of the experimental threshold values the coefficients ΔK_1, C_{th}^{+} and C_{th}^{-} for the analytical threshold value calculation are adapted. Therefore two algorithms are available in the program. The first one is a simple search algorithm, in which the search domain is divided into intervals. The second one is a gradient based algorithm. For both methods start values of the fitting coefficients must be set. Furthermore, the coefficients $a = 45$ mm, $a_0 = 0{,}0381$ mm, $\alpha_{CF} = 1{,}9$ and $\sigma_{max}/\sigma_F = 0{,}3$ are pretended. At the beginning the number of stress ratios is checked. If only positive stress ratios are available at least two otherwise three discrete threshold values are required. Since the crack propagation data only contains positive stress ratios C_{th}^{-} could not be analyzed, see equation (4) and (5). The adaption of the threshold value is visualized in Figure 2.

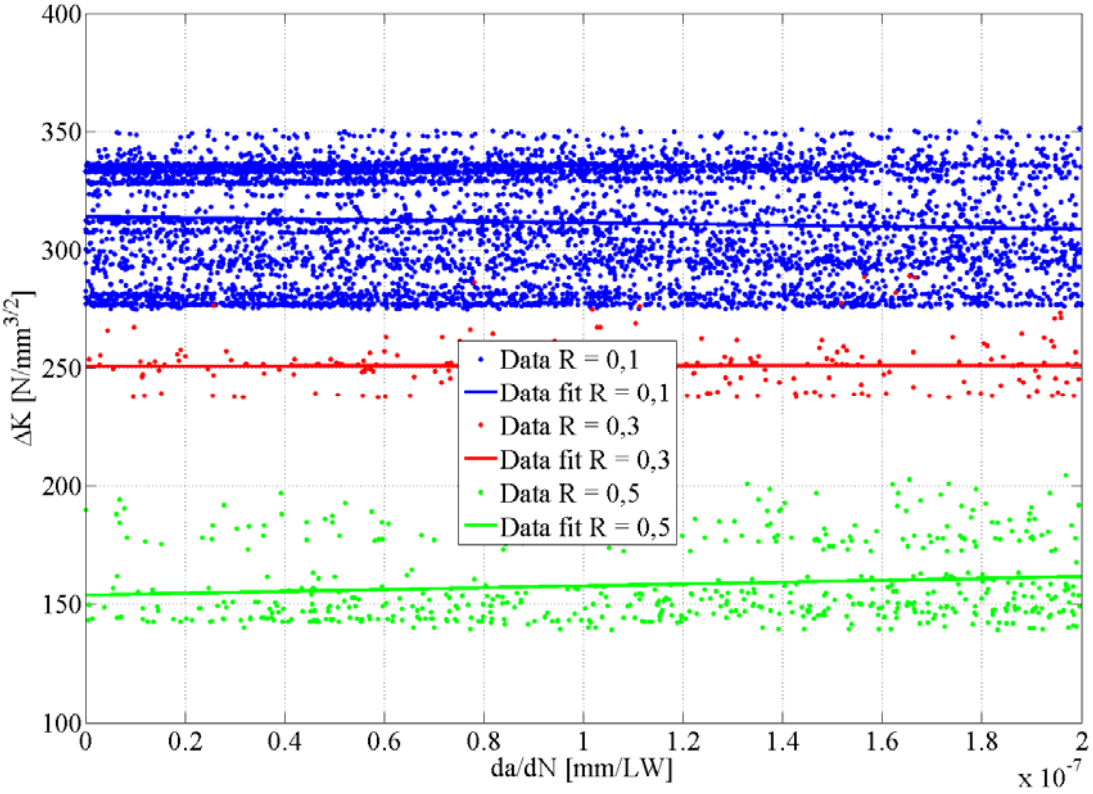

Figure 1: Determination of threshold values from crack propagation data by DLR-method

In the second step of the adaption the coefficients C, n, p and q are calculated by use of a search algorithm with nested intervals. Therefore, the four-dimensional search domain is divided into intervals. In every iteration loop a weighted effective error is calculated for every parameter combination. The parameter combination leading to a minimum error is used as center for the search domain of the next iteration loop. Additionally, the search domain is reduced, so that the intervals become smaller. Due to the nested intervals a convergence behavior of the algorithm is realized. Since error values of the threshold domain are much smaller compared to those of e.g. the K_C-domain the error values are weighted with the inverse of the crack propagation. For the adaption a fracture toughness $K_C = 4.200$ N/mm$^{3/2}$ is used. The results of the adaption together with the crack propagation data of 42CrMo4 are shown in Figure 3. The appropriated mean fitting coefficients are listed in Table 1 and consists with a probability of survival $P = 50$ %.

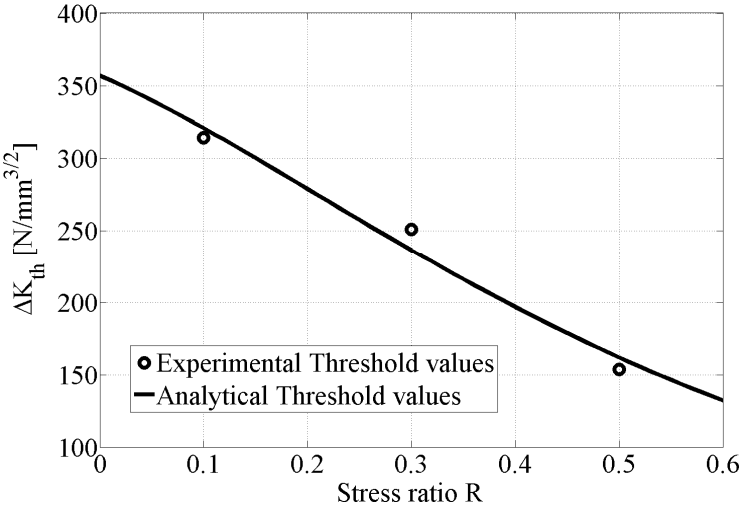

Figure 2: Experimental and analytical threshold values

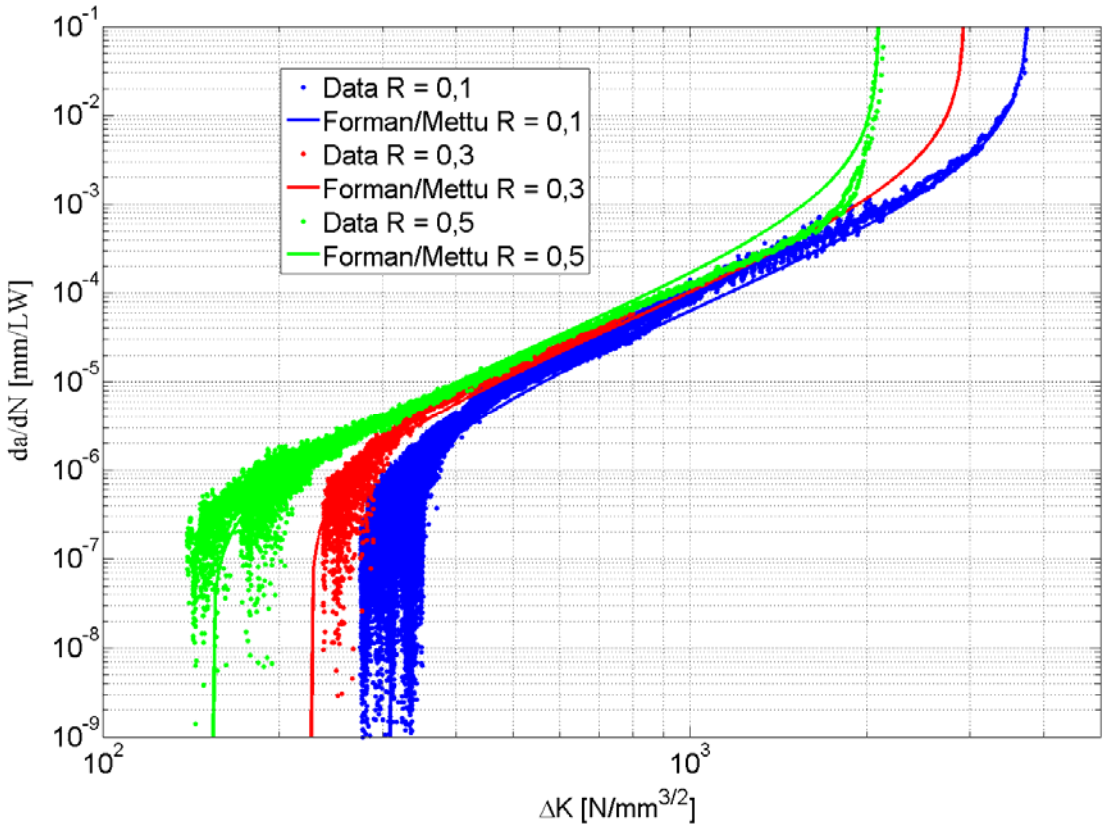

Figure 3: Crack propagation data of 42CrMo4 and the adaption by the *Forman/Mettu*-equation

3.2. Statistical analysis

The statistical analysis of the crack propagation data is divided in three steps. First for each stress ratio quantile curves are calculated for different probabilities of survival. Next, for each probability of survival the quantile curves are fitted by the *Forman/Mettu*-equation to obtain the fitting coefficients. Those are statistically analyzed in the last step to obtain their distribution functions.

To determine quantile curves the logarithm is taken from the crack propagation data. Afterwards the co-domain of the crack propagation data is divided into intervals. For each interval a polynomial regression function and a confidence interval for a pretended confidence probability are calculated. Therewith and for each domain discrete values are calculated for the mean and the upper and lower bound of the confidence interval. By use of interpolation functions the transition between the domains is smoothed and equidistant spaced values are calculated. Taking the antilogarithm the three quantile curves are obtained e.g. for a stress ratio $R = 0{,}1$, Figure 4. This procedure is also applied to the crack propagation data of the other stress ratios.

Every quantile curve corresponds to a probability of survival P. Taking the quantile curves for one probability of survival the fitting coefficients of the threshold value and the *Forman/Mettu*-equation are determined by use of the adaption program. For the statistical analysis of the crack propagation data of 42CrMo4 the probabilities of survival $P = 5\,\%$, $P = 50\,\%$ and $P = 95\,\%$ are used. The corresponding fitting coefficients are listed in Table 1. As can be seen the analytical crack propagation curve is shifted to top left with increasing probability of survival.

In the last step every fitting coefficient is statistically analyzed. This contains the choice of a distribution function and the calculation of the mean value and the standard deviation. For all fitting coefficients a normal distribution is used to describe the scattering, except of the coefficient C, Table 1. Here, a logarithmic normal distribution leads to the best results.

Table 1: Fitting coefficients of the *Forman/Mettu*-equation and their distribution functions

Parameter	ΔK_1 [N/mm$^{3/2}$]	C_{th}^{+}	C [mm/LW]	n	p	q	K_C [N/mm$^{3/2}$]
$P = 5$ %	71,0	3,1	$8{,}6 \cdot 10^{-12}$	2,5	0,65	1,1	4350
$P = 50$ %	55,75	3,4	$1{,}2 \cdot 10^{-11}$	2,4	0,8	0,9	4200
$P = 95$ %	42,0	3,83	$1{,}8 \cdot 10^{-11}$	2,3	1,0	0,7	4050
Distribution	normal	normal	log-normal	normal	normal	normal	normal
μ	56,3	3,44	-10,91	2,4	0,82	0,9	4200
s	8,81	0,22	0,0975	0,06	0,11	0,12	91,2

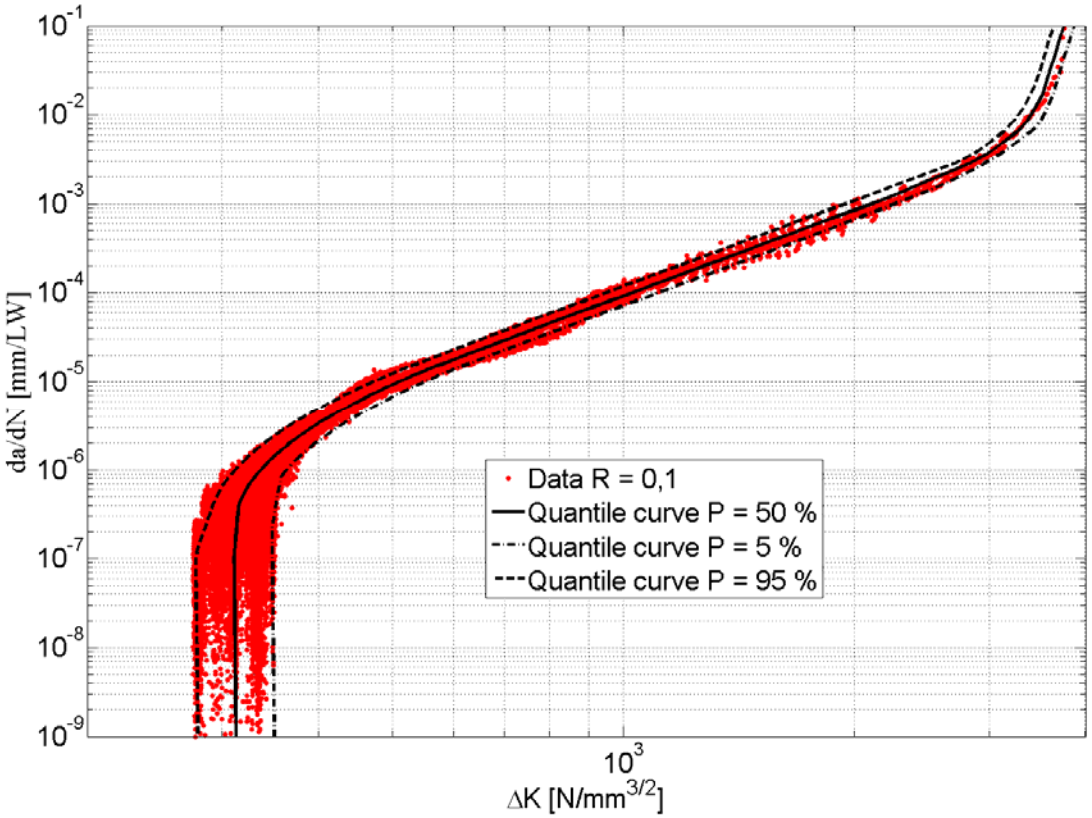

Figure 4: Crack propagation data of $R = 0{,}1$ and the adaption by quantile curves

4. Stochastic crack propagation simulation

To perform a sensitivity analysis of the fitting coefficients and to deduce statistical secured residual lifetimes stochastic crack propagation simulations were carried out by use of the analytical crack propagation software NASGRO 6.0 [9]. In a MATLAB script random input vectors of the fitting coefficients were generated using the determined distribution functions and a random number generator. To assure a probability of failure $P_A = 1$ % the input vectors contain of 10.000 elements, according to equation (12). For every element the MATLAB script generates an input file, starts the crack propagation simulation and reads in the residual lifetime.

The crack problem is a semi-elliptical surface crack in a hollow cylinder. The cylinder is loaded by a positive constant mean stress and a reverse bending stress. The amplitude of the bending stress is defined by a load frequency distribution. The initial crack size and the load were adapted, that the

frequency distribution is at least repeated 30 times to prevent load sequence effects [16]. The non-interaction material model was used.

4.1. Sensitivity analysis of the fitting coefficients on the residual lifetime

To identify the dependencies of the residual lifetime a sensitivity analysis was performed by use of the software Visual-XSel 12.0 [17]. The input factors are the seven fitting coefficients C, n, p, q, ΔK_1, C_{th}^+ and K_C. The output factor is the residual lifetime. To derive the sensitivities a regression analysis was performed by use of a quadratic model with interaction effects [17]. To increase the accuracy of the regression model the output factor was transformed by the natural logarithm, which followed from the Box-Cox-transformation. The reduced model, containing only the significant factors, explains 99,8 % of the data. The relative effects of the input factors are plotted in Figure 5. As can be seen the coefficients C, n and q are inversely proportional to the residual lifetime.

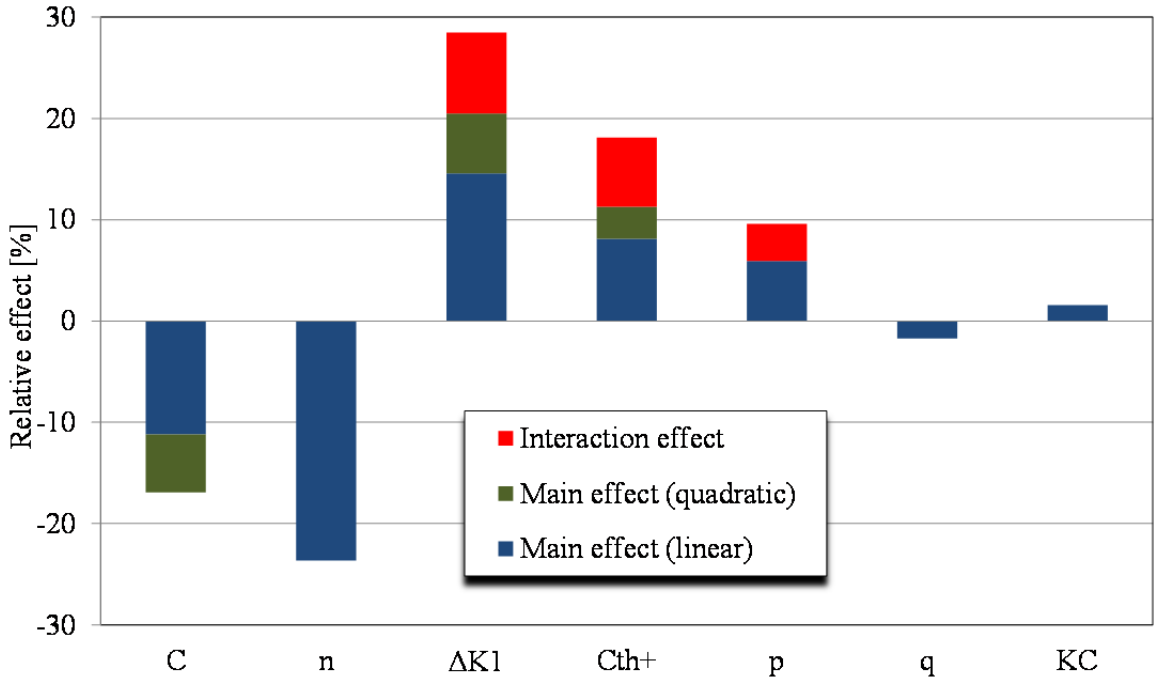

Figure 5: Relative effects of the fitting coefficients on the residual lifetime

Furthermore, it is obvious that the coefficients q and K_C are less significant and thus are not required as stochastic parameters in the stochastic crack propagation simulation. The coefficients C, n, ΔK_1, C_{th}^+ and p are significant. That means they have to be statistically analyzed and are required for the stochastic crack propagation simulation. Although, p has a small effect it cannot be neglected, due to the interaction effects between ΔK_1 and C_{th}^+.

The results of the sensitivity analysis are inconsistent to the statistical analysis of crack propagation data in literature [2, 10 – 14]. On the one hand it is not necessary to statistically analyze K_C in terms of a residual lifetime calculation. On the other hand a statistical analysis should include the coefficients n, p and C_{th}^+.

4.2. Determination of statistical secured residual lifetime

The stochastic crack propagation simulations with stochastic input vectors, created by a random number generator correspond to a basic Monte Carlo simulation. Additionally and based on the results of the sensitivity analysis a second stochastic crack propagation simulation was performed in

which only the five significant parameters were stochastically modeled. Afterwards, the residual lifetimes were statistically analyzed by use of the distribution fitting toolbox in MATLAB. The scattering of the residual lifetimes corresponds to a logarithmic normal distribution, Figure 6. By knowledge of the distribution function residual lifetimes can be calculated for arbitrary probabilities of failure. Furthermore it is able to determine the range of scattering

$$T_N = 1{:}\frac{N_{P=90\%}}{N_{P=10\%}} = 1{:}\frac{3{,}0 \cdot 10^6}{0{,}83 \cdot 10^5} = 1{:}3{,}6 \tag{13}$$

which is equal for both stochastic simulations. So the results of the sensitivity analysis are confirmed and the coefficients q and K_C are insignificant.

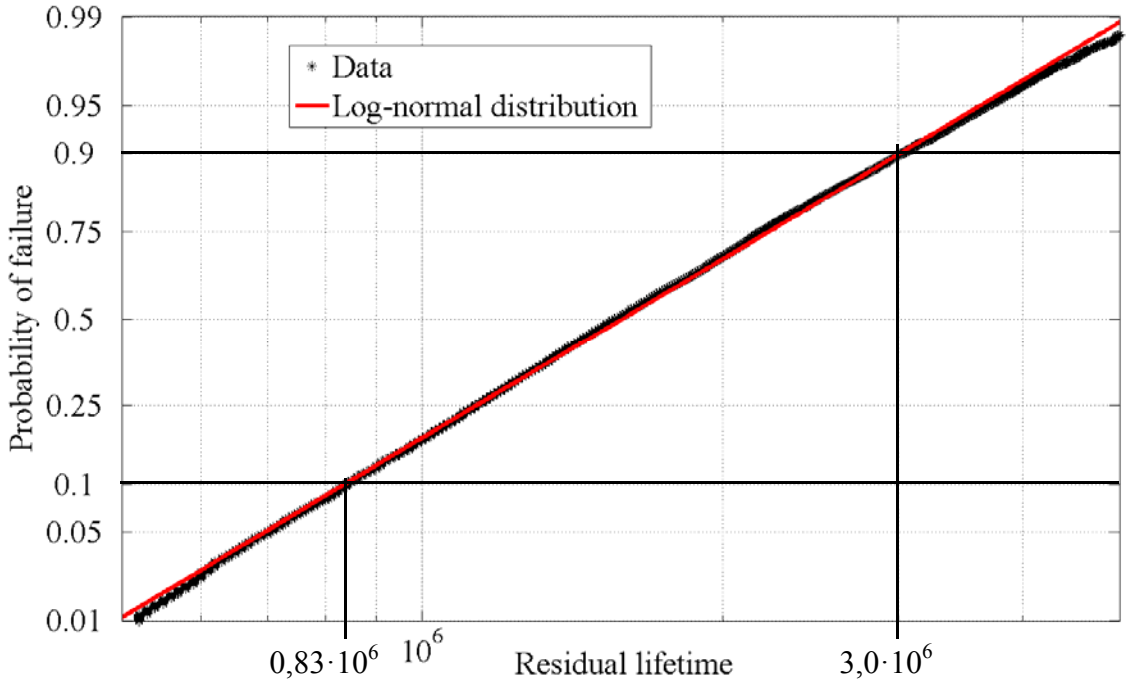

Figure 6: Residual lifetimes of stochastic crack propagation simulation with 7 stochastic parameters

5. Conclusion

In the current investigations crack propagation data of 42CrMo4 are analyzed. This includes the determination of fitting coefficients of the *Forman/Mettu*-equation which describes the crack propagation curve analytically. For the automated adaption of the crack propagation curve a MATLAB program was developed. Herewith, first the experimental threshold values are calculated and analytically described by use of a one criteria concept. Next the whole crack propagation curve is adapted by the *Forman/Mettu*-equation. Therefore, the limits of cyclic crack propagation (threshold value and fracture toughness) are required. For the determination of the optimum fitting coefficients a search algorithm with nested intervals is used. Problematically in this context is the calculation of an error value by reason that the co-domain reaches about eight decades.

Furthermore, the crack propagation data is statistically analyzed. By dividing the co-domain in several intervals regression functions and confidence intervals are calculated for each domain. Therewith, discrete quantile curves for pretended probabilities of survival are calculated using interpolation functions. From the adaption of quantile curves with constant probability of survival by the *Forman/Mettu*-equation the corresponding fitting coefficients are determined. The statistical analysis of these fitting coefficients leads to its distribution functions.

To obtain statistically secured residual lifetimes stochastic crack propagation simulations are

performed by use of MATLAB and the analytical crack propagation software NASGRO. Therefore, the significant parameters are stochastically modeled. To identify the significant parameters of the *Forman/Mettu*-equation a sensitivity analysis was performed. The results are inconsistence to the statistical analysis of crack propagation data used in literature. For instance the fracture toughness is insignificant to the scattering of the residual lifetime and can be neglected in the stochastic crack propagation simulation. As simulation method the basic Monte Carlo simulation was used. The stochastic input vectors are determined by a random number generator. After the simulation the residual lifetimes are statistically analyzed. Thus, it is possible to obtain residual lifetimes for pretended probabilities of survival.

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